ON THE ANNIHILATOR INTERSECTION GRAPH OF A COMMUTATIVE RING

M. Vafaei
A. Tehranian[*]
Department of Mathematics
Science and Research Branch
Islamic Azad University (IAU)
Tehran
Iran
dmvafaei@gmail.com
tehranian@srbiau.ac.ir

R. Nikandish
Department of Basic Sciences
Jundi-Shapur University of Technology
Dezful
Iran
r.nikandish@ipm.ir

Abstract. Let R be a commutative ring with identity and $A(R)$ be the set of ideals with non-zero annihilator. The annihilator intersection graph of R is defined as the graph $\mathbb{AIG}(R)$ with the vertex set $A(R)^* = A(R) \setminus \{0\}$ and two distinct vertices I and J are adjacent if and only if $\mathrm{Ann}(IJ) \neq \mathrm{Ann}(I) \cap \mathrm{Ann}(J)$. It follows that the annihilating-ideal graph $\mathbb{AG}(R)$ (a well-known graph with the same vertices and two distinct vertices I, J are adjacent if and only if $IJ = 0$) is a subgraph of $\mathbb{AIG}(R)$. It is proved that $\mathbb{AIG}(R)$ is connected with diameter at most two and with girth at most four, if $\mathbb{AIG}(R)$ contains a cycle. Moreover, we characterize all rings whose annihilator intersection graphs are complete or star. Furthermore, we study the affinity between annihilator intersection graph and annihilating-ideal graph associated with a ring.

Keywords: Annihilator intersection graph, annihilating ideal graph, star graph, girth.

1. Introduction

Assigning a combinatorics object to an algebraic structure leads to arising interesting algebraic and combinatorics problems. Therefore, the study of graphs associated with rings has became one of the active area in this field. There are a lot of papers which apply combinatorial methods to obtain algebraic results in ring theory (see for instance [2], [5], [6] and [7]). Moreover, for the most recent study in this field see [1], [4], [10] and [11].

Throughout this paper R is a commutative ring with unity. The sets of all zero-divisors, nilpotent elements, unit elements, minimal prime ideals, maximal

$*$. Corresponding author

ideals of R are denoted by $Z(R)$, $\mathrm{Nil}(R)$, $U(R)$, $\mathrm{Min}(R)$ and $\mathrm{Max}(R)$, respectively. For a subset A of a ring R we let $A^* = A \setminus \{0\}$. For every ideal I of R, we denote the *annihilator* of I by $\mathrm{Ann}(I)$. The set of ideals of R with non-zero annihilator is denoted by $\mathbb{A}(R)$ and $I \in \mathbb{A}^*(R)$ is called an *annihilating ideal of R*. The ring R is said to be *reduced* if it has no non-zero nilpotent element. For any undefined notation or terminology in ring theory, we refer the reader to [3, 8].

Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. By $\overline{G}$, $\mathrm{diam}(G)$ and $\mathrm{gr}(G)$, we mean the complement, the diameter and the girth of G, respectively. If $u, v \in V(G)$, then $d(u, v)$ means the distance between u, v. A complete bipartite graph with part sizes m and n is denoted by $K_{m,n}$. If the size of one of the parts is 1, then the graph is said to be a *star graph*. The graph $H = (V_0, E_0)$ is a subgraph of G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called an *induced subgraph* by V_0, denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{\{u, v\} \in E \,|\, u, v \in V_0\}$. Also G is *empty* if it has no edges. For any undefined notation or terminology in graph theory, we refer the reader to [12].

Let R be a commutative ring with $1 \neq 0$. Authors in [6], introduced the *annihilating-ideal graph* of R, denoted by $\mathbb{AG}(R)$, as the graph with the vertex set $\mathbb{A}^*(R)$ and two distinct vertices I and J are adjacent if and only if $IJ = 0$. For a ring R, we define another kind of undirected graph with the vertex set $\mathbb{A}^*(R)$ and two distinct vertices I and J are adjacent if and only if $\mathrm{Ann}(IJ) \neq \mathrm{Ann}(I) \cap \mathrm{Ann}(J)$. This graph is called *annihilator intersection graph* and denoted by $\mathbb{AIG}(R)$. It is not hard to see that $\mathbb{AG}(R)$ is a subgraph of $\mathbb{AIG}(R)$ and so it is interesting to explore the properties of $\mathbb{AIG}(R)$. In this paper, we study some connections between the graph-theoretic properties of $\mathbb{AIG}(R)$ and some algebraic properties of rings. Moreover, we investigate the affinity between annihilator intersection graph and annihilating-ideal graph associated with a ring. Especially, we focus on the conditions under which the annihilator intersection graph is identical to the annihilating-ideal graph.

2. Fundamental properties of $\mathbb{AIG}(R)$

In this section, we study fundamental properties of $\mathbb{AIG}(R)$. It is shown that $\mathbb{AIG}(R)$ is always a connected graph and $\mathrm{diam}(\mathbb{AIG}(R)) \leq 2$. Moreover, we prove that if $\mathbb{AIG}(R)$ contains a cycle, then $\mathrm{gr}(\mathbb{AIG}(R)) \leq 4$. We begin with a lemma containing several useful properties of $\mathbb{AIG}(R)$.

