TRIPLE POSITIVE SOLUTIONS FOR A THIRD-ORDER THREE-POINT BOUNDARY VALUE PROBLEM WITH SIGN-CHANGING GREEN'S FUNCTION

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Abstract. In this paper, we discuss the existence of triple positive solutions for a third-order three-point boundary value problem

$$\begin{cases} u'''(t) = f(t, u(t)), & t \in (0,1), \\ u'(0) = 0, \quad u(1) = \alpha u(\eta), & u''(\eta) = 0, \end{cases}$$

where $0 < \alpha < 1$, $\max\{\frac{1+2\alpha}{1+4\alpha}, \frac{1}{2-\alpha}\} < \eta < 1$. We first study the associated Green's function and obtain some useful properties. Our main tool is the fixed point theorem due to Avery and Peterson. It is to be observed that although the associated Green's function is sign-changing, the solution obtained is still positive. The results of this paper are new. An example demonstrates the main results.

Keywords: Third-order boundary value problem, triple positive solutions, sign-changing Green's function, cone.

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1. Introduction

In recent years, the existence and multiplicity of positive solutions of the boundary value problems with sign-changing Green's function have received much attention from many authors(see [8-13,15-19]). Specially, in [15,17], Sun and Zhao considered a class of BVP with an indefinitely signed Green's function of the form

$$\begin{cases} u'''(t) = f(t, u(t)), & t \in [0, 1], \\ u'(0) = u(1) = u''(\eta) = 0. \end{cases}$$

where $\eta \in (\frac{1}{2}, 1)$. Under the premise of $u'(1) \leq 0$, $t \in [0, 1]$ is established, by using the Guo-Krasnoselskii and Leggett-Williams fixed point theorems, the authors established the existence of single or multiple results of positive solutions.

In [11], Li, Sun and Kong considered the following BVP with an indefinitely signed Green's function

$$\begin{cases} u'''(t) = a(t)f(t, u(t)), & t \in (0, 1), \\ u'(0) = u(1) = 0, \quad u''(\eta) + \alpha u(0) = 0. \end{cases}$$

where

$$\alpha \in [0, 2), \eta \in [\frac{\sqrt{121 + 24\alpha}}{3(4 + \alpha)}, 1).$$

By means of the Guo-Krasnosel'skii fixed point theorem, existence results of positive solutions was obtained.

It is to be observed that there are other types of works on sign-changing Green's functions which prove the existence of sign-changing solutions, positive in some cases; see Infante and Webb's papers [5-7], Liu's papers [1,2], our papers [3,4].

Inspired by those papers, here we study the problem

$$\begin{cases} u'''(t) = f(t, u(t)), & t \in (0, 1), \\ u'(0) = 0, \quad u(1) = \alpha u(\eta), \quad u''(\eta) = 0, \end{cases}$$

Where $0 < \alpha < 1$,

$$\max\left\{\frac{1 + 2\alpha}{1 + 4\alpha}, \frac{1}{2 - \alpha}\right\} < \eta < 1, f : [0, 1] \times [0, +\infty) \to (0, +\infty)$$

is continuous, we give the following assumptions:

(H_0) For $u \in [0, +\infty)$, the mapping $t \mapsto f(t, u)$ is nonincreasing;

(H_1) For $t \in [0, 1]$, the mapping $u \mapsto f(t, u)$ is nondecreasing.

As is known to all, in general, in order to obtain positive solution of the boundary value problems, the Green's functions are positive. It is worth to notice that the Green's function is sign-changing in this paper. Inspired by the work of our papers [3,4], we try to establish some criteria for the existence of triple positive solutions to the problem (1.1). It is also noted that our method here is different from that of the references [8-13,15-17,19]. This will takes lots of difficulties for us to obtain the existence of positive solutions. To overcome it, we first study the associated Green's function and obtain some useful properties. Meanwhile, by applying the fixed point theorem due to Avery and Peterson, we obtain the existence of triple positive solutions to the problem (1.1) by making full use of the boundary conditions and the concavity and convexity of $u(t)$ skillfully. To the best of our knowledge, no paper has appeared in the literature which discusses the problem. The main purpose of this paper is to fill this gap. An example demonstrates the main results.

