SOME PROPERTIES OF A NEW KIND OF DOWNWARD SETS IN CERTAIN BANACH SPACES*

Sh. Al-Sharif†
M. Al-Qahtani
Department of Mathematics
Yarmouk University
Irbid
Jordan
sharifa@yu.edu.jo
mesfer-alqhtani@hotmail.com

Abstract. Let X be a Banach lattice with strong unit. In this paper, we give some characterizations of certain kind of downward sets in the sequence space $\ell^{\infty}(X)$. Further some results on best approximation of those sets are presented.
Keywords: Downwards sets, proximinal sets, Banach lattices.

1. Introduction

A vector lattice is an ordered vector space such that $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$ exist for all $x, y \in X$. Vector lattices are also called Riesz spaces or linear lattices, [9]. The most obvious example of a vector lattice is the set of real numbers, $\mathbb{R}$ with all the usual operations. A normed linear lattice X is a real normed vector lattice such that

$$|x| \leq |y| \Rightarrow \|x\| \leq \|y\| \text{ for any } x, y \in X,$$

where, $|x| := sup\,\{x, -x\}$ for each $x \in X$. If $(X, \leqq)$ is a normed ordered vector space, recall that an element in X, denoted by 1, is called a strong unit if $\|1\|$ $= 1$ and for each $x \in X$, there exists $0 < \lambda \in \mathbb{R}$ such that $x \leq \lambda 1$. Using the strong unit 1 a norm on X is defined by

$$\|x\| = \inf\{\lambda > 0 : |x| \leq \lambda 1\}$$

for all $x \in X$. It is clear that for all $x \in X$,

$$|x| \leq \|x\|\, 1. \tag{1.1}$$

Using (1.1), the closed unit ball of X, $B(x, r) = \{y \in X : \|y - x\| \leq r\}$, with center x and radius r can be written as

$$B(x, r) = \{y \in X : x - r1 \leq y \leq x + r1\}. \tag{1.2}$$

∗. This research is supported in part by Yarmouk University
†. Corresponding author

Certain kind of sets in Banach lattices that is called downward sets plays an important role in some part of mathematical economics and game theory. Recall that a subset W of a Banach lattice X is said to be downward, if $(w \in W, x \leq w)$ implies that $x \in W$. The set of the form $\{w \in \mathbb{R}^n : w \leq x\}$, where $x \in \mathbb{R}^n$ is a simple example of a downward set. For more on Banach Lattices we refer the reader to [7, 8, 9].

Convex sets in normed linear spaces and their best approximation properties has many important applications in science. However, since convexity in somehow is a restrictive assumption, so there is a need to study the best approximation by elements of some kind of non convex sets. In [6], Rubinov and Singer developed a theory of best approximation by elements of so-called normal sets in the finite-dimensional coordinate space $\mathbb{R}^n$ endowed with the max-norm. Martinez-Legaz, Rubionv and Singer in [3] have developed a theory of best approximation of downward subsets of the space $\mathbb{R}^n$. While the problem of best approximation by elements of downward sets in a Banach lattice was studied in [4, 5], the problem of best approximation in vector valued functions such as $\ell^p(X)$, $1 \leq p \leq \infty$, where X is a Banach lattice has never been considered.

It is the aim of this paper to give some characterization of some kind of downward sets in the space of bounded sequences $\ell^\infty(X)$ endowed with the max norm in terms of a coupling function. Further we study the problem of best approximation of those kind of sets. Indeed we precisely study proximity of $\ell^\infty(W)$ in $\ell^\infty(X)$, where X is a Banach lattice and W is a downward subset of X.

Throughout of this paper, X is a Banach Lattice with a strong unit and $\mathbb{N}$ is the set of all positive integers. Moreover the interior, the closure and the boundary of the subset W of X will be denoted by *intW*, *clW* and *bd*(W) respectively.

