POSITIVE IMPLICATIVE ENERGETIC SUBSETS OF BCK-ALGEBRAS

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Abstract. The notion of PI-energetic subsets in BCK-algebras is introduced, and several properties are investigated. Characterizations of I-energetic subsets and PI-energetic subsets are discussed, and conditions for a subset to be an I-energetic subset and a PI-energetic subset are provided. Relations between I-energetic subsets and PI-energetic subsets are considered, and conditions for an I-energetic subset to be a PI-energetic subset are given. Using PI-energetic subset, a positive implicative ideal is constructed. The condensational property of PI-energetic subset is established.
Keywords: S-energetic subset, I-energetic subset, PI-energetic subset

1. Introduction

Jun et al. [2] introduced the notions of S-energetic subsets and I-energetic subsets in BCK/BCI-algebras, and investigated several properties.

In this paper, we introduce the notion of PI-energetic subsets in BCK-algebras, and investigate several properties. We consider characterizations of I-energetic subsets and PI-energetic subsets. We provide conditions for a subset to be an I-energetic subset and a PI-energetic subset. We discuss relations between I-energetic subsets and PI-energetic subsets. We give conditions for an I-energetic subset to be a PI-energetic subset. Using PI-energetic subset, we make a positive implicative ideal. We establish the condensational property of PI-energetic subset.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

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An algebra $(X;*,0)$ of type $(2,0)$ is called a *BCI-algebra* if it satisfies the following conditions

(I) $(\forall x,y,z\in X)\ (((x*y)*(x*z))*(z*y)=0)$,

(II) $(\forall x,y\in X)\ ((x*(x*y))*y=0)$,

(III) $(\forall x\in X)\ (x*x=0)$,

(IV) $(\forall x,y\in X)\ (x*y=0, y*x=0\ \Rightarrow\ x=y)$.

If a BCI-algebra X satisfies the following identity

(V) $(\forall x\in X)\ (0*x=0)$,

then X is called a *BCK-algebra.* Any BCK/BCI-algebra X satisfies the following conditions:

(2.1) $$(\forall x\in X)\,(x*0=x)\,,$$

(2.2) $$(\forall x,y,z\in X)\,(x\le y\ \Rightarrow\ x*z\le y*z,\ z*y\le z*x)\,,$$

(2.3) $$(\forall x,y,z\in X)\,((x*y)*z=(x*z)*y)\,,$$

(2.4) $$(\forall x,y,z\in X)\,((x*z)*(y*z)\le x*y)$$

where $x\le y$ if and only if $x*y=0$. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x*y\in S$ for all $x,y\in S$. A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies

(2.5) $$0\in I,$$

(2.6) $$(\forall x\in X)\,(\forall y\in I)\,(x*y\in I\ \Rightarrow\ x\in I)\,.$$

Every ideal I of a BCK/BCI-algebra X satisfies the following condition:

(2.7) $$(\forall x,y\in X)\,(x\in I,\ y\le x\ \Rightarrow\ y\in I)\,.$$

A subset I of a BCK-algebra X is called a *positive implicative ideal* (see [3]) of X if it satisfies (2.5) and

(2.8) $$(\forall x,y,z\in X)\,((x*y)*z\in I,\ y*z\in I\ \Rightarrow\ x*z\in I)\,.$$

Observe that every positive implicative ideal is an ideal, but the converse is not true (see [3]).

We refer the reader to the books [1, 3] for further information regarding BCK/BCI-algebras.

3. Energetic subsets

In what follows, let X denote a BCK-algebra unless otherwise specified.

Definition 3.1 ([2]). A nonempty subset A of X is said to be S-*energetic* if it satisfies

$$(\forall a, b \in X)\,(a * b \in A \;\Rightarrow\; \{a, b\} \cap A \neq \emptyset)\,. \tag{3.1}$$

Definition 3.2 ([2]). A nonempty subset A of X is said to be I-*energetic* if it satisfies

$$(\forall x, y \in X)\,(y \in A \;\Rightarrow\; \{x, y * x\} \cap A \neq \emptyset)\,. \tag{3.2}$$

Proposition 3.3. *Every I-energetic subset A of X which does not contain zero element 0 satisfies the following properties:*

$$(\forall x, y \in X)\,(x \leq y,\; x \in A \;\Rightarrow\; y \in A)\,. \tag{3.3}$$

