SYMMETRIC METRIC SPACE

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Abstract. We first introduce the concept of symmetric metric space, then we prove that for any point symmetric metric space there is one and only one fixed point, lastly, we list some properties of the finite point symmetric metric spaces.

Keywords: Symmetric metric space, isometric map, fixed point, finite metric space, point symmetric map.

1. Introduction

In mathematics, a metric space is a set where a notion of distance (called a metric) between elements of the set is defined.

The metric space which most closely corresponds to our intuitive understanding of space is the 3-dimensional Euclidean space. In fact, the notion of "metric" is a generalization of the Euclidean metric arising from the four long-known properties of the Euclidean distance. The Euclidean metric defines the distance between two points as the length of the straight line segment connecting them. Other metric spaces occur for example in elliptic geometry and hyperbolic geometry, where distance on a sphere measured by angle is a metric, and the hyperboloid model of hyperbolic geometry is used by special relativity as a metric space of velocities.

A metric space is an ordered pair (M, ρ) where M is a set and ρ is a metric on M, i.e., a function

$$\rho : M \times M \to \mathbf{R}$$

such that for any $x, y, z \in M$, the following holds:

1) $\rho(x, y) \geq 0$ (non-negative),
2) $\rho(x, y) = 0$, iff $x = y$, (identity of indiscernibles),
3) $\rho(x, y) = d(y, x)$ (symmetry)
4) $\rho(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality) .

We suggest the readers to read [2] to learn more about metric space.

In [1] Wang and Bai researched linear structure on translation spaces. They introduced an interesting concept which is called translation space. In this paper, we use the same method of [1] to introduce a point symmetric metric space and research it.

The paper is organized as following. We first introduce a special metric space which is called point symmetric metric space, then we prove that for any point symmetric metric space there is one and only one fixed point, lastly, we list some properties of the finite point symmetric metric spaces.

2. Point symmetric metric space

Let's first give the definition of point symmetric map and point symmetric metric space.

Definition 1. Let (X, ρ) be a metric space, $f : X \to X$ be an isometric map, i.e., f is bijective and for any $x, y \in X$, we have that $\rho(f(x), f(y)) = \rho(x, y)$. If there exists $x_0 \in X$ such that, for any $x \in X$, we have that

$$\rho(x, x_0) = \rho(f(x), x_0) = \frac{1}{2}\rho(x, f(x)),$$

then f is called a x_0-symmetric map of X. A metric space (X, ρ) with a x_0-symmetric map of X, is called a x_0-symmetric metric space with map f.

Example 1. Let $X = [-1, 1]$ be a real interval, define $f : X \to X$ by $f(x) = -x$, for any $x \in X$. Then f is a 0-symmetric map of X.

In fact, for any $x \in X$, we have that

$$\rho(x, 0) = |x - 0| = |-x - 0| = \rho(f(x), 0) = \frac{1}{2}|x - (-x)| = \frac{1}{2}\rho(f(x), x).$$

Example 2. Let (X, ρ) be a Banach space over the field of real numbers, for any $a, b \in X$, define $\rho(a, b) = ||a - b||$.

Choose any $a \in X$, define $f_a : X \to X$ as $f_a(x) = 2a - x$, $\forall x \in X$.

Then f is bijective and for any $x, y \in X$, we conclude that $\rho(f_a(x), f_a(y)) = ||f_a(x) - f_a(y)|| = ||(2a - x) - (2a - y)|| = ||x - y|| = \rho(x, y)$. and $\rho(x, a) = ||x - a|| = ||(2a - x) - a|| = \rho(f_a(x), a) = \frac{1}{2}||x - (2a - x)|| = \frac{1}{2}\rho(x, f_a(x))$. Thus f is an a-symmetric map of X.

3. Fixed point of a point symmetric map

In this section we will research the fixed point of a point symmetric map of a metric space.

Proposition 1. Let (X, ρ) be a metric space, f be a x_0-symmetric map of X and also be a x_1-symmetric map of X, then $x_0 = x_1$.

Proof. For any $x \in X$, we have that $\rho(x, x_0) = \rho(f(x), x_0) = \frac{1}{2}\rho(x, f(x))$, and $\rho(x, x_1) = \rho(f(x), x_1) = \frac{1}{2}\rho(x, f(x))$. So $\rho(x_0, x_1) \leq \rho(x_0, x) + \rho(x, x_1) = 2\rho(x_0, x)$.

Thus, for any $x \in X$, we have that $\rho(x_0, x_1) \leq 2\rho(x_0, x)$. So $\rho(x_0, x_1) \leq 2\rho(x_0, x_0) = 0$ which conclude that $x_0 = x_1$. □

Proposition 2. Let (X, ρ) be a metric space, f be a x_0-symmetric map of X, then x_0 is a fixed point of f, i.e., $f(x_0) = x_0$.

Proof. For any $x \in X$, we have that $\rho(x, x_0) = \rho(f(x), x_0) = \frac{1}{2}\rho(x, f(x))$. Thus $\rho(x_0, x_0) = \rho(f(x_0), x_0) = \frac{1}{2}\rho(x_0, f(x_0))$. And so $\rho(f(x_0), x_0) = 0$ which conclude that $f(x_0) = x_0$. □

Proposition 3. Let (X, ρ) be a metric space, f be a x_0-symmetric map of X, then x_0 is the only fixed point of f.

