THE MAXIMUM PRINCIPLE OF TSALLIS ENTROPY IN A COMPLEX DOMAIN

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Abstract. In this paper, we extend the concept of Tsallis entropy in a complex domain. Based on this extension, we define some new classes of analytic functions (type Schwarz function). Other geometric properties are illustrated in the sequel. Our technique is delivered by the Maxwell Lemma and Jack Lemma.
Keywords: Fractional calculus, fractional entropy, analytic function, subordination and superordination.

1. Introduction

The theory of operators concerns with the study of different classes of functions on function spaces, establishing with differential operators and integral operators. The operators may be unfilled abstractly by their features, such as compact, bounded or closed operators. This study leads to two types of operators: linear operators and nonlinear operators. The study itself depends on the topological or geometrical properties of the spaces of functions. An amount of motivating physical schemes stands to entropic formulas that are additional commoner than the ordinary entropy. In 1988, Tsallis introduced a new type of fractional entropy. The Tsallis Entropy has been utilized beside with the Principle of maximum entropy to develop the Tsallis distribution. This entropy has been employed in many fields such as thermodynamics, chaos, statistical mechanics and information theory. For continuous probability distributions, the entropy is defined by the form:

$$\mathcal{T}_\mu[\phi] = \frac{1}{\mu-1}\left(1 - \int (\phi(x))^\mu \, dx\right), \quad \mu \neq 1,$$
$$or \mathcal{T}_\mu[\phi](x) = \frac{1}{\mu-1}\left(1 - \int_0^x (\phi(u))^\mu \, du\right),$$

where $\phi(x)$ is a probability density function.

Here, we extend the definition into the unit disk $U := \{z : |z| < 1\}$ by applying the analytic function $\varphi(z), z \in U$ (type Schwarz function $|\varphi(z)| < |z|$,)

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where it is normalized by $\varphi(0) = 0, \varphi'(0) = 1$. The class of all normalized functions is denoted by $\mathcal{A}_\varphi$. For integrals of the second type defined by [1]-[3]:

$$\mathcal{J}_\mu[\varphi](z) = \int_0^z (\varphi'(t))^\mu dt = \frac{1 - \mathcal{J}_\mu[\varphi](z)}{\mu - 1} \quad \mu \neq 1,\, z \in U.$$

In this note, we suggest the differential entropy, because this case enables us to study many geometric properties and information regarding the classes of analytic functions. Receiving information about the properties of a function from properties of its derivatives indicates a significant part in many areas of mathematical analysis.

The function φ has enough information in each geometric classes; such as the starlikeness $\Re(z\varphi'(z)/\varphi(z)) > \lambda, \lambda \in [0, 1)$ and convexity $\Re(1 + z\varphi''(z)/\varphi'(z)) > \lambda, \lambda \in [0, 1)$, if the complex entropy satisfies $\Re(\mathcal{T}_\mu[\varphi](z)) > 0$, which is equivalent to $\Re(\rho(z)) > 0, \mu > 1$, where $\rho(z) := 1 - \mathcal{J}_\mu[\varphi](z)$. The set of information for each geometric class can be realized by the conclusion

$$\mathcal{I}_\varphi^\mu = \Big\{\varphi \in A_\varphi : 0 < \Re\Big(\mathcal{T}_\mu[\varphi](z)\Big) < \frac{\mu}{\mu - 1},\, z \in U,\, \mu \neq 1\Big\}.$$

Our discussion is based on the Maxwell Lemma as well as Jack Lemma respectively

Lemma 1.1 ([4])**.** *If ϵ is real and ρ is analytic in the unit disk, then*

$$\Re\big(\rho(z) + \epsilon z\rho'(z)/\rho(z)\big) > 0 \Longrightarrow \Re\big(\rho(z)\big) > 0.$$

Lemma 1.2 ([5])**.** *Let $\psi(z)$ be analytic in U with $\psi(0) = 0$. Then if $|\psi(z)|$ approaches its maximization when $|z| = r$ at a point $z_0 \in U$, then $z_0\psi'(z_0) = \kappa\psi(z_0)$, where $\kappa \geq 1$ is a real number.*

Moreover, we need the subordination concept in the sequel, which is defined as follows: Assume that $\alpha(z)$ and $\beta(z)$ are two analytic functions in U. Then $\alpha(z)$ is said to be subordinate to $\beta(z)$ if there exists an analytic function $\psi(z)$ in U satisfying $\psi(0) = 0, |\psi(z)| < 1 (z \in U)$ and $\alpha(z) = \beta(\psi(z))$. This subordination is noted by $\alpha(z) \prec \beta(z), z \in U$.

