A NOTE ON STRONGLY FULLY STABLE BANACH ALGEBRA MODULES RELATIVE TO AN IDEAL

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Abstract. Let A be a unital algebra, a Banach algebra module M is strongly fully stable Banach A-module relative to ideal K of A, if for every submodule N of M and for each multiplier $\theta : N \to M$ such that $\theta(N) \subseteq N \cap KM$. In this paper, we adopt the concept of strongly fully stable Banach Algebra modules relative to an ideal which generalizes that of fully stable Banach Algebra modules and we study the properties and characterizations of strongly fully stable Banach A-module relative to ideal K of A.

Keywords: A-module, Banach A-module, fully stable Banach A-module, strongly fully stable Banach A-module relative to ideal.

1. Introduction

In [1], a non-empty set A is an algebra if, $(A, +, \cdot)$ is a vector space over a field $F, (A, +, \circ)$ is a ring and $(\alpha a) \circ b = \alpha(a \circ b) = a \circ (\alpha b)$, for every $a, b \in A$, $\alpha \in F$. A ring R is an algebra $< R, +, \cdot, -, 0 >$ where $+$ and $\cdot$ are two binary operations, $-$ is unary and 0 is nullary element satisfying, $< R, +, -, 0 >$ is an abelian group, $< R, \cdot >$ is a semigroup and $x.(y + z) = (x.y) + (x.z)$ and $(x + y).z = (x.z) + (y.z)$ (see [2]).

Following [1], let A be an algebra, recall that a Banach space E is a Banach left A-module if E is a left A-module, and $\|a.x\| \leq \|a\|\|x\|(a \in A, x \in E)$. A map from a left Banach A-module X into a left Banach A-module Y (A is not necessarily commutative) is said a multiplier (homomorphism) if it satisfies $T(a.x) = a.Tx$, for all $a \in A, x \in X$ (see [3]). In [4], a submodule N of an R-module M is said to be stable, if $f(N) \subseteq N$ for each R-homomorphism $f : N \to M$. M is called a fully stable module, each submodule of M is stable. Following [5], a Banach algebra module M is called fully stable Banach A-module if for every submodule N of M and for each multiplier $\theta : N \to M$ such that $\theta(N) \subseteq N$. In this paper the concept of strongly full stability relative to ideal for Banach A-modules has been introduced. A Banach algebra module M is called strongly fully stable Banach A-module relative to ideal K of A if for every submodule N of M and for each multiplier $\theta : N \to M$ such that $\theta(N) \subseteq N \cap KM$.

Structure of fully stable Banach A-module relative to an ideal in term of their elements is considered see (2.2) Studying Baer criterion gives another characterization of fully stable Banach A-module relative to ideal K of A, corollary (2.8).

2. Main results

In [6], a left Banach A-module X is n-generated for $n \in N$ if there exists $x_1, \ldots, x_n \in X$ such that each $x \in X$ can represented as $x = \sum_{k=1} a_k.x_k$ for some $a_1, \ldots, a_n \in A$. A module which is 1-generated is called a cyclic module.

Definition 2.1. Let X be Banach A-module, X is called fully stable Banach A-module relative to ideal K of A, if for every submodule N of X and for each multiplier $\theta : N \to X$ such that $\theta(N) \subseteq N \cap KX$. It is clear that every fully stable Banach A-module is fully stable Banach A-module relative to an ideal. Moreover, every fully stable Banach A-modules is strongly fully stable Banach A-modules relative to ideal, therefore X is strongly fully stable Banach A-modules relative to ideal, if and if for every 1-generated submodule L of X and for each multiplier $\theta : L \to X$ such that $\theta(L) \subseteq L \cap KX$.

Let X be a Banach A-modules and K be a non-zero ideal of A. If M is fully stable Banach A-modules and $X = KX$ then X is strongly fully stable Banach A-modules relative to K, since for each 1-generated submodule N of X and A-homomorphism $f : N \to X$, $f(N) \subseteq N = N \cap X = N \cap KX$.