Proof. Let $x \leq y$ for $x \in A$ and $y \in X$. Then

$$\{y, 0\} \cap A = \{y,\; x * y\} \cap A \neq \emptyset.$$

Since $0 \notin A$, it follows that $y \in A$. □

Theorem 3.4. *For a nonempty subset A of X which does not contain zero element 0, the following are equivalent:*

(1) *A is I-energetic.*

(2) $(\forall x, y, z \in X)\,(z * y \leq x,\; z \in A \;\Rightarrow\; \{x, y\} \cap A \neq \emptyset)\,.$

Proof. Assume that A is I-energetic and let $z * y \leq x$ for $z \in A$ and $x, y \in X$. Then $\{y,\; z * y\} \cap A \neq \emptyset$ by (3.2). Hence $y \in A$ or $z * y \in A$. If $y \in A$, then clearly $\{x, y\} \cap A \neq \emptyset$. If $z * y \in A$, then $\{x, 0\} \cap A = \{x,\; (z * y) * x\} \cap A \neq \emptyset$ by (3.2). Since $0 \notin A$, it follows that $x \in A$ and so $\{x, y\} \cap A \neq \emptyset$.

Conversely, suppose that (2) is valid and let $x \in A$. Since $x * (x * y) \leq y$, it follows that $\{y, x * y\} \cap A \neq \emptyset$. Therefore A is I-energetic. □

Theorem 3.5. *If a nonempty subset A of X satisfies the following condition:*

$$(\forall x, y, z \in X)\,(x * y \in A \;\Rightarrow\; \{z,\; ((x * y) * y) * z\} \cap A \neq \emptyset)\,, \tag{3.4}$$

then A is I-energetic.

Proof. Assume that $x \in A$. Then $x * 0 = x \in A$ by (2.1), which implies from (2.1) and (3.4) that

$$\{z,\; x * z\} \cap A = \{z,\; ((x * 0) * 0) * z\} \cap A \neq \emptyset$$

for all $z \in X$. Hence A is I-energetic. □

Lemma 3.6 ([2])**.** *Let A be a nonempty subset of X with $0 \notin A$. If A is I-energetic, then $X \setminus A$ is an ideal of X.*

Using Theorem 3.5 and Lemma 3.6, we have the following corollary.

Corollary 3.7. *Let A be a nonempty subset of X with $0 \notin A$. If A satisfies the condition* (3.4)*, then $X \setminus A$ is an ideal of X.*

Definition 3.8. A nonempty subset A of X is said to be *positive implicative energetic* (briefly, *PI-energetic*) if it satisfies

$$(3.5) \qquad (\forall x, y, z \in X)\,(x * z \in A \;\Rightarrow\; \{(x * y) * z, y * z\} \cap A \neq \emptyset)\,.$$

Example 3.9. (1) Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	2	0
3	3	1	1	0	1
4	4	4	4	4	0

It is routine to verify that $A := \{2, 4\}$ is a PI-energetic subset of X.

(2) Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	1	0
2	2	2	0	2	0
3	3	3	3	0	3
4	4	4	4	4	0

Then $A_1 := \{1, 2, 4\}$, $A_2 := \{2, 3, 4\}$ and $A_3 := \{2, 4\}$ are PI-energetic subsets of X.

Example 3.10. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	2	0
3	3	1	3	0	1
4	4	4	4	4	0

The set $A := \{0, 3\}$ is not a PI-energetic subset of X since $3 * 2 = 3 \in A$ but

$$\{(3*1)*2, 1*2\} \cap A = \emptyset.$$

Proposition 3.11. *Let A be a PI-energetic subset of X. If A does not contain zero element 0, then the following implication is valid.*

$$(\forall x, y \in X)\,(x*y \in A \;\Rightarrow\; (x*y)*y \in A)\,. \tag{3.6}$$

Proof. Let $x, y \in X$ be such that $x*y \in A$. Then

$$\{0,\ (x*y)*y\} \cap A = \{y*y,\ (x*y)*y\} \cap A \neq \emptyset$$

by (III) and (3.5). Since $0 \notin A$, it follows that $(x*y)*y \in A$. □

We provide conditions for a subset to be PI-energetic.