2. The main finding

Our aim is to achieve the property of the set $\mathcal{I}_\varphi^\mu$.

Theorem 2.1. *Let $\varphi \in \mathcal{A}_\varphi, \mu \geq 2$, and $1 < \lambda < 2$. If φ satisfies*

$$0 < \Re\Big(1 + \frac{z\varphi''(z)}{\varphi'(z)}\Big) < \frac{\lambda(2 - \lambda)}{2(1 + \lambda)}, \tag{1}$$

then $\varphi \in \mathcal{I}_\varphi^\mu$. Moreover, φ is starlike in the open unit disk.

Proof. First, we show that $\Re(\mathcal{T}_\mu[\varphi](z)) < \frac{\mu}{\mu-1}$. Define a function ψ as follows:

$$\psi(z) = \mathcal{T}_\mu[\varphi](z) - \frac{1}{\mu-1},\ \mu \neq 1$$

such that $\psi'(z) = (\mathcal{T}_\mu[\varphi](z))'$ and $\frac{z\varphi'(z)}{\varphi(z)} = \frac{\lambda(1-\psi(z))}{\lambda-\psi(z)}, \psi(z) \neq \lambda$. It is clear that ψ is analytic because φ is analytic in the open disk U. In addition, $\psi(0) = 0$, we need only to show that $|\psi(z)| < 1$. Also, implicitly it is containing λ and μ. A computation implies that

$$\begin{aligned}
&\Re\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) \\
&= \Re\left(\frac{\lambda(1-\psi(z))}{\lambda-\psi(z)} - \frac{z\psi'(z)}{1-\psi(z)} + \frac{z\psi'(z)}{\lambda-\psi(z)}\right) \\
&= \Re\left(\frac{\lambda(1-[\mathcal{T}_\mu[\varphi](z) - \frac{1}{\mu-1}])}{\lambda - [\mathcal{T}_\mu[\varphi](z) - \frac{1}{\mu-1}]} - \frac{z(\mathcal{T}_\mu[\varphi](z))'}{1-[\mathcal{T}_\mu[\varphi](z) - \frac{1}{\mu-1}]} + \frac{z(\mathcal{T}_\mu[\varphi](z))'}{\lambda - [\mathcal{T}_\mu[\varphi](z) - \frac{1}{\mu-1}]}\right) \\
&= \Re\left(\frac{\lambda(1+\frac{1}{\mu-1} - \mathcal{T}_\mu[\varphi](z))}{\lambda + \frac{1}{\mu-1} - \mathcal{T}_\mu[\varphi](z)} - \frac{z(\mathcal{T}_\mu[\varphi](z))'}{1+\frac{1}{\mu-1} - \mathcal{T}_\mu[\varphi](z)} + \frac{z(\mathcal{T}_\mu[\varphi](z))'}{\lambda + \frac{1}{\mu-1} - \mathcal{T}_\mu[\varphi](z)}\right) \\
&< \frac{\lambda(2-\lambda)}{2(1+\lambda)}.
\end{aligned}$$

In view of Lemma 1.2, there exists a complex number $z_0 \in U$ such that $\psi(z_0) = e^{i\theta}$ and $z_0\psi'(z_0) = \kappa\psi(z_0) = \kappa e^{i\theta}, \kappa \geq 1$. Therefore, we obtain

$$\begin{aligned}
&1 + \frac{z_0\varphi''(z_0)}{\varphi'(z_0)} \\
&= \frac{\lambda(1+\frac{1}{\mu-1} - \mathcal{T}_\mu[\varphi](z_0))}{\lambda + \frac{1}{\mu-1} - \mathcal{T}_\mu[\varphi](z_0)} - \frac{z_0(\mathcal{T}_\mu[\varphi](z_0))'}{1+\frac{1}{\mu-1} - \mathcal{T}_\mu[\varphi](z_0)} + \frac{z_0(\mathcal{T}_\mu[\varphi](z_0))'}{\lambda + \frac{1}{\mu-1} - \mathcal{T}_\mu[\varphi](z_0)} \\
&= \frac{\lambda(1-e^{i\theta})}{\lambda - e^{i\theta}} - \frac{\kappa e^{i\theta}}{1-e^{i\theta}} + \frac{\kappa e^{i\theta}}{\lambda - e^{i\theta}} = \frac{\lambda + (\kappa-\lambda)e^{i\theta}}{\lambda - e^{i\theta}} - \frac{\kappa e^{i\theta}}{1-e^{i\theta}}.
\end{aligned}$$