Following [7] for a nonempty subset M in a left Banach A-module θ, the annihilater ann_A of M is $ann_A(M) = \{a \in A; a.x = 0 \text{ for all } x \in M\}$. In [6], Let X be a Banach A-module, $N_x = \{N_x | n \in N, x \in X\}$ and $P_y = \{p_y | p \in P, y \in X\}$, $ann_A N_x = \{a \in A, a.n_x = 0, \forall n_x \in N_x\}$ and $ann_A P_y = \{a \in A, a.p_y = 0, \forall p_y \in P_y\}$. The following proposition gives another characterization of strongly fully stable Banach A-modules relative to an ideal.

Proposition 2.2. *X is fully stable Banach A-module if and only if for each $x, y \in X$ and N_x, P_y subsets of $X, y \notin N_X \cap KX$ implies $ann_A(N_x) \subsetneq ann_A(P_Y)$.*

Proof. Suppose that X is fully stable Banach A-module relative to ideal K of A, there exists $x, y \in X$ such that $y \notin N_x \cap KX$ and $ann_A(N_x) \subseteq ann_A(P_y)$. Define $\theta :< N_x > \to X$ by $\theta(a.n_x) = a.p_y$, for all $a \in A$, if $a.n_x = 0$ then $a \in ann_A(N_x) \subseteq ann_A(P_y)$. This implies that $a.p_y = 0$, hence θ is well define. It is clear θ that is a multiplier ,because X is strongly fully stable relative to an ideal, there exists an element $t \in A$ such that $\theta(m_x) = tm_x$, for each $m_x \in N_x$. In particular, $p_y = \theta(n_x) = tn_x \in N_x \cap KX$. Which is a contradiction. Thus X is strongly fully stable Banach module relative to an ideal. Conversely, assume that there is a subset N_x of X and a multiplier $\theta :< N_x > \to X$ such that $\theta(N_x) \subsetneq$

$N_x \cap KX$ then there exists an element $m_x \in N_x$ such that $\theta(m_x) \notin N_x \cap KX$. Let $s \in ann_A(N_x)$ therefore $sn_x = 0$, $s\theta(m_x) = \theta(stn_x) = \theta(tsn_x) = \theta(0) = 0$. Hence $ann_A(N_x) \subseteq ann_A(\theta(m_x))$. Which is a contradiction. □

Corollary 2.3. *Let X be a strongly fully stable Banach A-module relative to an ideal K of A. Then for each $x, y \in X$, $ann_A(P_y) = ann_A(N_x)$ implies $N_x \cap KX = P_y \cap KX$.*

Proof. Assume that there are two elements x, y in X such that

$$ann_A(N_x) = ann_A(P_y)$$

and $N_x \cap KX \neq P_y \cap KX$. Then without loss of generality there is an element z_x in N_x not in P_y. By proposition (2.2) we have $ann_A(P_y) \nsubseteq ann_A(Z_x)$ but $ann_A(N_x) \subseteq ann_A(Z_x)$, hence $ann_A(P_y) \nsubseteq ann_A(N_x)$ which is a contradiction.□

Definition 2.4. A submodule N of Banach A-module is called pure submodule if $KN = N \cap KX$ for each ideal K of A.

When the submodule of strongly fully stable Banach A-module relative to ideal have been partial answer in the following proposition.

Proposition 2.5. *Let X be a strongly fully stable Banach A-module relative to a non-zero ideal K of A. Then every pure submodule is strongly fully stable Banach A-module relative to an ideal.*

Proof. Let N be pure submodule of X. For each submodule L of N and a multiplier $f : L \to N$, put $g = i \circ f : L \to X$ (where i is the inclusion mapping of N to X), then by assumption $f(L) = g(L) \subseteq KX$, and since $f(L) \subseteq N$. Hence $f(L) \subseteq L \cap KX \cap N$. Since N is pure submodule of X then $N \cap KX = KN$, for each ideal K of A, therefore $f(L) \subseteq L \cap KN$. Thus N is strongly fully stable Banach A-module relative to K. □

Definition 2.6. A Banach A-module X is said to satisfy Baer criterion relative to an ideal K of A, if each submodule of X satisfies Baer criterion, that is, for every 1-generated submodule N of X and A-multiplier $\theta : N \to X$, there exists an element a in A such that $\theta(n) = an \in KX$ for all $n \in N$.