Theorem 3.12. *Let A be a nonempty subset of X which does not contain zero element 0. If A satisfies the condition (3.4), then A is PI-energetic.*

Proof. Let $x, z \in X$ be such that $x*z \in A$. If A is not PI-energetic, then there exists $y \in X$ such that

$$\{y*z,\ (x*y)*z\} \cap A = \emptyset.$$

Hence $y*z \in X \setminus A$ and $(x*y)*z \in X \setminus A$. Since

$$((x*z)*z)*(y*z) \leq (x*y)*z$$

and $X \setminus A$ is an ideal of X, it follows from (2.7) that $((x*z)*z)*(y*z) \in X \setminus A$. Thus

$$\{y*z,\ ((x*z)*z)*(y*z)\} \cap A = \emptyset,$$

which is contradictory to the condition (3.4). Therefore

$$\{(x*y)*z,\ y*z\} \cap A \neq \emptyset$$

whenever $x*z \in A$ for all $x, y, z \in X$, and so A is PI-energetic. □

Theorem 3.13. *For any nonempty subset A of X, if $X \setminus A$ satisfies the condition (2.8), then A is PI-energetic.*

Proof. Assume that A is not PI-energetic. Then for any $x, z \in X$ with $x*z \in A$, there exists $y \in X$ such that $\{(x*y)*z,\ y*z\} \cap A = \emptyset$. It follows that

$$(x*y)*z \in X \setminus A \text{ and } y*z \in X \setminus A.$$

Since $X \setminus A$ satisfies the condition (2.8), we have $x*z \in X \setminus A$, that is, $x*z \notin A$. This is a contradiction, and so A is a PI-energetic subset of X. □

Corollary 3.14. *For any nonempty subset A of X, if $X \setminus A$ satisfies the condition (2.8), then A is I-energetic.*

Corollary 3.15. *For any nonempty subset A of X, if $X \setminus A$ is a positive implicative ideal of X, then A is PI-energetic and so I-energetic.*

The converse of Corollary 3.15 is not true in general as seen in the following example.

Example 3.16. Consider a BCK-algebra $X = \{0, 1, 2, 3\}$ with the following Cayley table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	3	0

It is routine to verify that $A := \{0, 1\}$ is a PI-energetic subset of X, but $X \setminus A = \{2, 3\}$ is not a positive implicative ideal of X.

We consider relations between an I-energetic subset and a PI-energetic subset.

Theorem 3.17. *Every PI-energetic subset is I-energetic.*

Proof. Let A be a PI-energetic subset of X. Assume that $x \in A$. Since $x * 0 = x \in A$, it follows from (2.1) and (3.5) that $\{y,\ x*y\} \cap A = \{y*0,\ (x*y)*0\} \cap A \neq \emptyset$, for all $y \in X$. Hence A is an I-energetic subset of X. □

The converse of Theorem 3.17 is not true as seen in the following example.

Example 3.18. Let $X = \{0, 1, 2, 3, 4\}$ be a BCK-algebra with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

The set $A := \{1, 2, 3\}$ is an I-energetic subset of X, but it is not PI-energetic since $3 * 2 = 1 \in A$ but $\{(3 * 2) * 2, 2 * 2\} \cap A = \emptyset$.

We know that the I-energetic subset $A := \{1, 2, 3\}$ in Example 3.18 does not satisfy the condition (3.6). We consider conditions for an I-energetic subset to be PI-energetic.

Theorem 3.19. *Let A be an I-energetic subset of X which does not contain zero element 0. If A satisfies the condition (3.6), then A is PI-energetic.*

Proof. Let $x, z \in X$ be such that $x * z \in A$. We first show that

$$(x * z) * (y * z) \in A \;\Rightarrow\; (x * y) * z \in A \tag{3.7}$$

for all $x, y, z \in X$. Let $(x * z) * (y * z) \in A$ and assume that $(x * y) * z \notin A$, that is, $(x * y) * z \in X \setminus A$. Since

$$((x * (y * z)) * z) * z \leq (x * y) * z \tag{3.8}$$

and $X \setminus A$ is an ideal of X (see Lemma 3.6), we have $((x * (y * z)) * z) * z \in X \setminus A$ by (2.7). It follows from (2.3) and (3.6) that

$$(x * z) * (y * z) = (x * (y * z)) * z \in X \setminus A, \tag{3.9}$$

which is a contradiction. Hence (3.7) is valid. Now suppose that $\{y * z,\ (x * y) * z\} \cap A = \emptyset$ for some $y \in X$. Then $y * z \in X \setminus A$ and $(x * y) * z \in X \setminus A$. It follows from (3.7) that $(x * z) * (y * z) \in X \setminus A$ and so that $x * z \in X \setminus A$. This is a contradiction, and so $\{y * z,\ (x * y) * z\} \cap A \neq \emptyset$ whenever $x * z \in A$ for all $x, y, z \in X$. Therefore A is a PI-energetic subset of X. □

Corollary 3.20. *Every PI-energetic subset A which does not contain zero element* 0 *satisfies the condition* (3.7).