2. Preliminaries and lemmas

Let ϑ and θ be nonnegative continuous convex functionals on K, κ be a nonnegative continuous concave functional on K, and ψ be a nonnegative continuous functional on K. Then for positive real numbers c, d, e and q, we define the following convex sets:

$$\begin{aligned}
&K(\vartheta, q) = \{u \in K : \vartheta(u) < q\};\\
&K(\vartheta, \kappa, d, q) = \{u \in K : d \leq \kappa(u), \vartheta(u) \leq q\};\\
&K(\vartheta, \theta, \kappa, d, e, q) = \{u \in K : d \leq \kappa(u), \theta(u) \leq e, \vartheta(u) \leq q\};\\
&R(\vartheta, \psi, c, q) = \{u \in K : c \leq \psi(u), \vartheta(u) \leq q\}.
\end{aligned}$$

Lemma 2.1 ([14]). *Let K be a cone in a Banach space E. Let ϑ and θ be nonnegative continuous convex functionals on K, κ be a nonnegative continuous concave functional on K, and ψ be a nonnegative continuous functional on K satisfying $\psi(\lambda u) \leq \lambda\psi(u)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers M and q,*

$$\alpha(u) \leq \psi(u) \ \text{ and } \ \|u\| \leq M\vartheta(u), \tag{2.1}$$

for all $u \in \overline{K(\vartheta, q)}$. Suppose $T : \overline{K(\vartheta, q)} \to \overline{K(\vartheta, q)}$ is completely continuous and there exist positive numbers c, d and e with $c < d$ such that

(C_1) *$\{u \in K(\vartheta, \theta, \kappa, d, e, q) : \kappa(u) > q\} \neq \emptyset$ and $\kappa(Tu) > d$ for $u \in K(\vartheta, \theta, \kappa, d, e, q)$;*

(C_2) *$\kappa(Tu) > d$, for $u \in K(\vartheta, \kappa, d, q)$ with $\theta(Tu) > e$;*

(C_3) *$0 \notin R(\vartheta, \psi, c, q)$ and $\psi(Tu) < c$ for $u \in R(\vartheta, \psi, c, q)$, with $\psi(u) = c$.*

Then T has at least three fixed points u_1, u_2 and $u_3 \in \overline{K(\vartheta, q)}$ such that

$$\vartheta(u_i) \leq q \text{ for } i = 1, 2, 3, d < \kappa(u_1), c < \psi(u_2) \text{ with } \kappa(u_2) < d \text{ and } \psi(u_3) < c.$$

Lemma 2.2. *We assume that* (H_0) *holds, then for* $y \in C[0,1]$*, the BVP*

$$\begin{aligned} &u'''(t) = y(t), t \in (0,1), \\ &u'(0) = 0, \quad u(1) = \alpha u(\eta), \quad u''(\eta) = 0, \end{aligned}$$

has the following expression of Green's function:

$$\begin{cases} G_1(t,s), & s \in [0,\eta], \\ G_2(t,s), & s \in [\eta,1], \end{cases}$$

where

$$G_1(t,s) = \begin{cases} \dfrac{2s(1-\alpha\eta) - 2ts(1-\alpha)}{2(1-\alpha)}, & s \le t, \\ \dfrac{-(1-\alpha)t^2 - (1-\alpha)s^2 + 2(1-\alpha\eta)s}{2(1-\alpha)}, & t \le s, \end{cases}$$

$$G_2(t,s) = \begin{cases} \dfrac{(1-\alpha)(t-s)^2 - (1-s)^2}{2(1-\alpha)}, & s \le t, \\ \dfrac{-(1-s)^2}{2(1-\alpha)}, & t \le s, \end{cases}$$

Proof. The proof follows by direct calculations, we omitted it here.

Lemma 2.3. *Suppose* (H_0) *holds. Then* $G(t,s)$ *has the following properties:*

(*i*) $G_1(t,s) \ge 0$ *and* $G_2(t,s) \le 0$*, for all* $t \in [0,1]$*;*

(*ii*) $G(t,s_1) \ge -G(t,s_2)$*,* $t \in [0,1]$*,* $s_1 \in [\tau,\eta]$*,* $s_2 \in [\eta,1]$*, where* $\max\{\alpha\eta, \frac{1-\eta}{2\alpha}\} \le \tau < 2\eta - 1$*.*

Proof. (i) Since $\frac{\partial}{\partial t} G_1(t,s) \le 0$, for $t \in [0,1]$, we have

$$G_1(t,s) \ge G_1(1,s) = \frac{2\alpha(1-\eta)s}{2(1-\alpha)} \ge 0, \quad \forall t \in [0,1]$$