Lemma 2.1 *Let R be a ring and $I, J \in \mathbb{A}^*(R)$. Then the following statements hold:*

(1) *If $I - J$ is not an edge of $\mathbb{AIG}(R)$, then* $\mathrm{Ann}(I) = \mathrm{Ann}(J)$.

(2) *If $I - J$ is an edge of $\mathbb{AG}(R)$, then $I - J$ is an edge of $\mathbb{AIG}(R)$.*

(3) *If $I - J$ is not an edge of $\mathbb{AIG}(R)$, for some distinct vertices $I, J \in \mathbb{A}^*(R)$, then there exists a vertex $K \in \mathbb{A}^*(R) \setminus \{I, J\}$ such that $I - K - J$ is a path in $\mathbb{AG}(R)$.*

Proof. (1) Suppose that $\mathrm{Ann}(I) \neq \mathrm{Ann}(J)$. Without loss of generality, assume that $x \in \mathrm{Ann}(I) \setminus \mathrm{Ann}(J)$. Thus $x \notin \mathrm{Ann}(I) \cap \mathrm{Ann}(J)$ but $x \in \mathrm{Ann}(IJ)$, a contradiction. Thus $\mathrm{Ann}(I) = \mathrm{Ann}(J)$.

(2) Suppose that $I - J$ is an edge of $\mathbb{AG}(R)$, for some distinct vertices $I, J \in \mathbb{A}^*(R)$. Then $IJ = (0)$ and hence $\mathrm{Ann}(IJ) = R$. Since $I \neq (0)$ and $J \neq (0)$, $\mathrm{Ann}(I) \neq R$ and $\mathrm{Ann}(J) \neq R$. Thus $I - J$ is an edge of $\mathbb{AIG}(R)$.

(3) Suppose that $I - J$ is not an edge of $\mathbb{AIG}(R)$, for some distinct vertices $I, J \in \mathbb{A}^*(R)$. By Part 1, $\mathrm{Ann}(I) = \mathrm{Ann}(J)$. Moreover, $\mathrm{diam}(\mathbb{AG}(R)) \leq 3$, by [6, Theorem 2.1]. Hence we need only to show that $d_{\mathbb{AG}(R)}(I, J) \neq 3$. If $d_{\mathbb{AG}(R)}(I, J) = 3$, then $\mathrm{Ann}(I) \nsubseteq \mathrm{Ann}(J)$ and $\mathrm{Ann}(J) \nsubseteq \mathrm{Ann}(I)$, a contradiction. Therefore, $d_{\mathbb{AG}(R)}(I, J) = 2$. □

Since $\mathbb{AG}(R)$ is a subgraph of $\mathbb{AIG}(R)$, in view of Part (3) in the preceding lemma and [6, Theorem 2.1], we have the following result.

Theorem 2.1 *Let R be a ring. Then $\mathbb{AIG}(R)$ is connected and $\mathrm{diam}(\mathbb{AIG}(R)) \leq 2$. Moreover, if $\mathbb{AIG}(R)$ contains a cycle, then $\mathrm{gr}(\mathbb{AIG}(R)) \leq 4$.*

The next theorem shows that $\mathrm{gr}(\mathbb{AIG}(R)) = 4$ may occur. First we need the following lemmas.

Lemma 2.2 *Let R be a non-reduced ring. Then every non-zero nilpotent ideal of R is adjacent to all other vertices in $\mathbb{AIG}(R)$. In particular, the induced subgraph by nilpotent ideals is a complete subgraph of $\mathbb{AIG}(R)$.*

Proof. Assume that I is a non-zero nilpotent ideal of R that is not adjacent to J, for some $J \in \mathbb{A}^*(R) \setminus \{I\}$. Part (1) of Lemma 2.1 implies that $\mathrm{Ann}(I) = \mathrm{Ann}(J)$. If $I^2 = (0)$, then $I \subseteq \mathrm{Ann}(I) = \mathrm{Ann}(J)$ and thus $IJ = (0)$, a contradiction. Let $n \geq 3$ be the least positive integer such that $I^n = (0)$. Since $\mathrm{Ann}(I) = \mathrm{Ann}(J)$, $I^{n-1}J = (0)$ and $I^{n-2}J \neq (0)$. This implies that $I^{n-2}(IJ) = (0)$ but $I^{n-2}I = I^{n-1} \neq (0)$ and $I^{n-2}J \neq (0)$. This means that I is adjacent to J, a contradiction. □

A simple question is posed: If $I \in V(\mathbb{AIG}(R))$ is adjacent to every other vertex, then is it nilpotent? The following example shows that the answer is negative.

Example 2.1 Let $D = \mathbb{Z}_2[X, Y, Z]$, $I = (X^2, Y^2, XY, XZ, YZ)D$ be an ideal of D, and let $R = D/I$. Also, let $x = X + I$, $y = Y + I$ and $z = Z + I$ be elements of R. Then $\mathrm{Nil}(R) = R(x, y)$ and $Z(R) = R(x, y, z)$. It is clear that the set $\{Rx, Ry, \mathrm{Nil}(R), Z(R)\}$ is a clique and $\mathbb{AIG}(R) = K_4 \vee \overline{K}_\infty$.

To prove Theorem 2.2, the following lemma is needed.

Lemma 2.3 [7, Conjecture 1.11] *Let R be a reduced ring with more than two minimal prime ideals. Then* $\mathrm{gr}(\mathbb{AG}(R)) = 3$.