2. Characterization of downward sets in $\ell^\infty(X)$

For a Banach space X, let $\ell^\infty(X)$ denotes the space of all sequences $x = (x_i)$, $x_i \in X$, with $\|x\|_\infty = \sup_{i\in\mathbb{N}} \|x_i\| < \infty$. If W is a downward subset of X, by $\ell^\infty(W)$ we denote the subset of all sequences $w = (w_i)$, $w_i \in W$, with $\|w\|_\infty = \sup_{i\in\mathbb{N}} \|w_i\| < \infty$. In this section we characterize some kind of downward sets in the sequence space $\ell^\infty(X)$ in terms of a coupling function. We start by defining a partial order relation "$\leq$" on $\ell^\infty(X)$, where X is a Banach lattice with strong unit "1" as follows:

Definition 1. *For $x = (x_n), y = (y_n) \in \ell^\infty(X)$, we say that $x \leq y$ if and only if $x_n \leq y_n$ for all n.*

Proposition 2. *A relation $\leq$ is a partial order in $\ell^\infty(X)$.*

Proof. Follows from the definition. □

Proposition 3. *If X is a Banach lattice with strong unit 1, then $\ell^\infty(X)$ is a Banach lattice with strong unit $(1, 1, ..., 1, ...)$.*

Proof. Let $(x_n) \in \ell^\infty(X)$. Then, $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$. Since for all $n, x_n \in X$, and 1 is the strong unit of X, there exists $\lambda_n > 0$, such that $x_n \le \lambda_n 1$. With no less of generality, we can choose λ_n to be $\|x_n\|$. If $\lambda = \sup_{n\in\mathbb{N}} |\lambda_n| = \|(x_n)\|_\infty < \infty$, then, for all $n, x_n \le \|x_n\| 1 \le (\sup_{n\in\mathbb{N}} \|x_n\|)1 = \lambda 1$. Hence, $(x_n) \le (\lambda 1, \lambda 1, ..., \lambda 1, ...) = \lambda(1, 1, ..., 1, ...) = \lambda 1$. □

Proposition 4. *W is a downward set in X if and only if $\ell^\infty(W)$ is a downward set in $\ell^\infty(X)$.*

Proof. Let $x = (x_n) \in \ell^\infty(W)$ and $w = (w_n) \in \ell^\infty(X)$, such that $w \le x$. Then for all n, $w_n \le x_n$. But W is downward set in X and $x_n \in W$ for all n, it follow that $w_n \in W$ for all n, and $w \in \ell^\infty(W)$.

Conversely, let $w \in W, x \in X$, such that $x \le w$, consider the sequence $u = (x, x, x, ...) \in \ell^\infty(W)$, $v = (w, w, w, ...) \in \ell^\infty(X)$. Since $\ell^\infty(W)$ is a downward set and $u \le v$ it follows that $x \in W$. □

Proposition 5. *If G is a closed subset of X, then $\ell^\infty(G)$ is a closed subset of $\ell^\infty(X)$.*

Proof. Let $\left(x_n^k\right), k \ge 1$ be a sequence of $\ell^\infty(G)$, such that $\left(x_n^k\right) \to (x_n)$. Since for all $n, x_n^k \in G$ and G closed, it follows that, $x_n \in G$ for all n. Hence $(x_n) \in \ell^\infty(G)$. □

Theorem 6. *Let W be a closed downward subset of X and $(x_n) \in \ell^\infty(X)$. Then the following are true:*

(a) If $(x_n) \in \ell^\infty(W)$, then $(x_n - \lambda_n 1) \in int(\ell^\infty(W))$, for all $\epsilon > 0$, and all $(\lambda_n) \in \ell^\infty(\mathbb{R})$ with $\inf_{n\in\mathbb{N}} \lambda_n \ge \epsilon$.

(b) $int(\ell^\infty(W)) = \{(x_n) \in \ell^\infty(X) : (x_n + \epsilon 1) \in \ell^\infty(W)$ for some $\epsilon > 0\}$.

Proof. (a) For $\epsilon > 0$ and $(x_n) \in \ell^\infty(W)$, let, $\lambda_n \in \ell^\infty(\mathbb{R})$ with $\inf(\lambda_n) \ge \epsilon$ and

$$V = \left\{(y_n) \in \ell^\infty(X) : \|(y_n) - (x_n - \lambda_n 1)\|_\infty < \epsilon\right\},$$

be an open neighborhood for $(x_n - \lambda_n 1)$ in $\ell^\infty(X)$. Then, for all, n

$$\|y_n - (x_n - \lambda_n 1)\| \le \sup_n \|y_n - (x_n - \lambda_n 1)\| = \|(y_n) - (x_n - \lambda_n 1)\|_\infty < \epsilon.$$