By $\frac{\partial}{\partial t} G_2(t,s) \ge 0$, for $t \in [0,1]$, we get

$$G_2(t,s) \le G_2(1,s) = -\frac{\alpha(1-s)^2}{2(1-\alpha)} \le 0, \quad \forall t \in [0,1].$$

(ii) For $s_1 \in [\tau,\eta]$ and $t \in [0,1]$,

$$G(t,s_1) = G_1(t,s_1) \ge G_1(1,s_1) \ge \frac{2\alpha\tau(1-\eta)}{2(1-\alpha)}$$

and for $s_2 \in [\eta,1]$ and $t \in [0,1]$,

$$G(t,s_2) = G_2(t,s_2) \ge G_2(0,s_2) = -\frac{(1-s)^2}{2(1-\alpha)} \ge -\frac{(1-\eta)^2}{2(1-\alpha)}$$

Then,

$$\frac{G(t,s_1)}{-G(t,s_2)} \ge \frac{2\alpha\tau}{1-\eta} \ge 1, \text{ for } t \in [0,1], s_1 \in [\tau,\eta], s_2 \in [\eta,1].$$

We complete the proof. □

3. The main results and proofs

Let real Banach space $C[0,1]$ be equipped with the norm $\|u\| = \max_{0\le t\le 1}|u(t)|$. Denote

$$X = \{u \in C[0,1] : \min_{0\le t\le 1} u(t) \ge 0, u'(0) = 0 \text{ and } u(1) = \alpha u(\eta), u''(\eta) = 0,$$

and define the cone $K \subset X$ and the operator $T : C[0,1] \to C[0,1]$ by

$$K = \{u \in X : u(t) \text{ is concave on } [0,\eta], u(t) \text{ is convex on } [\eta,1]\},$$

and

$$(Tu)(t) = \int_0^1 G(t,s)f(s,u(s))ds.$$

Lemma 3.1. *If $u \in K$, thus*

$$u(t) \ge h(t)u(\eta), \quad \text{for } t \in [0,\eta] \text{ and } u(t) \le h(t)u(\eta), \text{ for } t \in [\eta,1],$$

where

$$\begin{cases} \dfrac{t}{\eta}, & t \in [0,\eta], \\ \dfrac{1-\alpha\eta-(1-\alpha)t}{1-\eta}, & t \in [\eta,1]. \end{cases}$$

Proof. Since $u \in K$, then $u(t)$ is concave on $[0,\eta]$, convex on $[\eta,1]$. Thus, we have

$$u(t) \ge u(0) + \frac{u(\eta)-u(0)}{\eta}t \ge \frac{t}{\eta}u(\eta), \quad \text{for} t \in [0,\eta],$$

and

$$\begin{aligned} u(t) &\le u(1) + \frac{u(1)-u(\eta)}{1-\eta}(t-1) = \frac{t-\eta}{1-\eta}\alpha u(\eta) + \frac{1-t}{1-\eta}u(\eta) \\ &= \frac{1-\alpha\eta-(1-\alpha)t}{1-\eta}u(\eta), \text{ for } t \in [\eta,1]. \end{aligned}$$

We complete the proof. □

Lemma 3.2. *Suppose* (H_0) *and* (H_1) *hold. Thus*

$$\int_\tau^\eta G(t,s)f(s,u(s))ds \ge -\int_\eta^1 G(t,s)f(s,u(s))ds. \tag{3.1}$$

Proof. Since $x \in [0,1-\eta]$, thus $\eta - \delta x \in [\tau,\eta], \eta + x \in [\eta,1]$, where $\delta = \frac{\eta-\tau}{1-\eta}$, τ as in Lemma 2.3. By Lemma 3.1, we have

$$h(\eta-\delta x) = 1 - \frac{\delta x}{\eta} \text{ and } h(\eta+x) = 1 - \frac{1-\alpha}{1-\eta}x, \text{ for } x \in [0,1-\eta].$$

By $\max\{\alpha\eta, \frac{1-\eta}{2\alpha}\} < \tau < 2\eta - 1$, we get

$$h(\eta - \delta x) \geq h(\eta + x), \quad \text{for all } x \in [0, 1-\eta]. \tag{3.2}$$