The following proposition and its corollary give another characterization of strongly fully stable Banach A-module relative to ideal.

Proposition 2.7. *Let X be a Banach A-module. Then Baer criterion holds for 1-generated submodules of X if and only if $ann_X(ann_A(N_x)) \subseteq N_x \cap KX$, for each $x \in X$.*

Proof. Assume that Baer criterion holds for 1-generated submodule of X. Let $y \in ann_X(ann_A(N_x))$ and define $\theta :< N_x >\to X$ by $\theta(a.n_x) = a.p_y$, for all $a \in A$. Let $a_1.n_x = a_2.n_x$, thus $(a_1 - a_2)n_x = 0$, where $a_1 - a_2 \in ann_A(N_x)$, so $(a_1 - a_2) \in ann_A(P_y)$. Therefore $(a_1 - a_2)p_y = 0$, then $a_1p_y = a_2p_y$, hence θ is well define. It is clear that clear θ is an A -multiplier. By the assumption, there exists an element $t \in A$ such that $\theta(m_x) = tm_x \in KX$ for each $m_x \in N_x$. This implies that, in particular, $p_y = \theta(n_x) = tn_x \in KX$, therefore $ann_X(ann_A(N_x)) \subseteq N_x \cap KX$, hence $ann_X(ann_A(N_x)) = N_x \cap KX$. Conversely, assume that $ann_X(ann_A(N_x)) = N_x \cap KX$. For each $N_x \subseteq X$. Then for each A-multiplier $\theta : N_x \to X$, and $s \in ann_A(N_x)$, we have $s\theta(n_x) = \theta(sn_x) = 0$. Thus $\theta(n_x) \in ann_X(ann_A(N_x)) = N_x \cap KX$, then $\theta(n_x) = tn_X \in KX$ for some $t \in A$, thus Baer criterion holds. □

Corollary 2.8. *X is strongly fully stable Banach A-module relative to ideal K of A if and only if $ann_X(ann_A(N_x)) \subseteq N_x \cap KX$, for each $x \in X$.*

In [9], let A be a unital Banach algebra. A-module X is called quasi α-injective if, $\varphi : N \to X$ is A-module homomorphism (multiplier) such that $\|\varphi\| \leq 1$, there exists A-module homomorphism (multiplier) $\theta : X \to X$, such that $\theta \circ i = \varphi$ and $\|\theta\| \leq \alpha$, where i is an isometry (A-module isomorphism is an isometry A-multiplier) from submodule N of X. We shall say that X is quasi injective if it is quasi α-injective for some α.

The concept of strongly quasi α-injective relative to an ideal K of A has been introduce.

Definition 2.9. Let A be a unital Banach algebra. A-module X is called strongly quasi α-injective relative to an ideal K of A if, $\varphi : N \to X$ is A-module homomorphism (multiplier) such that $\|\varphi\| \leq 1$, there exists A-module homomorphism (multiplier) $\theta : X \to X$, such that $(\theta \circ i)(n) = \varphi(n) \in KX$ and $\|\theta\| \leq \alpha$ where i is an isometry from submodule N of X to X. We shall say that X is strongly quasi injective relative to ideal if it is strongly quasi αinjective relative to ideal for some α.

The following proposition give the relation between strongly quasi α-injective Banach A-module relative to ideal and strongly fully stable Banach A-module relative to an ideal K of A has given

Proposition 2.11. *Let X be Banach A-module and K be a non-zero ideal of algebra A. If X is strongly fully stable Banach A-module relative to ideal then X is strongly quasi injective Banach A-module relative to ideal.*

Proof. Let N be a submodule of X and $f : N \to X$ be any A-module homomrphism. Since X is a fully stable Banach A-module relative to K, then $f(N) \subseteq N \cap KX$, thus there exist $t \in A$ such that $f(n) = tn$. Define $g : X \to X$ by $g(x) = tx$, it is clear that g is a well defined A-module homomorphism (multipler). Now $f(x) = g(x) = tx \in KX$, and for each $y \in N$, $(f \circ i)(y) - g(y) = f(y) - g(y) \in KX$, where i is isometry, and $\|g\| \leq \alpha$ for some α therefore X is strongly quasi injective Banach A-module relative to ideal. □

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