Hence, $|y_n - (x_n - \lambda_n 1)| \le \|y_n - (x_n - \lambda_n 1)\| < \epsilon$. Using (1.2)

$$\begin{aligned} -\epsilon 1 &< y_n - (x_n - \lambda_n 1) < \epsilon 1 \\ -\epsilon 1 + x_n - \lambda_n 1 &< y_n < \epsilon 1 + (x_n - \lambda_n 1) = x_n + (\epsilon - \lambda_n) 1 < x_n. \end{aligned}$$

Since W is a downward set it follows that $y_n \in W$ for all n. Consequently $(y_n) \in \ell^\infty(W)$ and $V \subset \ell^\infty(W)$. Hence $(x_n - \lambda_n 1) \in \text{int}(\ell^\infty(W)$. Notice that, (λ_n) can be chosen so that $\lambda_n = \epsilon\ \forall n$.

(b) Let $(x_n) \in \operatorname{int}(\ell^\infty(W))$. Then there exists $\epsilon_\circ > 0$, such that the closed ball $B((x_n), \epsilon_\circ) \subseteq \ell^\infty(W)$. That is

$$B((x_n), \epsilon_\circ) = \left\{ (y_n) \in \ell^\infty(X) : \sup_n \|y_n - x_n\| \leq \epsilon_\circ \right\} \subseteq \ell^\infty(W).$$

Hence

$$\begin{aligned} B((x_n), \epsilon_\circ) = \{(y_n) \in \ell^\infty(X) : \|y_n - x_n\| &\leq \sup_n \|y_n - x_n\| \\ &= \|(y_n) - (x_n)\|_\infty \leq \epsilon_\circ\}. \end{aligned}$$

Consequently using (1.2) we get

$$B((x_n), \epsilon_\circ) = \{(y_n) \in \ell^\infty(X) : x_n - \epsilon_\circ 1 \leq y_n \leq x_n + \epsilon_\circ 1\} \subseteq \ell^\infty(W),$$

and $(\epsilon_\circ 1 + x_n) \in \ell^\infty(W)$.

Conversely, suppose that there exists $\epsilon > 0$, such that $(x_n + \epsilon 1) \in \ell^\infty(W)$. Then, by part (a), we get $(x_n) = (x_n + \epsilon 1 - \epsilon 1) \in \operatorname{int}(\ell^\infty(W))$, which completes the proof. □

Corollary 7. *Let W be a closed downward subset of X and $(w_n) \in \ell^\infty(W)$. Then, $(w_n) \in bd(\ell^\infty(W))$ if and only if $(\lambda 1 + w_n) \notin \ell^\infty(W)$ for all $\lambda > 0$.*

Proof. Suppose that $(\lambda 1 + w_n) \in \ell^\infty(W)$ for some $\lambda > 0$. Then

$$(w_n) = (w_n + \lambda 1 - \lambda 1) \in \operatorname{int}(\ell^\infty(W)),$$

which is a contradiction, since $(w_n) \in \operatorname{bd}(\ell^\infty(W))$. Hence, $(\lambda 1 + w_n) \notin \ell^\infty(W)$, for all $\lambda > 0$.

Conversely, suppose that $(w_n) \in \operatorname{int}(\ell^\infty(W))$. Then by Theorem 6, $(\lambda 1 + w_n) \in \ell^\infty(W)$, for some $\lambda > 0$. This is a contradiction, since $(\lambda 1 + w_n) \notin \ell^\infty(W)$. Hence $(w_n) \notin \operatorname{int}(\ell^\infty(W))$. But $(w_n) \in \ell^\infty(W)$, it follows that $(w_n) \in \operatorname{bd}(\ell^\infty(W))$. □

Now, we will define what we call it a coupling ψ function that will be used later to characterize some kind of downward sets as follows:

$$\psi : \ell^\infty(X) \times \ell^\infty(X) \to \ell^\infty(\mathbb{R}) \tag{2.1}$$

$$\psi((x_n), (y_n)) = (\Phi(x_n, y_n)),$$

where, $\Phi(x_n, y_n) = \sup\{\lambda \in \mathbb{R} : \lambda 1 \leq x_n + y_n\}$, for all $(x_n), (y_n) \in \ell^\infty(X)$. Since 1 is a strong unit of X, it follows that the set $\{\lambda \in \mathbb{R} : \lambda 1 \leq x_n + y_n\}$ is non-empty and bounded above (by the number $\|x_n + y_n\|$). Clearly this set is closed.