Proposition 3.21. *Let A be a nonempty subset of X which does not contain zero element* 0*. If A satisfies the condition* (3.7)*, then the condition* (3.4) *is valid.*

Proof. Let $x * y \in A$ for $x, y \in X$. Assume that $\{z,\ ((x * y) * y) * z\} \cap A = \emptyset$ for some $z \in X$. Then $z \in X \setminus A$ and $((x * z) * y) * y = ((x * y) * y) * z \in X \setminus A$. It follows from (III), (2.1), (2.3) and (3.7) that $(x * y) * z = ((x * z) * y) * (y * y) \in X \setminus A$. Since $X \setminus A$ is an ideal of X, we have $x * y \in X \setminus A$, which is a contradiction. Therefore (3.4) is valid. □

Corollary 3.22. *Let A be an I-energetic subset of X which does not contain zero element* 0*. If A satisfies the condition* (3.6)*, then A also satisfies the condition* (3.4).

For any nonempty subset A of X, consider a set

$$E_a := \{x \in X \mid x * a \in A\}. \tag{3.10}$$

Theorem 3.23. *Let A be an I-energetic subset of X. Then A is PI-energetic if and only if E_a is an I-energetic subset of X for all $a \in X$.*

Proof. Assume that A is PI-energetic and let $y \in E_a$. Then $y * a \in A$, and so $\{x * a,\ (y * x) * a\} \cap A \neq \emptyset$ by (3.5). Thus $x * a \in A$ or $(y * x) * a \in A$, that is, $x \in E_a$ or $y * x \in E_a$. It follows that $\{x,\ y * x\} \cap E_a \neq \emptyset$. Therefore E_a is an I-energetic subset of X for all $a \in X$.

Conversely, suppose that E_a is an I-energetic subset of X for all $a \in X$. Let $x * z \in A$ for $x, z \in X$. Then $x \in E_z$, and so $\{y,\ x * y\} \cap E_z \neq \emptyset$. It follows that $y \in E_z$ or $x * y \in E_z$ and so that $y * z \in A$ or $(x * y) * z \in A$. Hence $\{y * z,\ (x * y) * z\} \cap A \neq \emptyset$, and therefore A is PI-energetic. □

Given a PI-energetic subset A of X, we provide a condition that $X \setminus A$ is a positive implicative ideal of X.

Theorem 3.24. *Let A be a nonempty subset of X with $0 \notin A$. If A is PI-energetic, then $X \setminus A$ is a positive implicative ideal of X.*

Proof. Since $0 \notin A$, we have $0 \in X \setminus A$. Let $x, y, z \in X$ be such that $y * z \in X \setminus A$ and $(x * y) * z \in X \setminus A$. Assume that $x * z \in A$. Then $\{(x * y) * z,\ y * z\} \cap A \neq \emptyset$ by (3.5), which implies that $y * z \in A$ or $(x * y) * z \in A$. This is a contradiction, and so $x * z \in X \setminus A$. This shows that $X \setminus A$ is a positive implicative ideal of X. □

Corollary 3.25. *Let A be a nonempty subset of X with $0 \notin A$. If A is PI-energetic, then $X \setminus A$ is an ideal and hence a subalgebra of X.*

Let A be an I-energetic subset and B a PI-energetic subset of X such that $A \subseteq B$ and $0 \notin B$. Assume that $(x * z) * (y * z) \in A$ for $x, y, z \in X$. Then $(x * z) * (y * z) \in B$ and so $(x * y) * z \in B$ by Corollary 3.20. If $(x * y) * z \notin A$, then $((x * ((x * y) * z)) * y) * z = ((x * y) * z) * ((x * y) * z) = 0 \in X \setminus B$ by (2.3) and (III). Since $X \setminus B$ is a positive implicative ideal of X, it follows that $((x*z)*(y*z))*((x*y)*z) = ((x*((x*y)*z))*z)*(y*z) \in X \setminus B \subseteq X \setminus A$. Since $X \setminus A$ is an ideal of X, we have $(x * z) * (y * z) \in X \setminus A$ which is a contradiction. Therefore $(x * y) * z \in A$, and so A satisfies the condition (3.7). Consequently, A is a PI-energetic subset of X.

We summarize this as a theorem, so called the Condensational Property of PI-energetic set.

Theorem 3.26 (Condensational Property of PI-energetic set)**.** *Let A and B be I-energetic subsets of X such that $A \subseteq B$ and $0 \notin B$. If B is PI-energetic, then so is A.*

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