Proof. Since R is reduced, by [9, Corollary 2.4], $Z(R) = \cup_{\mathfrak{p}\in \mathrm{Min}(R)}\mathfrak{p}$. Suppose that $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_3$ are three distinct minimal prime ideals. If $x \in \mathfrak{p}_1 \setminus \mathfrak{p}_2 \cup \mathfrak{p}_3$, then $\mathrm{Ann}(x) \subseteq \mathfrak{p}_2 \cap \mathfrak{p}_3$. Let $0 \neq y \in \mathrm{Ann}(x)$. Since $Rx\mathfrak{p}_2 \neq (0)$, we may choose $a \in Rx \cap \mathfrak{p}_2$. Also, since R is reduced, $Rx \cap \mathrm{Ann}(x) = (0)$. This implies that $a \notin \mathrm{Ann}(x)$. Since $a, y \in \mathfrak{p}_2$, we have $a + y = z \in \mathfrak{p}_2$ and so $\mathrm{Ann}(z) \neq (0)$. By [9, Corollary 2.2], $hz = 0$, for some $h \notin \mathfrak{p}_2$. This means that $Rh \neq Ra$ and $Rh \neq Ry$. Now, we show that $\mathrm{Ann}(z) = \mathrm{Ann}(y) \cap \mathrm{Ann}(a)$. To see this, it is enough to show that $\mathrm{Ann}(z) \subseteq \mathrm{Ann}(y) \cap \mathrm{Ann}(a)$. Let $fz = 0$. Then $f(a + y) = fa + fy = 0$. If $fa = 0$ (or $fy = 0$), then $fy = 0$ (or $fa = 0$) and thus $f \in \mathrm{Ann}(y) \cap \mathrm{Ann}(a)$. So we let $fa \neq 0$ and $fy \neq 0$. Since $fa + fy = 0$, $0 \neq fa = -fy \in Rx \cap \mathrm{Ann}(x)$. Since R is reduced, $Rx \cap \mathrm{Ann}(x) = (0)$, a contradiction. Hence $\mathrm{Ann}(z) = \mathrm{Ann}(y) \cap \mathrm{Ann}(a)$ and thus $Ra - Ry - Rh - Ra$ is a cycle of length 3. □

Theorem 2.2 *Let R be a ring. Then the following statements are equivalent:*

(1) $\mathrm{gr}(\mathbb{AIG}(R)) = 4$.

(2) *R is reduced,* $|\mathrm{Min}(R)| = 2$ *and R contains no minimal ideal.*

(3) $\mathbb{AG}(R) = K_{\infty,\infty}$.

(4) $\mathbb{AIG}(R) = K_{\infty,\infty}$.

Proof. (1) $\Rightarrow$ (2) If R is non-reduced, then by Lemma 2.2, $\mathrm{gr}(\mathbb{AIG}(R)) \in \{3, \infty\}$, a contradiction. Hence R is reduced. It follows from Lemma 2.3 that $|\mathrm{Min}(R)| = 2$. Moreover, if R contains a minimal ideal, then $R = R_1 \times R_2$, where R_1 and R_2 are reduced rings. The equality $|\mathrm{Min}(R)| = 2$ implies that one of the rings is a field and the other one is an integral domain. Now, we can easily check that $\mathrm{gr}(\mathbb{AIG}(R)) = \infty$, a contradiction. Thus R contains no minimal ideal.

(2) $\Rightarrow$ (3) Let $\mathrm{Min}(R) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$. Since R is reduced, $Z(R) = \mathfrak{p}_1 \cup \mathfrak{p}_2$ and $\mathfrak{p}_1 \cap \mathfrak{p}_2 = (0)$, by [9, Corollary 2.4]. Let A, B be the sets of all non-zero ideals contained in $\mathfrak{p}_1, \mathfrak{p}_2$, respectively. It is not hard to see that $\mathbb{AIG}(R) = K_{|A|,|B|}$. As R contains no minimal ideal, $|A| = |B| = \infty$.

(3) $\Rightarrow$ (4) Suppose that $\mathbb{AG}(R) = K_{\infty,\infty}$ with two parts V_1 and V_2. Let $I, J \in V_i$, for some $1 \leq i \leq 2$. We need only to show that I is not adjacent to J in $\mathbb{AIG}(R)$. With no loss of generality, we can assume that $I, J \in V_1$. If $IJ \in V_1$, then $\mathrm{Ann}(IJ) = \mathrm{Ann}(I) \cap \mathrm{Ann}(J)$ and thus I is not adjacent to J. If $IJ \notin V_1$, then $IJ \in V_2$ and thus $IIJ = I^2J = (0)$. By [7, Theorem 2.3], R is reduced and thus $\mathrm{Ann}(I) = \mathrm{Ann}(I^2)$, for every ideal I of R. This together with $I^2J = (0)$ imply that $IJ = (0)$, a contradiction.

(4) $\Rightarrow$ (1) is clear. □

In Theorem 2.3, all rings R with star $\mathbb{AIG}(R)$ are characterized.

Theorem 2.3 *Let R be a ring. Then $\mathbb{AIG}(R)$ is a star graph if and only if one of the following statements holds:*

(1) *$R = F \times D$, where F is a field and D is an integral domain.*

(2) *R has exactly two non-zero proper ideals.*

(3) *$\mathrm{Nil}(R)$ is a minimal and prime ideal of R.*

Proof. First suppose that $\mathbb{AIG}(R)$ is a star graph. We consider the following cases.