Let $s = \eta - \delta x$, $\forall x \in [0, 1-\eta]$. From Lemmas 2.3 and 3.1, (3.2) and (H_1), we have

$$\begin{aligned}
\int_\tau^\eta G(t,s)f(s,u(s))ds &= \delta \int_0^{1-\eta} G(t,\eta-\delta x)f(\eta-\delta x, u(\eta-\delta x))dx \\
&\geq -\int_0^{1-\eta} G(t,\eta+x)f(\eta-\delta x, h(\eta-\delta x)u(\eta))dx \\
&\geq -\int_0^{1-\eta} G(t,\eta+x)f(\eta, h(\eta+x)u(\eta))dx.
\end{aligned} \tag{3.3}$$

On the other hand, let $s = \eta + x$, $x \in [0, 1-\eta]$, by Lemma 3.1, one has

$$\begin{aligned}
-\int_\eta^1 G(t,s)f(s,u(s))ds &= -\int_0^{1-\eta} G_2(t,\eta+x)f(\eta+x, u(\eta+x))dx \\
&\leq -\int_0^{1-\eta} G(t,\eta+x)f(\eta, h(\eta+x)u(\eta))dx.
\end{aligned} \tag{3.4}$$

Thus, in view of (3.3) and (3.4), we know that (3.1) holds.

Lemma 3.3. *If $u \in K$, then $u(t) \geq \gamma\|u\|$, for $t \in [\alpha\eta, \tau]$, where $\gamma = \frac{\eta-\tau}{\eta}$.*

Proof. Since $u \in K$, thus $u''(t) \leq 0$, for $t \in [0,\eta]$ and $u''(t) \geq 0$, for $t \in [\eta, 1]$, in view of $u'(0) = 0$, $u(1) = \alpha u(\eta) < u(\eta)$, we have

$$u'(t) \leq u'(0) = 0, \quad \text{for } t \in [0, \eta] \tag{3.5}$$

and

$$u(t) \leq \max\{u(\eta), u(1)\} = u(\eta), \quad \text{for } t \in [\eta, 1]. \tag{3.6}$$

In view of (3.5), we have

$$u(t) \leq u(0), \ \text{for } t \in [0, \eta]. \tag{3.7}$$

By (3.6) and (3.7), we get

$$\|u\| = \max_{0 \leq t \leq 1} |u(t)| = u(0).$$

If $t \in [\alpha\eta, \tau]$, then by the concavity of $u(t)$, we have

$$u(t) \geq u(0) + \frac{u(\eta) - u(0)}{\eta} t \geq \frac{\eta - \tau}{\eta}\|u\|.$$

Lemma 3.4. *Assumes* (H$_0$) *and* (H$_1$) *hold, then* $T : K \to K$ *is completely continuous.*

Proof. If $u \in K$, we have by Lemma 3.2 that

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s)f(s,u(s))ds \\ &= \left(\int_0^\tau G(t,s)f(s,u(s))ds + \int_\tau^\eta G(t,s)f(s,u(s))ds \right.\\ &\left. + \int_\eta^1 G(t,s)f(s,u(s))ds\right) \geq 0.\end{aligned}$$

Moreover, $(Tu)'(0) = 0$, $(Tu)(1) = \alpha(Tu)(\eta)$, $(Tu)''(\eta) = 0$. Thus, $T : K \to X$.

On the other hand, by $(Tu)'''(t) \geq 0$, for $t \in [0,1]$, which together with $(Tu)''(\eta) = 0$ implies that

$$(Tu)''(t) \leq 0, \quad \text{for } t \in [0,\eta] \text{ and } (Tu)''(t) \geq 0, \quad \text{for } t \in [\eta, 1],$$

This show that $T : K \to K$. It can be shown that $T : K \to K$ is completely continuous by Arzela-Ascoli theorem.

Let the nonnegative continuous concave functional κ on K, the nonnegative continuous convex functionals θ, ϑ and the nonnegative continuous functional ψ be defined on the cone K by

$$\kappa(u) = \min_{\alpha\eta \leq t \leq \tau} |u(t)|, \text{ for } u \in K,$$

and

$$\vartheta(u) = \theta(u) = \psi(u) = \max_{0 \leq t \leq 1} |u(t)|, \quad \text{for } u \in K.$$

For convenience, we denote

$$N = \int_0^\eta G_1(0,s)ds, \quad M = \int_{\alpha\eta}^\tau G_1(\tau,s)ds.$$

Theorem 3.1. *Suppose that* (H$_0$) *and* (H$_1$) *hold. If* $0 < c < d < \gamma q$ *and the following conditions hold:*

(H_2) $f(t,u) < \frac{q}{N}$, *for all* $t \in [0,\eta]$, $u \in [0,q]$,

(H_3) $f(t,u) > \frac{d}{M}$, *for all* $t \in [\alpha\eta, \tau]$, $u \in [d, \frac{d}{\gamma}]$,

(H_4) $f(t,u) < \frac{c}{N}$, *for all* $t \in [0,\eta]$, $u \in [0,c]$.

