COMMON FIXED POINT THEOREMS FOR FUZZY MAPPINGS IN b-METRIC SPACE

Aqeel Shahzad
Department of Mathematics and Statistics
Riphah International University
Islamabad - 44000
Pakistan
aqeel4all84@gmail.com
Abdullah Shoaib*
Department of Mathematics and Statistics
Riphah International University
Islamabad - 44000
Pakistan
abdullahshoaib15@yahoo.com
Qasim Mahmood
Department of Mathematics and Statistics
Riphah International University
Islamabad - 44000
Pakistan
qasim_math@yahoo.com

Abstract. In this paper we establish some fixed point results for fuzzy mapping in a complete b-metric space. Our results unify, extend and generalize several results in the existing literature. An example is also given to support our results.

Keywords: Fixed point, complete b-metric space, fuzzy mapping, Hausdorff metric space.

1. Introduction and preliminaries

Fixed point theory plays an important role in the various fields of mathematics. It provides very important tools for finding the existence and uniqueness of the solutions. The Banach contraction theorem has an important role in fixed point theory and became very papular due to iterations which can be easily implemented on the computers. The idea of fuzzy set was first laid down by Zadeh [9]. Later on Weiss [8] and Butnariu [3] obtained many fixed point results for fuzzy mapping in metric spaces. Afterward, Heilpern [4] initiated the idea of fuzzy contraction mappings and proved a fixed point theorem for fuzzy contrac-

$*$. Corresponding author

tion mappings which is a fuzzy analogue of Nadler's fixed point theorem for multivalued mappings [6]. Further work on fuzzy mappings can be seen in [7].

In this paper, we obtain a fixed point and a common fixed point for fuzzy mapping in complete b-metric space. An example is also given which supports the obtained results.

Here, the obtained results for fuzzy mapping in b-metric space under certain contractive conditions are helpful for Hausdorff dimensions computing which are helpful in high energy physics to understand e^{∞}-spaces. In high energy physics these results are also helpful for solving the arising geometric problems due to the involvement of fuzzy sets.

Definition 1.1 ([2]). Let X be any nonempty set and $b \geq 1$ be any given real number. A function $d : X \times X \to \mathbb{R}^+$ is called a b-metric, if it satisfies the following conditions for all $x, y, z \in X$:

i) $d(x, y) = 0$ if and only if $x = y$,

ii) $d(x, y) = d(y, x)$,

iii) $d(x, z) \leq b[d(x, y) + d(y, z)]$.

Then, the pair (X, d) is called a b-metric space.

Definition 1.2 ([5]). Let (X, d) be a b-metric space and $\{x_n\}$ be a sequence in X. Then,

i) $\{x_n\}$ is called a convergent sequence if and only if there exists $x \in X$, such that for all $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that for all $n \geq n(\epsilon)$, we have $d(x_n, x) < \epsilon$. So, we write $\lim_{n\to\infty} x_n = x$.

ii) $\{x_n\}$ is called a Cauchy sequence if and only if for all $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that for each $m, n \geq n(\epsilon)$, we have $d(x_n, x_m) < \epsilon$.

iii) (X, d) is called complete if every Cauchy sequence in X converges to a point $x \in X$ such that $d(x, x) = 0$.

Definition 1.3 ([6]). Let (X, d) be a metric space. We define the Hausdorff metric on $CB(X)$ induced by d. Then,

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(A, y)\},$$

for all $A, B \in CB(X)$, where $CB(X)$ denotes the family of closed and bounded subsets of X and

$$d(x, B) = \inf\{d(x, a) : a \in B\},$$

for all $x \in X$.