Case 1. R is a reduced ring. Suppose that the vertex $I \in \mathbb{A}^*(R)$ is adjacent to every other vertex. If $I \neq I^2$, then $I - I^2$ is an edge of $\mathbb{AIG}(R)$. Since R is a reduced ring, $\mathrm{Ann}(I) = \mathrm{Ann}(I^2) = \mathrm{Ann}(I^3)$ and thus I is not adjacent to I^2, a contradiction. Thus $I = I^2$. We claim that I is a minimal ideal. Let $J \subseteq I$, where J is an ideal of R. If $J \neq I$, then $J\mathrm{Ann}(I) = (0)$ implies that J is adjacent to $\mathrm{Ann}(I)$, a contradiction. So the claim is proved and thus $R \cong Ra \times R(1-a)$, by [13, 2.3 and 2.7]. We may assume that $R \cong R_1 \times R_2$ with $R_1 \times (0)$ is adjacent to every other vertex. If R_1 has a non-trivial ideal, say I, then $I \times (0)$ is adjacent to $(0) \times R_2$, a contradiction. Thus R_1 is a field. Similarly, if $xy = 0$, for some $x, y \in R_2$, then $R_2x - R_2y$ is an edge of $\mathbb{AIG}(R)$, a contradiction. Hence $Z(R_2) = (0)$ and so $R = F \times D$, where F is a field and D is an integral domain.

Case 2. R is a non-reduced ring. If $Z(R) = \mathrm{Nil}(R)$, then by Lemma 2.2, $\mathbb{AIG}(R)$ is a complete graph and so it has exactly two non-zero proper ideals. Let $Z(R) \neq \mathrm{Nil}(R)$. By Lemma 2.2, $\mathrm{Nil}(R)$ is a minimal ideal of R. We show that $\mathrm{Nil}(R)$ is a prime ideal of R. Let $IJ \subseteq \mathrm{Nil}(R)$, $I \nsubseteq \mathrm{Nil}(R)$ and $J \nsubseteq \mathrm{Nil}(R)$. Since $\mathbb{AIG}(R)$ is a star graph, $IJ \neq (0)$ and thus $IJ = \mathrm{Nil}(R)$. This implies that $IIJ = I^2J = (0)$ and so $I^2 = \mathrm{Nil}(R)$, a contradiction.

The converse is clear. □

The following example provides an infinite star annihilator intersection graph.

Example 2.2 Let $R = \mathbb{Z}_2[X, Y]/(XY, X^2)$ and let $x = X + (XY, X^2)$, $y = Y + (XY, X^2)$. Then $\mathrm{Nil}(R) = \{0, x\}$ is a minimal and prime ideal of R and $Z(R) = R(x, y)$. It is clear that $\mathbb{AIG}(R) = K_1 \vee \overline{K}_\infty$.

To prove Theorem 2.4, the following lemmas is needed.

Lemma 2.4 *Let R be a non-reduced ring. Suppose that $\mathbb{AIG}(R)$ is a star graph. Then R is indecomposable.*

Proof. Let $R = R_1 \times R_2$ and $\mathrm{Nil}(R_1) \neq (0)$, where R_i is a ring, for $1 \leq i \leq 2$. If $a \in \mathrm{Nil}(R_1)^*$, then $R_1a \times R_2 - R_1a \times (0) - R_1 \times (0) - R_1a \times R_2$ forms a path of length three, a contradiction. So R is indecomposable. □

Next, we give more details on non-reduced ring with star $\mathbb{AIG}(R)$.

Lemma 2.5 *Let R be a non-reduced ring with $|\mathbb{A}^*(R)| > 2$. If $\mathbb{AG}(R)$ is a star graph, then the following statements are equivalent:*

(1) $\mathbb{AIG}(R) \neq \mathbb{AG}(R)$.

(2) $\mathbb{AIG}(R)$ *is a complete graph.*

(3) *There are at least two vertices of $\mathbb{AIG}(R)$ which are adjacent to every other vertex.*

Proof. (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are obvious. We have only to prove (1) $\Rightarrow$ (2). Suppose that H is adjacent to every other vertex of $\mathbb{AG}(R)$. Thus $H = \mathrm{Ann}(Z(R))$ and $\mathrm{Ann}(J) = H$, for every $J \in \mathbb{A}^*(R)\backslash\{H\}$. Let $I, J \in \mathbb{A}^*(R)\backslash\{H\}$ and $I \neq J$. We need only show that I is adjacent to J. Since $\mathbb{AIG}(R) \neq \mathbb{AG}(R)$, $K_1 - K_2$ is an edge of $\mathbb{AIG}(R)$ that is not an edge of $\mathbb{AG}(R)$, for some $K_1, K_2 \in \mathbb{A}^*(R) \setminus \{H\}$. Hence $K_1K_2 \neq (0)$ and $\mathrm{Ann}(K_1K_2) \neq \mathrm{Ann}(K_1) \cap \mathrm{Ann}(K_2)$. This means that $K_1K_2 = H$. Since $IK_1K_2 = IH = (0)$, $IK_1 = H$. As $JIK_1 = JH = (0)$, we deduce that $IJ = H$. This implies that $\mathrm{Ann}(IJ) = Z(R)$ and $\mathrm{Ann}(I) = \mathrm{Ann}(J) = H$. Therefor, I is adjacent to J. □

The following example may explain Lemma 2.5 better.

