ON PÁL-TYPE INTERPOLATION II

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Abstract. In this paper, we study the convergence of Pál-type interpolation on two sets of non-uniformly distributed zeros on the unit circle, which are obtained by projecting vertically the nodes of the real line.

Keywords: Pál-type Interpolation, explicit forms, convergence.

1. Introduction

In 1975, L.G. Pál [5] introduced a different type of Hermite interpolation by prescribing the function values at one set of points, whereas its first order derivative values at another set of points. He obtained a unique polynomial of degree at most $2n-1$ satisfying the interpolating properties.

After that, many mathematicians [1,3] have taken such problem on a different set of nodes *viz.* finite interval, infinite interval or unit circle. Author [2] had also considered some Pál-type Interpolation on the real line and unit circle and established the convergence theorem for them.

In this paper, we consider two pairwise disjoint set nodes $\xi_n = \{t_k\}_{k=0}^{2n-1}$ and $Z_n = \{z_k\}_{k=1}^{2n}$, which are vertically projected zeros of two different polynomials onto the unit circle. On these sets of nodes , we consider the Pál-type Interpolation and obtain a convergence theorem for that interpolatory polynomial. In section 2, we give some preliminaries and in section 3, we describe the problem and in sections 4 and 5, we give explicit representation and estimation of interpolatory polynomials respectively. In section 6, we give the convergence of such polynomials.

2. Preliminaries

In this section, we shall give some well known results, which we shall use in our present paper.

The differential equation satisfied by $\Pi_n(x)$ is

$$(2.1)\qquad \left(1-x^2\right)\ \Pi_n''(x)+n(n-1)\,\Pi_n(x)=0,$$

$$(2.2)\qquad W(z)=K_n\ \Pi_n(x)\,z^n,$$

$$(2.3)\qquad H(z)=K_n^*\ \Pi_n'(x)\ \ z^{n-1},$$

$$(2.4)\qquad R(z)=\left(z^2-1\right)H(z)\,,$$

we shall require the fundamental polynomials of Lagrange interpolation based on Z_n and ξ_n, respectively

$$(2.5)\qquad L_k(z)=\frac{R(z)}{(z-t_k)\,R'(t_k)},\ k=0\,(1)\,2n-1,$$

$$(2.6)\qquad l_k(z)=\frac{W(z)}{(z-z_k)\,W'(z_k)},\ k=1\,(1)\,2n,$$

$$(2.7)\qquad W'(z_k)=\frac{K_n}{2}\ \Pi_n'(x_k)\ \left(z_k^2-1\right)\ z_k^{n-2},\ \ k=1\,(1)\,2n,$$

$$(2.8)\qquad W'(t_k)=K_n\ n\ \Pi_n(u_k)\ \ t_k^{n-1}\ \ k=0\,(1)\,2n-1,$$

$$(2.9)\qquad H'(z_k)=K_n^*\ \ (n-1)\ \ \Pi_n'(x_k)\,z_k^{n-2},k=1\,(1)\,2n,$$

$$(2.10)\qquad H'(t_k)=\frac{K_n^*}{2}\ \Pi_n''(u_k)\ \left(t_k^2-1\right)\ t_k^{n-1},k=0\,(1)\,2n-1,$$

$$(2.11)\qquad I_{1j}(z)=\int_0^z t^{n-j-1}\ W(t)\ \ dt,j=0,1$$

such that $I_{1j}(-1)=(-1)^{n-j}\ I_{1j}(1)$.

We shall also use the following well-known inequalities (see [6])

$$(2.12)\qquad |P_n(x)|\le 1,$$

$$(2.13)\qquad |\Pi_n(x)|\le\left(\frac{2n}{\pi}\right)^{\frac{1}{2}},$$

$$(2.14)\qquad \left(1-x^2\right)^{\frac{1}{4}}|P_n(x)|\le\left(\frac{2}{n\pi}\right)^{\frac{1}{2}}.$$

If u_k be the zeros of $P_n'(x)$, then

$$(2.15)\qquad P_n(u_k)>\frac{1}{\sqrt{8\pi k}}.$$

Let $x_k=\cos\theta_k$, $(k=1,2,..,n)$ be the zeros of n^{th} Legendre polynomial $P_n(x)$, with $1>x_1>x_2>....>-1$, then

$$(2.16)\qquad \begin{cases}\left(1-x_k^2\right)\ge k^2n^{-2}, & k=1,2,...,\left[\frac{n}{2}\right]\\ \left(1-x_k^2\right)\ge(n-k+1)^2\,n^{-2}, & k=\left[\frac{n}{2}\right]+1,...n\end{cases}$$

$$(2.17)\qquad \begin{cases}|P_n'(x_k)|\ge ck^{-\frac{3}{2}}n^2, & k=1,2,...,\left[\frac{n}{2}\right]\\ |P_n'(x_k)|\ge c\,(n-k+1)^{-\frac{3}{2}}\,n^2, & k=\left[\frac{n}{2}\right]+1,...n.\end{cases}$$

For more details, see [6].