A fuzzy set in X is a function with domain X and values in $[0,1]$, $F(X)$ is the collection of all fuzzy sets in X. If A is a fuzzy set and $x \in X$, then the function value $A(x)$ is called the grade of membership of x in A. The α-level set of fuzzy set A, is denoted by $[A]_\alpha$, and defined as:

$$\begin{aligned} [A]_\alpha &= \{x : A(x) \geq \alpha\}, \quad \text{where} \quad \alpha \in (0,1], \\ [A]_0 &= \overline{\{x : A(x) > 0\}}. \end{aligned}$$

Let X be any nonempty set and Y be a metric space. A mapping T is called a fuzzy mapping, if T is a mapping from X into $F(Y)$. A fuzzy mapping T is a fuzzy subset on $X \times Y$ with membership function $T(x)(y)$. The function $T(x)(y)$ is the grade of membership of y in $T(x)$. For convenience, we denote the α-level set of $T(x)$ by $[Tx]_\alpha$ instead of $[T(x)]_\alpha$ [1].

Definition 1.4 ([1])**.** A point $x \in X$ is called a fuzzy fixed point of a fuzzy mapping $T : X \to F(X)$ if there exists $\alpha \in (0,1]$ such that $x \in [Tx]_\alpha$.

Lemma 1.5 ([1])**.** Let A and B be nonempty closed and bounded subsets of a metric space (X,d). If $a \in A$, then

$$d(a,B) \leq H(A,B).$$

Lemma 1.6 ([1])**.** Let A and B be nonempty closed and bounded subsets of a metric space (X,d) and $0 < \alpha \in R$. Then, for $a \in A$, there exists $b \in B$ such that

$$d(a,b) \leq H(A,B) + \alpha.$$

2. Main results

Now, we present our main results.

Theorem 2.1. *Let (X,d) be a complete b-metric space with constant $b \geq 1$. Let $T : X \to F(X)$ be a fuzzy mapping and for $x \in X$, there exist $\alpha(x) \in (0,1]$ satisfying the following condition:*

$$\begin{aligned} H([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) \leq\ & a_1 d(x,[Tx]_{\alpha(x)}) + a_2 d(y,[Ty]_{\alpha(y)}) + a_3 d(x,[Ty]_{\alpha(y)}) \\ & + a_4 d(y,[Tx]_{\alpha(x)}) + a_5 d(x,y) \\ & + a_6 \frac{d(x,[Tx]_{\alpha(x)})(1 + d(x,[Tx]_{\alpha(x)}))}{1 + d(x,y)}, \end{aligned} \tag{2.1}$$

for all $x, y \in X$. Also, $a_i \geq 0$, where $i = 1, 2, \ldots 6$ with $ba_1 + a_2 + b(b+1)a_3 + b(a_5 + a_6) < 1$ and $\sum_{i=1}^{6} a_i < 1$. Then, T has a fixed point.

Proof. Let x_0 be any arbitrary point in X, such that $x_1 \in [Tx_0]_{\alpha(x_0)}$. Then, by Lemma 1.6 there exists $x_2 \in [Tx_1]_{\alpha(x_1)}$, such that

$$\begin{aligned}
d(x_1, x_2) &\leq H([Tx_0]_{\alpha(x_0)}, [Tx_1]_{\alpha(x_1)}) + (a_1 + ba_3 + a_5 + a_6) \\
&\leq a_1 d(x_0, [Tx_0]_{\alpha(x_0)}) + a_2 d(x_1, [Tx_1]_{\alpha(x_1)}) + a_3 d(x_0, [Tx_1]_{\alpha(x_1)}) \\
&\quad + a_4 d(x_1, [Tx_0]_{\alpha(x_0)}) + a_5 d(x_0, x_1) + \\
&\quad a_6 \frac{d(x_0, [Tx_0]_{\alpha(x_0)})(1 + d(x_0, [Tx_0]_{\alpha(x_0)}))}{1 + d(x_0, x_1)} + (a_1 + ba_3 + a_5 + a_6) \\
&\leq a_1 d(x_0, x_1) + a_2 d(x_1, x_2) + ba_3[d(x_0, x_1) + d(x_1, x_2)] \\
&\quad + a_5 d(x_0, x_1) + a_6 d(x_0, x_1) + (a_1 + ba_3 + a_5 + a_6) \\
d(x_1, x_2) &\leq \frac{a_1 + ba_3 + a_5 + a_6}{1 - (a_2 + ba_3)} d(x_0, x_1) \\
&\quad + \frac{(a_1 + ba_3 + a_5 + a_6)}{1 - (a_2 + ba_3)}. \qquad (2.2)
\end{aligned}$$