Example 2.3 Let $R = \mathbb{Z}_2[X,Y]/(X^2,Y^2)$. Then $\mathbb{AG}(R) = K_{1,4}$ and $\mathbb{AIG}(R) = K_5$. Also, $\mathbb{AG}(\mathbb{Z}_{16}) = K_{1,2}$ and $\mathbb{AIG}(\mathbb{Z}_{16}) = K_3$.

Theorem 2.4 *Let R be a non-reduced ring that is not an integral domain. Then the following statements are equivalent:*

(1) $\mathbb{AIG}(R)$ *is a star graph.*

(2) $\mathrm{gr}(\mathbb{AIG}(R)) = \infty$.

(3) $\mathbb{AIG}(R) = \mathbb{AG}(R)$ *and* $\mathrm{gr}(\mathbb{AG}(R)) = \infty$.

(4) *Either $Z(R) = \mathrm{Nil}(R)$ and $|\mathbb{A}^*(R)| = 2$ or $Z(R) \neq \mathrm{Nil}(R)$ and $\mathrm{Nil}(R)$ is a minimal and prime ideal of R.*

(5) *Either $\mathbb{AIG}(R) = K_{1,1}$ or $\mathbb{AIG}(R) = K_{1,\infty}$.*

(6) *Either $\mathbb{AG}(R) = K_{1,1}$ or $\mathbb{AG}(R) = K_{1,\infty}$ and $\mathbb{AIG}(R)$ is not complete.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3) Since R is a non-reduced ring, by Lemma 2.2, there exists a vertex of $\mathbb{AIG}(R)$ which is adjacent to every other vertex. This together with $\mathrm{gr}(\mathbb{AIG}(R)) = \infty$ and $\mathrm{diam}(\mathbb{AIG}(R)) \leq 2$ imply that $\mathbb{AIG}(R)$ is a star graph. Since $\mathbb{AG}(R)$ is a connected subgraph of $\mathbb{AIG}(R)$, $\mathbb{AIG}(R) = \mathbb{AG}(R)$ and so $\mathrm{gr}(\mathbb{AG}(R)) = \infty$.

(3) $\Rightarrow$ (4) Since R is a non-reduced ring, by Lemma 2.2, it is easily seen that $\mathbb{AIG}(R)$ is a star graph and thus by Theorem 2.3, the result holds.

(4) $\Rightarrow$ (5) By Theorem 2.3, we need only to show that if $Z(R) \neq \mathrm{Nil}(R)$, then $|V(\mathbb{AIG}(R))| = \infty$. If $|V(\mathbb{AIG}(R))| < \infty$, then R is an Artinian ring and thus by Lemma 2.4, R is a local ring. This contradicts $Z(R) \neq \mathrm{Nil}(R)$.

(5) $\Rightarrow$ (6) is clear.

(6) $\Rightarrow$ (1) By Lemma 2.5, $\mathbb{AIG}(R) = \mathbb{AG}(R)$ and thus $\mathbb{AIG}(R)$ is a star graph. □

The final results of this section are devoted to study rings R with complete $\mathbb{AIG}(R)$.

Theorem 2.5 *Let R be a reduced ring. Then $\mathbb{AIG}(R)$ is complete if and only if $\mathrm{Ann}(I) \neq \mathrm{Ann}(J)$, for every distinct pair $I, J \in \mathbb{A}^*(R)$.*

Proof. One implication is clear, by Part (1) of Lemma 2.1. To prove the converse, suppose that $\mathbb{AIG}(R)$ is complete. If $\mathrm{Ann}(I) = \mathrm{Ann}(J)$, for some $I, J \in \mathbb{A}^*(R)$, then $\mathrm{Ann}(IJ) \neq \mathrm{Ann}(I) = \mathrm{Ann}(J)$. Thus $rIJ = (0)$, $rI \neq (0)$ and $rJ \neq (0)$, for some $r \in R$. Since $rIJ = (0)$, $rI \subseteq \mathrm{Ann}(J) = \mathrm{Ann}(I)$ and thus $rII = rI^2 = (0)$. Since R is a reduced ring, $rI = (0)$, a contradiction. □

Corollary 2.1 *Let R be a reduced ring and $\mathbb{AIG}(R)$ is a complete graph. Then the following statements hold:*

(1) *For every $I \in \mathbb{A}^*(R)$ and every positive integer $n \geq 2$, $I = I^n$.*

(2) *$R = Z(R) \cup U(R)$.*

Proof. (1) Suppose that there exists $I \in \mathbb{A}^*(R)$, that $I \neq I^n$, for some positive integer $n \geq 2$. Since $\mathbb{AIG}(R)$ is a complete graph, $\mathrm{Ann}(I^{n+1}) \neq \mathrm{Ann}(I^n) \cap \mathrm{Ann}(I)$. As R is a reduced ring, $\mathrm{Ann}(I^{n+1}) = \mathrm{Ann}(I^n) = \mathrm{Ann}(I)$, a contradiction.