For each $(y_n) \in \ell^\infty(X)$, define the function $\psi_{(y_n)} : \ell^\infty(X) \to \ell^\infty(\mathbb{R})$ by

$$\psi_{(y_n)}((x_n)) = \psi((x_n), (y_n)) = (\Phi(x_n, y_n)). \tag{2.2}$$

Proposition 8. *The function ψ satisfies the following properties.*

(1) *For all $(x_n),(y_n)\in\ell^\infty(X)$, $-\infty\le\|\psi((x_n),(y_n))\|_\infty\le\|(x_n)+(y_n)\|_\infty$.*

(2) *$(\Phi(x_n,y_n)1)\le(x_n+y_n)$ for all $(x_n),(y_n)\in\ell^\infty(X)$.*

(3) *$\psi((x_n),(y_n))=\psi((y_n),(x_n))$ for all $(x_n),(y_n)\in\ell^\infty(X)$.*

(4) *$\psi((x_n),(-x_n))=(0,0,...,0,...)$ for all $(x_n)\in\ell^\infty(X)$.*

Proof.

$$
\begin{aligned}
-\infty\le\|\psi((x_n),(y_n))\|_\infty &= \sup_n\|\Phi(x_n,y_n)\| \\
&\le \sup_n\|x_n+y_n\| = \|(x_n+y_n)\|_\infty .
\end{aligned}
\tag{1}
$$

$$
(\Phi(x_n,y_n)1) = ((\sup\{\lambda\in\mathbb{R}:\lambda 1\le x_n+y_n\})1)\le(x_n+y_n). \tag{2}
$$

$$
\begin{aligned}
\psi((x_n),(y_n)) = (\Phi(x_n,y_n)) &= (\sup\{\lambda\in\mathbb{R}:\lambda 1\le x_n+y_n\}) \\
&= (\sup\{\lambda\in\mathbb{R}:\lambda 1\le y_n+x_n\}) = \psi((y_n),(x_n)).
\end{aligned}
\tag{3}
$$

$$
\psi((x_n),(-x_n)) = (\sup\{\lambda\in\mathbb{R}:\lambda 1\le x_n-x_n\}) = (0,0,...,0,...). \tag{4}
$$

□

A function $f:\ell^\infty(X)\to\ell^\infty(\mathbb{R})$ is said to be increasing, whenever $(x_n),(y_n)\in\ell^\infty(X)$, $[(x_n)\ge(y_n)\Rightarrow f((x_n))\ge f((y_n))]$, and plus-homogeneous if

$$
(f((x_n)+(\alpha_n 1)) = f((x_n))+(\alpha_n) \text{ for all } (x_n)\in\ell^\infty(X) \text{ and } (\alpha_n)\in\ell^\infty(\mathbb{R})).
$$

A function $f:\ell^\infty(X)\to\ell^\infty(\mathbb{R})$ is called topical if this function is increasing and plus-homogeneous.

Lemma 9. *The function $\psi_{(y_n)}$ defined by* (2.2) *is topical.*

Proof. (1) Let $(x_n),(z_n)\in\ell^\infty(X)$ with $(x_n)\le(z_n)$. Then, since $x_n\le z_n$ for all n, $\{\lambda\in\mathbb{R}:\lambda 1\le x_n+y_n\}\subset\{\lambda\in\mathbb{R}:\lambda 1\le z_n+y_n\}$. Hence,

$$
\begin{aligned}
\psi_{(y_n)}((x_n)) &= \psi((x_n),(y_n)) \\
&= (\sup\{\lambda\in\mathbb{R}:\lambda 1\le x_n+y_n\}) \\
&\le (\sup\{\lambda\in\mathbb{R}:\lambda 1\le z_n+y_n\}) \\
&= \psi_{(y_n)}((z_n)).
\end{aligned}
$$

(2) Let $(x_n)\in\ell^\infty(X)$ and $(\alpha_n)\in\ell^\infty(\mathbb{R})$ be arbitrary. Then

$$
\begin{aligned}
\psi_{(y_n)}((x_n)+(\alpha_n)1) &= \psi((x_n)+(\alpha_n)1,(y_n)) \\
&= (\sup\{\lambda\in\mathbb{R}:\lambda 1\le x_n+\alpha_n 1+y_n\}) \\
&= (\sup\{\lambda\in\mathbb{R}:(\lambda-\alpha_n)1\le x_n+y_n\}).
\end{aligned}
$$