Let

$$\tau = \frac{(a_1 + ba_3 + a_5 + a_6)}{1 - (a_2 + ba_3)} < \frac{1}{b}.$$

Then by (2.2), we have

$$d(x_1, x_2) \leq \tau d(x_0, x_1) + \tau.$$

Again by Lemma 1.6, $x_3 \in [Tx_2]_{\alpha(x_2)}$ such that

$$\begin{aligned}
d(x_2, x_3) &\leq H([Tx_1]_{\alpha(x_1)}, [Tx_2]_{\alpha(x_2)}) + \frac{(a_1 + ba_3 + a_5 + a_6)^2}{1 - (a_2 + ba_3)} \\
&\leq a_1 d(x_1, [Tx_1]_{\alpha(x_1)}) + a_2 d(x_2, [Tx_2]_{\alpha(x_2)}) + a_3 d(x_1, [Tx_2]_{\alpha(x_2)}) \\
&\quad + a_4 d(x_2, [Tx_1]_{\alpha(x_1)}) + a_5 d(x_1, x_2) + \\
&\quad a_6 \frac{d(x_1, [Tx_1]_{\alpha(x_1)})(1 + d(x_1, [Tx_1]_{\alpha(x_1)}))}{1 + d(x_1, x_2)} + \frac{(a_1 + ba_3 + a_5 + a_6)^2}{1 - (a_2 + ba_3)} \\
d(x_2, x_3) &\leq a_1 d(x_1, x_2) + a_2 d(x_2, x_3) + ba_3[d(x_1, x_2) + d(x_2, x_3)] + a_5 d(x_1, x_2) \\
&\quad + a_6 d(x_1, x_2) + \frac{(a_1 + ba_3 + a_5 + a_6)^2}{1 - (a_2 + ba_3)} \\
&\leq \frac{(a_1 + ba_3 + a_5 + a_6)}{1 - (a_2 + ba_3)} d(x_1, x_2) + \frac{(a_1 + ba_3 + a_5 + a_6)^2}{(1 - (a_2 + ba_3))^2} \\
d(x_2, x_3) &\leq \left(\frac{(a_1 + ba_3 + a_5 + a_6)}{1 - (a_2 + ba_3)}\right)^2 d(x_0, x_1) \\
&\quad + 2\left(\frac{(a_1 + ba_3 + a_5 + a_6)}{1 - (a_2 + ba_3)}\right)^2 \quad \text{by (2.2)} \\
d(x_2, x_3) &\leq \tau^2 d(x_0, x_1) + 2\tau^2.
\end{aligned}$$

Continuing the same way, we obtain a sequence $\{x_n\}$ such that $x_n \in [Tx_{n+1}]_{\alpha(x_{n+1})}$, we have

$$d(x_n, x_{n+1}) \leq \tau^n d(x_0, x_1) + n\tau^n. \tag{2.3}$$

Now, for any positive integers m, n $(n > m)$, we have

$$\begin{aligned} d(x_m, x_n) &\leq b[d(x_m, x_{m+1}) + d(x_{m+1}, x_n)] \\ &\leq b(d(x_m, x_{m+1})) + b[b\{d(x_{m+1}, x_{m+2}) + d(x_{m+2}, x_n)\}] \\ &\leq b(d(x_m, x_{m+1})) + b^2(d(x_{m+1}, x_{m+2})) + \cdots + b^{n-m}(d(x_{n-1}, x_n)) \\ &\leq b\tau^m d(x_0, x_1) + mb\tau^m + b^2\tau^{m+1}d(x_0, x_1) + b^2(m+1)\tau^{m+1} + \cdots \\ &\quad + b^{n-m}\tau^{n-1}d(x_0, x_1) + b^{n-m}(n-1)\tau^{n-1} \qquad \text{by (2.3)} \\ &\leq b\tau^m(1 + b\tau + \cdots + b^{n-m}\tau^{n-m-1})d(x_0, x_1) + \sum_{i=m}^{n-1} b^{i-m} i\tau^i \\ &\leq \frac{b\tau^m}{1 - b\tau} d(x_0, x_1) + \sum_{i=m}^{n-1} b^{n-m} i\tau^i. \end{aligned}$$