(2) Suppose that $a \in Z(R)^*$. By Part (1), $Ra = Ra^2$ and thus there is an $r \in R$ such that $a = ra^2$. Let $e = ra$ and so $e^2 = r^2a^2 = ra = e$. This implies that e is a non-trivial idempotent and hence $R = R_1 \times R_2$, where R_1 and R_2 are two rings. Now, we show that $R_i = Z(R_i) \cup U(R_i)$, for $i = 1, 2$. With no loss of generality, suppose that $x \in R_1 \setminus Z(R_1) \cup U(R_1)$. Then $Rx \neq Rx^2$, a contradiction, by Part (1). □

We close this section with the following example.

Example 2.4 Let $R_1 = \prod_{i \in \Lambda} \mathbb{Z}$ and $R_2 = \prod_{i \in \Lambda} \mathbb{Z}_2$, where $|\Lambda| \geq 2$. By Theorem 2.5, $\mathbb{AIG}(R_1)$ is not a complete graph whereas $\mathbb{AIG}(R_2)$ is a complete graph.

3. When $\mathbb{AIG}(R) = \mathbb{AG}(R)$? (non-reduced case)

In this section, we study non-reduced rings R whose $\mathbb{AG}(R)$ and $\mathbb{AIG}(R)$ are identical.

Theorem 3.1 *Let R be a non-reduced ring and $\mathbb{AIG}(R) = \mathbb{AG}(R)$. Then one of the following statements holds.*

(1) $\mathrm{Nil}(R)^2 = (0)$.

(2) *R is a local ring with exactly two non-zero proper ideals $Z(R)$ and $Z(R)^2$.*

Proof. Suppose that $\mathrm{Nil}(R)^2 \neq (0)$. By Lemma 2.2, I is adjacent to every other vertex, for some ideal $I \subseteq \mathrm{Nil}(R)$. The equality $\mathbb{AIG}(R) = \mathbb{AG}(R)$ implies that $Z(R)$ is a vertex $\mathbb{AIG}(R)$, by [6, Theorem 2.2]. Since $\mathrm{Nil}(R)^2 \neq (0)$, $I\mathrm{Nil}(R) \neq (0)$, for some ideal $I \subseteq \mathrm{Nil}(R)$ and thus $IZ(R) \neq (0)$, a contradiction unless $I = Z(R) = \mathrm{Nil}(R)$. This means that every vertex of $\mathbb{AIG}(R)$ is a nilpotent ideal of R and thus by Lemma 2.2, $\mathbb{AIG}(R)$ is a complete graph. Since $\mathbb{AIG}(R) = \mathbb{AG}(R)$, by [6, Theorem 2.7], R is a local ring with exactly two non-zero proper ideals $Z(R)$ and $Z(R)^2$. $\square$

Theorem 3.2 *Let R be a ring. If $\mathbb{AIG}(R) = \mathbb{AG}(R)$, then $\mathrm{Ann}(I) = \mathrm{Ann}(I^2)$, for every $I \nsubseteq \mathrm{Nil}(R)$.*

Proof. Assume that $\mathbb{AIG}(R) = \mathbb{AG}(R)$ and $I \nsubseteq \mathrm{Nil}(R)$. If $I = I^2$, then there is nothing to prove. So let $I \neq I^2$. Since $I^3 \neq (0)$, by Part (1) of Lemma 2.1, $\mathrm{Ann}(I) = \mathrm{Ann}(I^2)$. $\square$

Let R be a non-reduced ring and $\mathbb{AIG}(R) = \mathbb{AG}(R)$. We show that $|\mathrm{Min}(R)| = 1$. First we need a series of lemmas.

Lemma 3.1 *Let R be a ring and $I \nsubseteq \mathrm{Nil}(R)$ be an ideal of R. Then $\mathrm{Ann}(I) = \mathrm{Ann}(I^2)$ if and only if $\mathrm{Ann}(I) = \mathrm{Ann}(I^n)$ for every integer $n \geq 2$.*

Proof. One side is clear. To prove the converse first we show that $\mathrm{Ann}(I) = \mathrm{Ann}(I^3)$. If $x \in \mathrm{Ann}(I^3) \setminus \mathrm{Ann}(I)$, then $xI^3 = (0)$ and $xI \neq (0)$. Thus $xI \subseteq \mathrm{Ann}(I^2) = \mathrm{Ann}(I)$ and so $xI^2 = xI = (0)$, a contradiction. Hence $\mathrm{Ann}(I) = \mathrm{Ann}(I^3)$. Similarly, $\mathrm{Ann}(I) = \mathrm{Ann}(I^4)$ and so the result holds by induction on n. $\square$

Lemma 3.2 *Let R be a non-reduced ring and $\mathbb{AIG}(R) = \mathbb{AG}(R)$. Then $|\mathrm{Min}(R)| \leq 2$.*

Proof. Assume that $\mathbb{AIG}(R) = \mathbb{AG}(R)$. Suppose to the contrary, $\mathfrak{p}_1$, $\mathfrak{p}_2$ and $\mathfrak{p}_3$ are three distinct minimal prime ideals. Let $a \in \mathfrak{p}_1 \setminus \mathfrak{p}_2 \cup \mathfrak{p}_3$. Thus $\mathfrak{p}_2 \cup \mathfrak{p}_3 \nsubseteq \mathrm{Ann}(a)$ (as $\mathrm{Ann}(a) \subseteq \mathfrak{p}_2 \cap \mathfrak{p}_3$). So one may assume that $ab \neq 0$, for some $b \in \mathfrak{p}_2 \cup \mathfrak{p}_3 \setminus \mathfrak{p}_1$. With no loss of generality, assume that $b \in \mathfrak{p}_2 \setminus \mathfrak{p}_1$. Obviously,