3. The problem

Let $\{t_k\}_{k=0}^{2n-1}$ and $\{z_k\}_{k=1}^{2n}$ be two disjoint set of nodes obtained by projecting vertically the zeros of $\left(1-x^2\right)\Pi_n'(x)$ and $\Pi_n(x)$ onto the unit circle respectively, where

$$\Pi_n(x) = \left(1-x^2\right) P_{n-1}'(x), n = 2,3,... \tag{3.1}$$

$P_{n-1}(x)$ stands for $(n-1)^{th}$ Legendre polynomial.

Here we are interested to determine the convergence of interpolatory polynomial satisfying the conditions :

$$\begin{cases} R_n(t_k) = \alpha_k, & k = 0(1)2n-1 \\ R_n'(z_k) = 0, & k = 1(1)2n, \end{cases} \tag{3.2}$$

where $\alpha_k' s$ are arbitrary given complex numbers.

4. Explicit representation of interpolatory polynomial

We shall write $Q_n(z)$ satisfying (3.2) as

$$R_n(z) = \sum_{k=0}^{2n-1} \alpha_k A_k(z), \tag{4.1}$$

where $A_k(z)$ are unique polynomial of degree at most $4n-1$ determined by the following conditions:

For $k = 0(1)2n-1$

$$\begin{cases} A_k(t_j) = \delta_{jk}, & j = 0(1)2n-1 \\ A_k'(z_j) = 0, & j = 1(1)2n. \end{cases} \tag{4.2}$$

Theorem 4.1. *For* $k = 0(1)2n-1$

$$\begin{aligned} A_k(z) = {} & L_k(z)\frac{W(z)}{W(t_k)} \\ & + \frac{z^{-n+1}H(z)}{(t_k^2-1)W(t_k)H'(z_k)}\{N_k(z) + b_{10}I_{10}(z) + b_{11}I_{11}(z)\}, \end{aligned} \tag{4.3}$$

where

$$N_k(z) = -\int_0^z z^{n-1}(z^2-1)\frac{W'(z)+c_kW(z)}{(z-t_k)}dz, \quad c_k = -\frac{W'(t_k)}{W(t_k)}, \tag{4.4}$$

$$b_{10} = -\frac{N_k(1) + (-1)^{n+1} N_k(-1)}{2I_{10}(1)}, \tag{4.5}$$

$$b_{11} = -\frac{N_k(1) + (-1)^{n} N_k(-1)}{2I_{11}(1)}. \tag{4.6}$$

Proof. Consider (4.3), we can obtain (4.4)-(4.6) owing to conditions (4.2). The theorem follows. □

5. Estimation of fundamental polynomials

Lemma 5.1. *Let $L_k(z)$ be given by* (2.5) *Then we have*

$$\max_{|z|=1} \sum_{k=0}^{2n-1} |L_k(z)| \leqslant c \log n, \tag{5.1}$$

where c is a constant independent of n and z.

Lemma 5.2. *Let $l_k(z)$ be given by* (2.6) *Then, we have*

$$\max_{|z|=1} \sum_{k=1}^{2n} |l_k(z)| \leqslant c \log n, \tag{5.2}$$

where c is a constant independent of n and z.

Lemma 5.3. *Let $A_k(z)$ be defined in Theorem* 4.1*, then for* $|z| \leqslant 1$

$$\sum_{k=0}^{2n-1} |A_k(z)| \leqslant c n^{\frac{1}{2}} \log n, \tag{5.3}$$

where c is a constant independent of n and z.

Proof. Using Lemma 5.2 and inequalities (2.11)-(2.16), we get (5.3). □

6. Convergence

In this section, we prove the following theorem:

Theorem 5.1. *Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$, then the sequence $\{R_n\}$ defined by*

$$R_n(z) = \sum_{k=0}^{2n-1} f(t_k) A_k(z) \tag{6.1}$$

converges uniformly to $f(z)$.

To prove (6.1), we shall need the following:

Remark. Let $f(z)$ be continuous for $|z| \leq 1$ and $f' \in Lip\frac{1}{2}$, then the sequence $\{R_n\}$ converges uniformly to $f(z)$ provided

$$\omega_2\left(f, n^{-1}\right) = O(n^{-\frac{3}{2}}). \tag{6.2}$$

Jackson's Inequality. Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$, then there exists a polynomial $F_n(z)$ of degree at most $4n-1$ satisfying

$$|F_n(z) - f(z)| \leq c\omega_2\left(f, n^{-1}\right), z = e^{i\theta}, (0 \leq \theta < 2\pi). \tag{6.3}$$

Also an inequality due to O. Kiŝ [4] viz.

$$\left|F_n^{(m)}(z)\right| \leq cn^m \omega_2\left(f, n^{-1}\right), \quad \text{for } m \in I^+. \tag{6.4}$$

Proof of Theorem 5.1. Let $z = e^{i\theta}(0 \leq \theta < 2\pi)$, using (6.1)-(6.4) and Lemma 5.3, the theorem follows. □

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