Since $b\tau < 1$, it follows from Cauchy root test that $\Sigma b^{n-m} i\tau^i$ is convergent and hence $\{x_n\}$ is a Cauchy sequence. Since, (X, d) is complete. Then, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Now,

$$\begin{aligned} d(z, [Tz]_{\alpha(z)}) &\leq b\left[d(z, x_{n+1}) + d(x_{n+1}, [Tz]_{\alpha(z)})\right] \\ &\leq b\left[d(z, x_{n+1}) + H([Tx_n]_{\alpha(x_n)}, [Tz]_{\alpha(z)})\right]. \end{aligned}$$

Using (2.1), with $n \to \infty$ we get

$$(1 - b(a_2 + a_3))d(z, [Tz]_{\alpha(z)}) \leq 0.$$

So, we get

$$z \in [Tz]_{\alpha(z)}.$$

Hence, $z \in X$ is a fixed point. □

Theorem 2.2. *Let (X, d) be a complete b-metric space with constant $b \geq 1$. Let $S, T : X \to F(X)$ be two fuzzy mappings and for $x \in X$, there exist $\alpha_S(x), \alpha_T(x) \in (0, 1]$ satisfying the following condition:*

$$\begin{aligned} H([Tx]_{\alpha_T(x)}, [Sy]_{\alpha_S(y)}) &\leq a_1 d(x, [Tx]_{\alpha_T(x)}) + a_2 d(y, [Sy]_{\alpha_S(y)}) \\ &\quad + a_3 d(x, [Sy]_{\alpha_S(y)}) + a_4 d(y, [Tx]_{\alpha_T(x)}) \\ &\quad + a_5 d(x, y). \end{aligned} \tag{2.4}$$

for all $x, y \in X$. Also $a_i \geq 0$, where $i = 1, 2, \ldots 5$ with $(a_1 + a_2)(b + 1) + b(a_3 + a_4)(b + 1) + 2ba_5 < 2$ and $\sum_{i=1}^{5} a_i < 1$. Then, S and T have a common fixed point.

Proof. Let x_0 be any arbitrary point in X, such that $x_1 \in [Tx_0]_{\alpha(x_0)}$. Then, by Lemma 1.6 there exists $x_2 \in [Sx_1]_{\alpha(x_1)}$, such that

$$\begin{aligned}
d(x_1,x_2) &\leq H([Tx_0]_{\alpha(x_0)},[Sx_1]_{\alpha(x_1)}) + (a_1+ba_3+a_5)\\
&\leq a_1 d(x_0,[Tx_0]_{\alpha(x_0)}) + a_2 d(x_1,[Sx_1]_{\alpha(x_1)}) + a_3 d(x_0,[Sx_1]_{\alpha(x_1)})\\
&+ a_4 d(x_1,[Tx_0]_{\alpha(x_0)}) + a_5 d(x_0,x_1) + (a_1+ba_3+a_5)\\
&\leq a_1 d(x_0,x_1) + a_2 d(x_1,x_2) + ba_3[d(x_0,x_1)+d(x_1,x_2)]\\
&+ a_5 d(x_0,x_1) + (a_1+ba_3+a_5) \qquad (2.5)\\
d(x_1,x_2) &\leq \frac{a_1+ba_3+a_5}{1-(a_2+ba_3)} d(x_0,x_1) + \frac{(a_1+ba_3+a_5)}{1-(a_2+ba_3)}.
\end{aligned}$$