$\mathrm{Ann}(b) \subseteq \mathfrak{p}_1$. Also, it follows from [9, Theorem 2.1, p. 2], there exists an element $x \in \mathrm{Ann}(a^n)$ (for some positive integer n) such that $x \notin \mathfrak{p}_1$. This, together with Lemma 3.1, imply that $x \in \mathrm{Ann}(a) \setminus \mathrm{Ann}(b)$. Therefore, $\mathrm{Ann}(a) \neq \mathrm{Ann}(b)$. So by Part (1) of Lemma 2.1, $Ra - Rb$ is an edge of $\mathbb{AIG}(R)$ that is not an edge of $\mathbb{AG}(R)$, a contradiction. Hence $|\mathrm{Min}(R)| \leq 2$. □

Lemma 3.3 *Let R be a non-reduced ring. Then $\mathbb{AIG}(R) = \mathbb{AG}(R)$ if and only if one of the following statements holds.*

(1) *R is a local ring with exactly two non-zero proper ideals $Z(R)$ and $Z(R)^2$.*

(2) *If $IJ \neq (0)$, for some $I, J \in \mathbb{A}^*(R)$, then $\mathrm{Ann}(I) = \mathrm{Ann}(J)$ and $\mathrm{Ann}(I)$ is a prime ideal of R.*

Proof. Let $\mathbb{AIG}(R) = \mathbb{AG}(R)$ and (1) does not hold, for some ring R. Let $IJ \neq (0)$, for some $I, J \in \mathbb{A}^*(R)$. Then by Part (1) of Lemma 2.1, $\mathrm{Ann}(I) = \mathrm{Ann}(J)$. We show that $\mathrm{Ann}(I)$ is a prime ideal of R. Let $K_1K_2 \subseteq \mathrm{Ann}(I)$, $K_1 \nsubseteq \mathrm{Ann}(I)$ and $K_2 \nsubseteq \mathrm{Ann}(I)$. By Theorem 3.1, $\mathrm{Nil}(R)^2 = (0)$ and thus $K_1 \neq I$ or $K_2 \neq I$. With no loss of generality, one may assume that $K_2 \neq I$. It is easily seen that $K_2 - I$ is an edge of $\mathbb{AIG}(R)$ that is not an edge of $\mathbb{AG}(R)$, a contradiction. Hence $\mathrm{Ann}(I)$ is a prime ideal of R.

Conversely, assume that one of the conditions is satisfied. If condition (1) holds, there is nothing to prove. Hence assume that condition (2) holds. If $IJ = (0)$, for all $I, J \in \mathbb{A}^*(R)$, then $\mathbb{AG}(R)$ is complete and so $\mathbb{AIG}(R)$ is complete, i.e, $\mathbb{AIG}(R) = \mathbb{AG}(R)$. To complete the proof, we show that if $IJ \neq (0)$ for some $I, J \in \mathbb{A}^*(R)$, then $\mathrm{Ann}(IJ) = \mathrm{Ann}(I) = \mathrm{Ann}(J)$. By (2), $\mathrm{Ann}(I) = \mathrm{Ann}(J)$ and $\mathrm{Ann}(I)$ is a prime ideal of R. Let $x \in \mathrm{Ann}(IJ) \setminus \mathrm{Ann}(I)$. So $RxJI = (0)$, $RxJ \neq (0)$ and $RxI \neq (0)$. Thus $RxJ \subseteq \mathrm{Ann}(I)$, $Rx \nsubseteq \mathrm{Ann}(I)$ and $J \nsubseteq \mathrm{Ann}(I)$, a contradiction. This means that $\mathrm{Ann}(IJ) = \mathrm{Ann}(I) = \mathrm{Ann}(J)$ and hence $\mathbb{AIG}(R) = \mathbb{AG}(R)$. □

Now, we are ready to show that $|\mathrm{Min}(R)| = 1$, if R is a non-reduced ring with $\mathbb{AIG}(R) = \mathbb{AG}(R)$.

Theorem 3.3 *Let R be a non-reduced ring and $\mathbb{AIG}(R) = \mathbb{AG}(R)$. Then $|\mathrm{Min}(R)| = 1$.*

Proof. Suppose that $|\mathrm{Min}(R)| \neq 1$. By Lemma 3.2, $|\mathrm{Min}(R)| = 2$. By Lemma 2.2, I is adjacent to every other vertex, for some ideal $I \subseteq \mathrm{Nil}(R)$. Since $\mathbb{AIG}(R) = \mathbb{AG}(R)$, by [6, Theorem 2.2], we conclude that $Z(R)$ is a vertex $\mathbb{AIG}(R)$. Suppose that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are two distinct minimal prime ideals of R. We show that $Z(R) = \mathfrak{p}_1 \cup \mathfrak{p}_2$. Let $x \in Z(R) \setminus \mathfrak{p}_1 \cup \mathfrak{p}_2$. Since $RxZ(R) \neq (0)$, by Lemma 3.3, $\mathrm{Ann}(x)$ is a prime ideal of R. Obviously, $\mathrm{Ann}(x) \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2$, a contradiction. Hence $Z(R) = \mathfrak{p}_1 \cup \mathfrak{p}_2$ but in this case since $Z(R)$ is an ideal of R, $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ or $\mathfrak{p}_2 \subseteq \mathfrak{p}_1$ again a contradiction. Therefore, $|\mathrm{Min}(R)| = 1$. □

Theorem 3.4 *Let R be a non-reduced ring and $\mathfrak{p}$ be the only minimal prime ideal of R.*

(1) *If $\mathfrak{p} \neq Z(R)$, then the following statements are equivalent:*

  (i) $\mathbb{AG}(R) = K_{|A|} \vee \overline{K}_\infty$, *where A is the set of all nilpotent ideals of R.*

  (ii) $\mathfrak{p}Z(R) = (0)$.