Proof. Suppose that $x_1 \in X$ is another fixed point of f. Then $f(x_1) = x_1$.

From $\rho(x, x_0) = \rho(f(x), x_0) = \frac{1}{2}\rho(x, f(x))$, $\forall x \in X$, we know that $\rho(x_1, x_0) = \rho(f(x_1), x_0) = \frac{1}{2}\rho(x_1, f(x_1)) = 0$. And we conclude that $x_0 = x_1$. □

4. Further properties of a point symmetric map

Proposition 4. Let (X, ρ) be a metric space and f be a x_0-symmetric map of X, then for any $x_1 \in X$ such that $x_1 \neq x_0$ and for any positive natural number n, we have that $f^n(x_1) \neq x_0$.

Proof. For any $x_1 \in X$ such that $x_1 \neq x_0$, then from $\rho(x_1, x_0) = \rho(f(x_1), x_0) = \frac{1}{2}\rho(x_1, f(x_1))$, we know that $f(x_1) \neq x_0$.

Suppose that $f^n(x_1) = x_0$, then from the previous proposition, we conclude that $f^{n-1}(x_1) = x_0$, and consequently, $f^{n-2}(x_1) = x_0$, $f^{n-3}(x_1) = x_0$, ..., $f^2(x_1) = x_0$, $f(x_1) = x_0$.

This is a contradiction. □

Proposition 5. Let (X, ρ) be a metric space and f be a x_0-symmetric map of X, then for any $x_1 \in X$ such that $x_1 \neq x_0$ and for any positive natural number n, we have that $f^n(x_1) \neq f^{n-1}(x_1)$.

Proof. Let's suppose that $f^n(x_1) = f^{n-1}(x_1)$, then $f^{n-1}(x_1)$ is a fixed point of X. So $f^{n-1}(x_1) = x_0$, and this is a contradiction by the previous proposition. □

We will use $|X|$ to denote cardinality of set X.

Proposition 6. Let (X, ρ) be a x_0-symmetric metric space with map f and X is a finite set, $|X| \geq 2$, for any $x_1 \in X$ such that $x_1 \neq x_0$, then there exists two positive nature numbers m and n which are subjected to $|X| \geq m > n$ and $m - n \geq 2$ such that $f^m(x_1) = f^n(x_1)$.

Proof. Let's suppose $|X| \geq 2$. For any $x_1 \in X$ such that $x_1 \neq x_0$, then from

$$\rho(x_1, x_0) = \rho(f(x_1), x_0) = \frac{1}{2}\rho(x_1, f(x_1)),$$

we know that $f(x_1) \neq x_0$ and $f(x_1) \neq x_1$.

We construct a sequence $\{x_0\}_{i=0}^{|X|} \subseteq X$ as following:

$$x_i = f^i(x_1)(i = 1, 2, ..., |X|).$$

Because there are only $|X|$ elements in X and $\{x_0\}_{i=0}^{|X|}$ consists of $|X| + 1$ elements, so there must be two elements of $\{x_0\}_{i=0}^{|X|}$, for example, x_m and x_n, such that $x_m = x_n$, i.e., $f^m(x_1) = f^n(x_1)$. We can suppose that $m > n$, and obviously, $|X| > m > n$. From the previous proposition, we know that $m-n \geq 2$. □

Example 3. Set $X = C[0, 1]$, $x_0 = x_0(t) = 0$, $x_1 = x_1(t) = 1$, $f(x_1) = -1$, $f^2(x_1) = 2t - 1$ Then $\rho(x_1, x_0) = \rho(f(x_1), x_0) = \frac{1}{2}\rho(x_1, f(x_1)) = 1$, and $\rho(f(x_1), x_0) = \rho(f^2(x_1), x_0) = \frac{1}{2}\rho(f(x_1), f^2(x_1)) = 1$. This shows that $f^2(x_1) \neq x_1$.

From the previous example, we can declare the following remark.

Remark 1. Let (X, ρ) be a metric space, f be a x_0-symmetric map of X and g also be a x_0-symmetric map of X, then we can not obtain that $f = g$.

Proof. Let $X = \{x_0, x_1, x_2, x_3\} \subseteq C[0, 1]$, $x_0 = x_0(t) = 0$, $x_1 = x_1(t) = 1$, $x_2 = x_2(t) = -1$, $x_3 = 2t - 1$, define f and g as following:

$$\begin{array}{c|cccc} X & x_0 & x_1 & x_2 & x_3 \\ \hline f(X) & x_0 & x_2 & x_3 & x_1 \end{array}, \qquad \begin{array}{c|cccc} X & x_0 & x_1 & x_2 & x_3 \\ \hline g(X) & x_0 & x_3 & x_1 & x_2 \end{array},$$

then it is easy to verify that f and g are isometric maps of X. Moreover, they are x_0-symmetric maps of X. But $f \neq g$. □

5. Funding

This work is supported by Research Foundation of Chongqing Municipal Education Commission (KJ1710253, KJ1401010), and Chongqing Municipal Key Laboratory of Institutions of Higher Education (Grant No. C16).

References

[1] G. Wang and Y. Bai, *Linear structure on translation spaces,* Acta Mathematica Sinica, 48 (2005), 1-10.

[2] A. Wilansky, *Functional analysis*, New York: Blaisdell Publishing Company, 1964.

Accepted: 05.06.2017