Similarly, by symmetry we have

$$\begin{aligned}
d(x_2,x_1) &\leq H([Sx_1]_{\alpha(x_1)},[Tx_0]_{\alpha(x_0)}) + (a_2+ba_4+a_5)\\
&\leq a_1 d(x_1,[Sx_1]_{\alpha(x_1)}) + a_2 d(x_0,[Tx_0]_{\alpha(x_0)}) + a_3 d(x_1,[Tx_0]_{\alpha(x_0)})\\
&+ a_4 d(x_0,[Sx_1]_{\alpha(x_1)}) + a_5 d(x_1,x_0) + (a_2+ba_4+a_5)\\
&\leq a_1 d(x_1,x_2) + a_2 d(x_0,x_1) + ba_4[d(x_0,x_1)+d(x_1,x_2)]\\
&+ a_5 d(x_1,x_0) + (a_2+ba_4+a_5) \qquad (2.6)\\
d(x_2,x_1) &\leq \frac{a_2+ba_4+a_5}{1-(a_1+ba_4)} d(x_0,x_1) + \frac{(a_2+ba_4+a_5)}{1-(a_1+ba_4)}.
\end{aligned}$$

Adding (2.5) and (2.6), we get

$$\begin{aligned}
d(x_1,x_2) \leq{}& \frac{a_1+a_2+ba_3+ba_4+2a_5}{2-(a_1+a_2+ba_3+ba_4)} d(x_0,x_1)\\
&+ \frac{a_1+a_2+ba_3+ba_4+2a_5}{2-(a_1+a_2+ba_3+ba_4)}. \qquad (2.7)
\end{aligned}$$

Let

$$\tau = \frac{a_1+a_2+ba_3+ba_4+2a_5}{2-(a_1+a_2+ba_3+ba_4)} < \frac{1}{b}.$$

Then by (2.7), we have

$$d(x_1,x_2) \leq \tau d(x_0,x_1) + \tau$$

Again by Lemma 1.6, $x_3 \in [Tx_2]_{\alpha(x_2)}$ such that

$$\begin{aligned}
d(x_2,x_3) &\leq H([Sx_1]_{\alpha(x_1)},[Tx_2]_{\alpha(x_2)})\\
&+ \frac{(a_1+a_2+ba_3+ba_4+2a_5)^2}{2-(a_1+a_2+ba_3+ba_4)}\\
&\leq \tau^2 d(x_0,x_1) + 2\tau^2.
\end{aligned}$$

Continuing the same way, we obtain a sequence $\{x_n\}$ such that $x_{2n+1} \in [Tx_{2n}]_{\alpha(x_{2n})}$ and $x_{2n+2} \in [Sx_{2n+1}]_{\alpha(x_{2n+1})}$, with

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) &\leq H([Tx_{2n}]_{\alpha(x_{2n})}, [Sx_{2n+1}]_{\alpha(x_{2n+1})}) \\
&+ \frac{(a_1 + ba_3 + a_5)^{2n+1}}{(1 - (a_2 + ba_3))^{2n}} \\
&\leq a_1 d(x_{2n}, [Tx_{2n}]_{\alpha(x_{2n})}) + a_2 d(x_{2n+1}, [Sx_{2n+1}]_{\alpha(x_{2n+1})}) \\
&+ a_3 d(x_{2n}, [Sx_{2n+1}]_{\alpha(x_{2n+1})}) + a_4 d(x_{2n+1}, [Tx_{2n}]_{\alpha(x_{2n})}) \\
&+ a_5 d(x_{2n}, x_{2n+1}) \\
&+ \frac{(a_1 + ba_3 + a_5)^{2n+1}}{(1 - (a_2 + ba_3))^{2n}}
\end{aligned}$$

$$d(x_{2n+1}, x_{2n+2}) \leq \frac{a_1 + ba_3 + a_5}{1 - (a_2 + ba_3)} d(x_{2n}, x_{2n+1}) + \frac{(a_1 + ba_3 + a_5)^{2n+1}}{(1 - (a_2 + ba_3))^{2n+1}}. \tag{2.8}$$