  (iii) $\mathbb{AG}(R) = \mathbb{AIG}(R)$.

(2) *If $\mathfrak{p} = Z(R)$, then the following statements are equivalent:*

  (i) $\mathbb{AG}(R)$ *is a complete graph.*

  (ii) $\mathbb{AIG}(R)$ *is a complete graph.*

  (iii) $\mathbb{AG}(R) = \mathbb{AIG}(R)$.

Proof. (1) (i) $\Rightarrow$ (ii) Since $\mathbb{AG}(R) = K_{|A|} \vee \overline{K}_\infty$, every vertex of $K_{|A|}$ is adjacent to all other vertices but there is no adjacency between two arbitrary vertices of $\overline{K}_\infty$. This implies that $\mathrm{Ann}(Z(R)) = \mathrm{Nil}(R)$ and $IJ \neq (0)$, for every $I, J \in V(\overline{K}_\infty)$. Now we show that $\mathrm{Ann}(Z(R))$ is a prime ideal of R. For this, let $IJ \subseteq \mathrm{Ann}(Z(R))$, $I \nsubseteq \mathrm{Ann}(Z(R))$ and $J \nsubseteq \mathrm{Ann}(Z(R))$. If $IJ = (0)$, then $I - J$ is an edge of $\overline{K}_\infty$, a contradiction unless $I = J$. But in this end, we have $I^2 = 0$, and so $I \subseteq \mathrm{Ann}(Z(R))$, a contradiction. So $IJ \neq (0)$. Since $IJ \subseteq \mathrm{Ann}(Z(R))$ and $I \nsubseteq \mathrm{Ann}(Z(R))$, $KIJ = (0)$, $KI \neq (0)$ for some $K \in V(\mathbb{AG}(R))$. This implies that $J \in V(\mathbb{AG}(R))$, $IJJ = IJ^2 = (0)$, $J^2 \subseteq \mathrm{Ann}(Z(R))$. Hence $J^2J = J^3 = (0)$, a contradiction. Thus $\mathrm{Ann}(Z(R))$ is a prime ideal of R and thus $\mathfrak{p}Z(R) = (0)$.

(ii) $\Rightarrow$ (iii) Since $\mathrm{Ann}(Z(R))$ is a prime ideal of R, $\mathrm{Ann}(Z(R)) = \mathrm{Nil}(R)$ and for every $I \in \mathbb{A}^*(R)$ that $I \nsubseteq \mathrm{Nil}(R)$, $\mathrm{Ann}(I) = \mathrm{Nil}(R)$. Now, it is easy to see that $\mathbb{AG}(R)[A] = \mathbb{AIG}(R)[A]$ and $\mathbb{AG}(R)[B] = \mathbb{AIG}(R)[B]$ are two subgraphs such that $\mathbb{AIG}(R)[A]$ is complete, $\mathbb{AIG}(R)[B]$ is null and $\mathbb{AG}(R) = \mathbb{AIG}(R) = \mathbb{AIG}(R)[A] \vee \mathbb{AIG}(R)[B]$, where A is the set of all nilpotent ideals of R and $B = \mathbb{A}^*(R) \setminus A$.

(iii) $\Rightarrow$ (i) Since $\mathbb{AG}(R) = \mathbb{AIG}(R)$, by Lemma 2.2, $\mathrm{Ann}(Z(R)) = \mathrm{Nil}(R)$ and since $\mathrm{Nil}(R)$ is a prime ideal of R, we can easily get $\mathbb{AG}(R) = K_{|A|} \vee \overline{K}_\infty$.

(2) All cases are clear by Theorem 3.1 and Lemma 2.2. □

We close this paper with the following example.

Example 3.1 Let $D = \mathbb{Z}_2[X, Y, Z]$, $I = (X^2, Y^2, XY, XZ)D$ be an ideal of D, and let $R = D/I$. Also, let $x = X + I$, $y = Y + I$ and $z = Z + I$ be elements of R. Then $\mathrm{Nil}(R) = R(x, y)$ and $Z(R) = R(x, y, z)$. It is clear that $\mathrm{Nil}(R)$ is the only minimal prime ideal of R and $\mathrm{Nil}(R)Z(R) \neq (0)$. Thus by Theorem 3.4, $\mathbb{AG}(R) \neq \mathbb{AIG}(R)$. Also, by construction, $\mathbb{AIG}(R) = K_{|A|} \vee \overline{K}_\infty$ but since $\mathrm{Nil}(R)Z(R) \neq (0)$, $\mathbb{AG}(R) \neq K_{|A|} \vee \overline{K}_\infty$, where A is the set of all nilpotent ideals of R.

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