Since

$$\Re(\frac{1}{1-e^{i\theta}}) = \frac{1}{2}, \Re(\frac{1}{\lambda - e^{i\theta}}) = \frac{1}{2\lambda} + \frac{\lambda^2-1}{2\lambda(1+\lambda^2-2\lambda\cos(\theta))}$$

then, we attain

$$\Re\Big(1 + \frac{z_0\varphi''(z_0)}{\varphi'(z_0)}\Big) = \frac{1+\lambda}{2} + \frac{(\lambda^2-1)(1-\lambda+\kappa)}{2(1+\lambda^2-2\lambda\cos\theta)}.$$

For $\lambda \in (1,2)$, and $\kappa = 1$, we arrive at

$$\Re\Big(1 + \frac{z_0\varphi''(z_0)}{\varphi'(z_0)}\Big) \geq \frac{2-\lambda}{2(1+\lambda)} + \frac{(\lambda-1)(2-\lambda)}{2(1+\lambda)} = \frac{\lambda(2-\lambda)}{2(1+\lambda)},$$

which is a contradiction. Therefore, $|\psi(z)| < 1$, and $\Re(\mathcal{T}_\mu[\varphi](z)) < \frac{\mu}{\mu-1}$. Obviously,

$$\psi(z) = \mathcal{T}_\mu[\varphi](z) - \frac{1}{\mu-1} = \frac{\lambda(\frac{z\varphi'(z)}{\varphi(z)} - 1)}{\frac{z\varphi'(z)}{\varphi(z)} - \lambda} = \frac{\lambda(1 - \frac{z\varphi'(z)}{\varphi(z)})}{\lambda - \frac{z\varphi'(z)}{\varphi(z)}}. \tag{2}$$

Hence, we obtain $\frac{z\varphi'(z)}{\varphi(z)} \prec \frac{\lambda(1-z)}{\lambda-z}$. This conclude that φ is starlike in U. Now, we proceed to show that $\Re(\mathcal{T}_\mu[\varphi](z)) > 0$. Since φ is starlike, then we define a function $\rho(z)$ such that $\frac{z\varphi'(z)}{\varphi(z)} = \rho(z)$. If, we let $\Lambda(z) := \frac{z\varphi'(z)}{\varphi(z)} = \rho(z)$ then we receive

$$\frac{z\Lambda'(z)}{\Lambda(z)} + \Lambda(z) = 1 + \frac{z\varphi''(z)}{\varphi'(z)} = \rho(z) + \frac{z\rho'(z)}{\rho(z)}.$$

Hence, we obtain

$$\Re\Big(1 + \frac{z\varphi''(z)}{\varphi'(z)}\Big) = \Re\Big(\rho(z) + \frac{z\rho'(z)}{\rho(z)}\Big) > 0.$$

By applying Lemma 1.1, we have $\Re(\rho(z)) > 0$ and this implies that $\Re(\mathcal{T}_\mu[\varphi](z)) > 0$. This completes the proof. □

Corollary 2.1. *Let the assumptions of Theorem 2.1 hold. Then* $\int_0^z \frac{\varphi(\iota)}{\iota} d\iota$ *is convex.*

3. Applications

We illustrate some examples of functions in the set $\mathcal{I}_\varphi^\mu$ by using Theorem 2.1.

Example 3.1. Consider the function

$$\varphi(z) = \Big(\frac{1+\lambda}{2}\Big)(1-z)^{-\frac{1-\lambda}{1+\lambda}-1} - \Big(\frac{1+\lambda}{2}\Big), \quad z \in U.$$

Obviously, $\varphi(0) = 0$ and $\varphi'(0) = 1$, where $\varphi'(z) = (1-z)^{-\frac{1-\lambda}{1+\lambda}-2}, z \in U$. Now, we have

$$\begin{aligned} 0 < 1 + \Re\Big(\frac{z\varphi''(z)}{\varphi'(z)}\Big) &= 1 + (2 + \frac{1-\lambda}{1+\lambda})\Re\Big(\frac{z}{1-z}\Big), \quad z \in U \\ &= \frac{\lambda - 1}{2(1+\lambda)} < \frac{\lambda(2-\lambda)}{2(1+\lambda)}, \quad z \to^- 1, \lambda \in (1,2). \end{aligned}$$

Hence, in view of Theorem 2.1, $\varphi \in \mathcal{I}_\varphi^\mu$ and it is starlike in the open unit disk.