Similarly,

$$\begin{aligned}
d(x_{2n+2}, x_{2n+1}) &\leq H([Sx_{2n+1}]_{\alpha(x_{2n+1})}, [Tx_{2n}]_{\alpha(x_{2n})}) \\
&+ \frac{(a_2 + ba_4 + a_5)^{2n+1}}{(1 - (a_1 + ba_4))^{2n}} \\
&\leq a_1 d(x_{2n+1}, [Sx_{2n+1}]_{\alpha(x_{2n+1})}) + a_2 d(x_{2n}, [Tx_{2n}]_{\alpha(x_{2n})}) \\
&+ a_3 d(x_{2n+1}, [Tx_{2n}]_{\alpha(x_{2n})}) + a_4 d(x_{2n}, [Sx_{2n+1}]_{\alpha(x_{2n+1})}) \\
&+ a_5 d(x_{2n+1}, x_{2n}) + \frac{(a_2 + ba_4 + a_5)^{2n+1}}{(1 - (a_1 + ba_4))^{2n}}
\end{aligned}$$

$$d(x_{2n+2}, x_{2n+1}) \leq \frac{(a_2 + ba_4 + a_5)}{(1 - (a_1 + ba_4))} d(x_{2n}, x_{2n+1}) + \frac{(a_2 + ba_4 + a_5)^{2n+1}}{(1 - (a_1 + ba_4))^{2n}}. \tag{2.9}$$

By (2.8) and (2.9),

$$d(x_{2n+1}, x_{2n+2}) \leq \tau d(x_{2n}, x_{2n+1}) + \tau^{2n+1}.$$

Therefore,

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq \frac{a_1 + a_2 + ba_3 + ba_4 + 2a_5}{2 - (a_1 + a_2 + ba_3 + ba_4)} d(x_{n-1}, x_n) \\
&+ \left(\frac{a_1 + a_2 + ba_3 + ba_4 + 2a_5}{2 - (a_1 + a_2 + ba_3 + ba_4)} \right)^n \\
d(x_n, x_{n+1}) &\leq \tau d(x_{n-1}, x_n) + \tau^n
\end{aligned}$$

$$
\begin{aligned}
&\le \tau\left[\tau d(x_{n-2},x_{n-1})+\tau^{n-1}\right]+\tau^{n}\\
&= \tau^{2} d(x_{n-2},x_{n-1})+2\tau^{n}\\
&\le \cdots\cdots
\end{aligned}
$$

$$d(x_n,x_{n+1}) \le \tau^n d(x_0,x_1)+n\tau^n. \tag{2.10}$$

Now, for any positive integers m,n $(n>m)$, we have

$$
\begin{aligned}
d(x_m,x_n) &\le b[d(x_m,x_{m+1})+d(x_{m+1},x_n)]\\
&\le b(d(x_m,x_{m+1}))+b[b\{d(x_{m+1},x_{m+2})+d(x_{m+2},x_n)\}]\\
&\le b(d(x_m,x_{m+1}))+b^2(d(x_{m+1},x_{m+2}))+\cdots+b^{n-m}(d(x_{n-1},x_n))\\
&\le b\tau^m d(x_0,x_1)+mb\tau^m+b^2\tau^{m+1}d(x_0,x_1)+b^2(m+1)\tau^{m+1}+\cdots\\
&+b^{n-m}\tau^{n-1}d(x_0,x_1)+b^{n-m}(n-1)\tau^{n-1} \qquad \text{by (2.10)}\\
&\le b\tau^m(1+b\tau+\cdots+b^{n-m}\tau^{n-m-1})d(x_0,x_1)+\sum_{i=m}^{n-1} b^{i-m} i\tau^i\\
&\le \frac{b\tau^m}{1-b\tau}d(x_0,x_1)+\sum_{i=m}^{n-1} b^{n-m} i\tau^i.
\end{aligned}
$$

Since $b\tau<1$, it follows from Cauchy root test that $\Sigma b^{n-m} i\tau^i$ is convergent and hence $\{x_n\}$ is a Cauchy sequence in X. Since, (X,d) is complete. Then, there exists $z\in X$ such that $x_n\to z$ as $n\to\infty$.