Thus the BVP (1.1) *has at least three nonnegative solutions* (*positive on* $(0,1)$) u_1, u_2 *and* u_3 *such that*

$$\max_{0\le t\le 1}|u_i(t)|<q \text{ for } i=1,2,3, d<\min_{\alpha\eta\le t\le\tau}|u_1(t)|,$$
$$c<\max_{0\le t\le 1}|u_2(t)| \text{ with } \min_{\alpha\eta\le t\le\tau}|u_2(t)|<d, \quad \text{and } \max_{0\le t\le 1}|u_3(t)|<c.$$

Proof. Firstly, if $u\in\overline{K(\vartheta,q)}$, then, we may assert that $T:\overline{K(\vartheta,q)}\to\overline{K(\vartheta,q)}$ is completely continuous operator.

To see this, suppose $u\in\overline{K(\vartheta,q)}$, then $\vartheta(u)=\max_{0\le t\le 1}|u(t)|\le q$. It follows from (H_2) that

$$\begin{aligned}
\vartheta(Tu)&=\max_{0\le t\le 1}|Tu(t)|=\max_{0\le t\le 1}\int_0^1 G(t,s)f(s,u(s))ds\\
&=\max_{0\le t\le 1}\left(\int_0^\eta G(t,s)f(s,u(s))ds+\int_\eta^1 G(t,s)f(s,u(s))ds\right)\\
&\le\int_0^\eta \max_{0\le t\le 1}G(t,s)f(s,u(s))ds=\int_0^\eta G_1(0,s)f(s,u(s))ds\\
&<\frac{q}{N}\int_0^\eta G_1(0,s)ds=q.
\end{aligned}$$

Therefore, $T:\overline{K(\vartheta,q)}\to\overline{K(\vartheta,q)}$. This together with Lemma 3.4 implies that $T:\overline{K(\vartheta,q)}\to\overline{K(\vartheta,q)}$ is completely continuous operator.

To check condition (C_1) of Lemma 2.1, we let $u(t)=\frac{d}{\gamma}$, for $t\in[0,1]$. It is easy to verify that $u(t)=\frac{d}{\gamma}\in K(\vartheta,\theta,\kappa,d,\frac{d}{\gamma},q)$ and $\kappa(u)=\kappa(\frac{d}{\gamma})>d$, and so $\{u\in K(\vartheta,\theta,\kappa,d,\frac{d}{\gamma},q):\kappa(u)>d\}\ne\emptyset$. Thus, for all $u\in K(\vartheta,\theta,\kappa,d,\frac{d}{\gamma},q)$, we have that $d\le u(t)\le\frac{d}{\gamma}$, for $t\in[\tau_1,\tau]$ and $T(u)\in K$, from Lemma 3.3, (H_3), one has

$$\begin{aligned}
\min_{\alpha\eta\le t\le\tau}(Tu)(t)=(Tu)(\tau)&=\int_0^1 G(\tau,s)f(s,u(s))ds\\
&=\int_0^\tau G(\tau,s)f(s,u(s))ds+\int_\tau^\eta G(\tau,s)f(s,u(s))ds\\
&+\int_\eta^1 G(\tau,s)f(s,u(s))ds\\
&>\frac{d}{M}\int_{\alpha\eta}^\tau G(\tau,s)ds=d.
\end{aligned}$$

Consequently,

$$\min_{\alpha\eta\le t\le\tau}(Tu)(t)>d, \text{ for} u\in K(\vartheta,\theta,\kappa,d,\frac{d}{\gamma},q) \text{ with } d\le u(t)\le\frac{d}{\gamma}, t\in[\alpha\eta,\tau].$$

This explains condition (C_1) of Lemma 2.1 holds.