Example 3.2. Define the function

$$\varphi(z) = \Big(\frac{2(\lambda+1)}{\lambda^2+2}\Big)(1-z)^{-\frac{\lambda(\lambda-2)}{2(1+\lambda)}-1} - \Big(\frac{2(\lambda+1)}{\lambda^2+2}\Big), \quad z \in U.$$

Obviously, $\varphi(0) = 0$ and $\varphi'(0) = 1$, where $\varphi'(z) = (1-z)^{-\frac{\lambda(\lambda-2)}{2(1+\lambda)}-2}$. A calculation implies

$$\begin{aligned} 0 < 1 + \Re\Big(\frac{z\varphi''(z)}{\varphi'(z)}\Big) &= 1 + \big(2 + \frac{\lambda(\lambda-2)}{2(1+\lambda)}\big)\Re\Big(\frac{z}{1-z}\Big), \quad z \in U \\ &= \frac{\lambda(2-\lambda)}{4(1+\lambda)} < \frac{\lambda(2-\lambda)}{2(1+\lambda)}, \quad z \to^{-} 1, \lambda \in (1,2). \end{aligned}$$

Thus, in view of Theorem 2.1, $\varphi \in \mathcal{I}^{\mu}_{\varphi}$ and it is starlike in the open unit disk.

Example 3.3. Assume the function

$$\varphi(z) = \Big(\frac{\lambda+1}{1-\lambda}\Big)(1-z)^{-\frac{\lambda-2}{1+\lambda}-1} - \Big(\frac{\lambda+1}{1-\lambda}\Big), \quad z \in U.$$

Plainly, $\varphi(0) = 0$ and $\varphi'(0) = 1$, where $\varphi'(z) = (1-z)^{-\frac{\lambda-2}{1+\lambda}-2}$. A computation yields

$$\begin{aligned} 0 < 1 + \Re\Big(\frac{z\varphi''(z)}{\varphi'(z)}\Big) &= 1 + \big(2 + \frac{\lambda-2}{1+\lambda}\big)\Re\Big(\frac{z}{1-z}\Big), \quad z \in U \\ &= \frac{2-\lambda}{2(1+\lambda)} < \frac{\lambda(2-\lambda)}{2(1+\lambda)}, \quad z \to^{-} 1, \lambda \in (1,2). \end{aligned}$$

Thus, in view of Theorem 2.1, $\varphi \in \mathcal{I}^{\mu}_{\varphi}$ and it is starlike in the open unit disk.

Example 3.4. Consider the function

$$\varphi(z) = \Big(\frac{2(1+\lambda)}{1-\lambda}\Big)(1-z)^{-\frac{\lambda-2}{2(1+\lambda)}-1} - \Big(\frac{2(1+\lambda)}{1-\lambda}\Big), \quad z \in U.$$

clearly $\varphi(0) = 0$ and $\varphi'(0) = 1$, where $\varphi'(z) = (1-z)^{-\frac{\lambda-2}{2(1+\lambda)}-2}$. An operation gives

$$\begin{aligned} 0 < 1 + \Re\Big(\frac{z\varphi''(z)}{\varphi'(z)}\Big) &= 1 + \big(2 + \frac{\lambda-2}{2(1+\lambda)}\big)\Re\Big(\frac{z}{1-z}\Big), \quad z \in U \\ &= \frac{2-\lambda}{4(1+\lambda)} < \frac{\lambda(2-\lambda)}{2(1+\lambda)}, \quad z \to^{-} 1, \lambda \in (1,2). \end{aligned}$$

Thus, in view of Theorem 2.1, $\varphi \in \mathcal{I}^{\mu}_{\varphi}$ and it is starlike in the open unit disk.

4. Conclusion

An extended definition of Tsallis entropy in a complex domain is imposed based on the set of analytic functions. This definition leaded to define a new class of analytic function, may call the class of geometric information of analytic functions. We introduced a sufficient condition of the existing. Moreover, we studied some geometric properties (see Theorem 2.1 and Corollary 2.1). Recently, the author introduced classes of integral operators in the open unit disk [6]-[9].

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