Now, we prove $z\in X$ is a common fixed point of S and T.

$$
\begin{aligned}
d(z,[Sz]_{\alpha(z)}) &\le b\left[d(z,x_{2n+1})+d(x_{2n+1},[Sz]_{\alpha(z)})\right]\\
&\le b\left[d(z,x_{2n+1})+H([Tx_{2n}]_{\alpha(x_{2n})},[Tz]_{\alpha(z)})\right].
\end{aligned}
$$

Using (2.4), with $n\to\infty$ we get

$$(1-b(a_2+a_3))d(z,[Sz]_{\alpha(z)})\le 0.$$

So, we get

$$z\in [Sz]_{\alpha(z)}.$$

This implies that $z\in X$ is a fixed point for S. Similarly, we can show that $z\in[Tz]_{\alpha(z)}$. Hence, $z\in X$, is a common fixed point. □

Example 2.3. Let $X=[0,1]$ and $d(x,y)=|x-y|$, whenever $x,y\in X$, then (X,d) is a complete b-metric space. Define a fuzzy mapping $T:X\to F(X)$ by

$$T(x)(t)=\begin{cases}1, & 0\le t\le x/4\\ 1/2, & x/4<t\le x/3\\ 1/4, & x/3<t\le x/2\\ 0, & x/2<t\le 1\end{cases}$$

For all $x \in X$, there exists $\alpha(x) = 1$, such that$[Tx]_{\alpha(x)} = [0, \frac{x}{4}]$. Then,

$$\begin{aligned} H([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) \leq & \frac{1}{5}\left|x - \frac{x}{4}\right| + \frac{1}{10}\left|y - \frac{y}{4}\right| + \frac{1}{15}\left|x - \frac{y}{4}\right| \\ & + \frac{1}{20}\left|y - \frac{x}{4}\right| + \frac{1}{25}|x - y| \\ & + \frac{1}{30}\left(\frac{\left|x - \frac{x}{4}\right|\left(1 + \left|x - \frac{x}{4}\right|\right)}{1 + |x - y|}\right) \end{aligned}$$

Since, all the conditions of Theorem 2.1 are satisfied. Therefore, $0 \in X$ is the fixed point of T.

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References

[1] A. Azam, *Fuzzy Fixed Points of Fuzzy Mappings via a Rational Inequality*, Hacettepe Journal of Mathematics and Statistics, 40 (3) (2011), 421-431.

[2] H. Aydi, M. Bota, E. Karapınar and S. Mitrović, *A fixed point theorem for set-valued quasi-contractions in b-metric spaces*, Fixed Point Theory and Applications, 2012:88.

[3] D. Butnariu, *Fixed point for fuzzy mapping*, Fuzzy Sets and Systems, 7 (1982), 191-207.

[4] S. Heilpern, *Fuzzy mappings and fixed point theorems*, Journal of Mathematical Analysis and Applications, 83(2) (1981), 566-569.

[5] J. Joseph, D. Roselin and M. Marudai, *Fixed Point Theorem on Multi-Valued Mappings in b-metric spaces*, SpringerPlus, 5:217, (2016).

[6] S.B. Nadler, *Multivalued contraction mappings*, Pacific Journal of Mathematics, 30 (1969), 475-488.

[7] M. Rashid, A. Shahzad and A. Azam, *Fixed point theorems for L-fuzzy mappings in quasi-pseudo metric spaces*, Journal of Intelligent & Fuzzy Systems 32 (2017), 499-507.

[8] M.D. Weiss, *Fixed points and induced fuzzy topologies for fuzzy sets*, Journal of Mathematical Analysis and Applications, 50 (1975), 142-150.

[9] L.A. Zadeh, *Fuzzy Sets*, Information and Control, 8 (1965), 338-353.

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