Secondly, We verify that (C_2) of Lemma 2.1 is satisfied. By Lemma 3.3 and Lemma 3.4, we have

$$\min_{\alpha\eta\le t\le\tau}(Tu)(t)\ge\gamma\|Tu\|=\gamma\theta(Tu)>d,\quad\text{for } u\in K(\vartheta,\kappa,d,q)\text{ with }\theta(Tu)>\frac{d}{\gamma}$$

which shows that $\kappa(Tu)>d$, $u\in K(\vartheta,\kappa,d,q)$ with $\theta(Tu)>\frac{d}{\gamma}$.

Finally, we show condition (C_3) of Lemma 2.1 is also satisfied. Obviously, as $\psi(0)=0<c$, then $0\notin R(\vartheta,\psi,c,q)$. If $u\in R(\vartheta,\psi,c,q)$ with $\psi(u)=c$. Then in view of condition (H_4), we have

$$\begin{aligned}\psi(Tu)&=\max_{0\le t\le1}|Tu(t)|\\&=\max_{0\le t\le1}\int_0^1 G(t,s)f(s,u(s))ds\\&=\max_{0\le t\le1}\left(\int_0^\eta G(t,s)f(s,u(s))ds+\int_\eta^1 G(t,s)f(u(s))ds\right)\\&\le\int_0^\eta\max_{0\le t\le1}G(t,s)f(s,u(s))ds\\&=\int_0^\eta G_1(0,s)f(s,u(s))ds\\&<\frac{c}{N}\int_0^\eta G_1(0,s)ds=c.\end{aligned}$$

To sum up, all the conditions of Lemma 2.1 are satisfied, it follows from Lemma 2.1 that there exist three nonnegative solutions (positive on $(0,1)$) u_1, u_2 and u_3 for the BVP (1.1) such that

$$\begin{aligned}&\max_{0\le t\le1}|u_i(t)|<q\text{ for }i=1,2,3, d<\min_{\alpha\eta\le t\le\tau}|u_1(t)|,\\&c<\max_{0\le t\le1}|u_2(t)|\text{ with }\min_{\alpha\eta\le t\le\tau}|u_2(t)|<d,\quad\text{and }\max_{0\le t\le1}|u_3(t)|<c.\end{aligned}$$

4. Example

Consider the BVP

$$\begin{cases}u'''(t)=f(t,u(t)), & t\in(0,1),\\ u'(0)=0,\quad u(1)=\frac{1}{2}u(\frac{3}{4}),\quad u''(\frac{3}{4})=0, &\end{cases}$$

where

$$f(t,u)=\begin{cases}(1-t)u^2, & (t,u)\in[0,1]\times[0,1],\\(1-t)(97u-96), & (t,u)\in[0,1]\times[1,2],\\98(1-t)(u-1)^2, & (t,u)\in[0,1]\times[2,6],\\2450(1-t), & (t,u)\in[0,1]\times[6,+\infty).\end{cases}$$

It is easy to see that (H_0) and (H_1) are satisfied. Let $\tau = \frac{1}{2}$, an easy computation shows that:

$$\alpha\eta = \frac{3}{8}, \gamma = \frac{1}{3}, N = \frac{9}{32}, M = \frac{21}{512},$$

Thus, if we choose $c = 1$, $d = 2$, $q = 690$, then $f(t,u)$ satisfies the following conditions:

(H_2) $f(t,u) < \frac{q}{N} \approx 2453.3333$, for all $t \in [0, \frac{3}{4}]$, $u \in [0, 690]$,

(H_3) $f(t,u) > \frac{d}{M} \approx 48.7619$, for all $t \in [\frac{3}{8}, \frac{1}{2}]$, $u \in [2, 6]$,

(H_4) $f(t,u) < \frac{c}{N} \approx 3.5556$, for all $t \in [0, \frac{3}{4}]$, $u \in [0, 1]$.

Hence, all the conditions of Theorem 3.1 are satisfied. Thus, the BVP (4.1) has at least three nonnegative solutions (positive on $(0,1)$) u_1, u_2 and u_3 satisfying

$$\max_{0\le t\le 1} |u_i(t)| < 690, \quad \text{for } i = 1,2,3, 2 < \min_{\frac{3}{8}\le t\le \frac{1}{2}} |u_1(t)|,$$
$$1 < \max_{0\le t\le 1} |u_2(t)| \text{ with } \min_{\frac{3}{8}\le t\le \frac{1}{2}} |u_2(t)| < 2, \quad \text{and} \max_{0\le t\le 1} |u_3(t)| < 1.$$

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