Let $\lambda - \alpha_n = \beta$. Then $\lambda = \beta + \alpha_n$. Hence

$$\begin{aligned}\psi_{(y_n)}((x_n) + (\alpha_n)1) &= (\sup\{\beta + \alpha_n \in \mathbb{R} : \beta 1 \leq x_n + y_n\}) \\ &= (\sup\{\beta \in \mathbb{R} : \beta 1 \leq x_n + y_n, \}) + (\alpha_n) \\ &= \psi((x_n), (y_n)) + (\alpha_n) \\ &= \psi_{(y_n)}((x_n)) + (\alpha_n).\end{aligned}$$

□

Theorem 10. *The function $\psi_{(y_n)}$ is Lipschitz continuous in the ℓ^∞ norm.*

Proof. Let $(x_n), (z_n) \in \ell^\infty(X)$ be arbitrary. Since $|x_n - z_n| \leq \|(x_n) - (z_n)\|_\infty 1$, it follows that

$$z_n - \|(x_n) - (z_n)\|_\infty \leq x_n \leq z_n + \|(x_n) - (z_n)\|_\infty .$$

In view of (Lemma 9) we have

$$\psi_{(y_n)}((z_n)) - (\|(x_n) - (z_n)\|_\infty 1) \leq \psi_{(y_n)}((x_n)) \leq \psi_{(y_n)}((z_n)) + (\|(x_n) - (z_n)\|_\infty 1),$$

and hence

$$\left\|\psi_{(y_n)}((x_n)) - \psi_{(y_n)}((z_n))\right\|_\infty \leq \|(x_n) - (z_n)\|_\infty . \tag{2.3}$$

Therefore, $\psi_{(y_n)}$ is Lipschitz continuous. □

Corollary 11. *The function ψ defined in* (2.1) *is continuous in the ℓ^∞ norm.*

Proof. It follows directly from (2.3). □

Now we prove one of the main results in this paper

Theorem 12. *Let W be a closed downward subset of X and $(y_k^\circ) \in \ell^\infty(W)$. If $S = \{k \in \mathbb{N}, y_k^\circ \in bd\,(W)\} \neq \phi$, then,*

(a) $(y_n^\circ) \in bd\,(\ell^\infty(W))$.

(b) $\Phi\,(w_k, -y_k^\circ) \leq 0$, *for all $k \in S$ and all $(w_n) \in \ell^\infty(W)$.*

Proof. (a) Let $(y_n^\circ) \in \ell^\infty(W)$ and $B\,(y_n^\circ, \epsilon)$ be any neighborhood of (y_n°) . Then if

$$B\,(y_n^\circ, \epsilon) = \{(x_n) \in \ell^\infty(X) : \|(x_n) - (y_n^\circ)\|_\infty < \epsilon\} ,$$

it follows that for all n,

$$\|x_n - y_n^\circ\| < \|(x_n) - (y_n^\circ)\|_\infty = \sup_n \|x_n - y_n^\circ\| < \epsilon.$$

So, for $k \in S, \|x_k - y_k^\circ\| < \epsilon$. Since $y_k^\circ \in bd(W)$, any neighborhood of y_k° contains a point $u_k \in W$ and a point $z_k \notin W$. Now consider the sequence u given by, $u =$

$\left(y_1^\circ, y_2^\circ, ..., y_{k-1}^\circ, u_k, y_{k+1}^\circ, ...\right) \in \ell^\infty(W)$ and, $z = \left(y_1^\circ, y_2^\circ, ..., y_{k-1}^\circ, z_k, y_{k+1}^\circ, ...\right) \notin \ell^\infty(W)$. Then,

$$\begin{aligned} \|u_k - y_k^\circ\| &< \epsilon \text{ and } \|z_k - y_k^\circ\| < \epsilon \Rightarrow \|u_k\| \le \|y_k^\circ\| + \epsilon \text{ and} \\ \|z_k - y_k^\circ\| &< \epsilon \Rightarrow \|z_k\| \le \|y_k^\circ\| + \epsilon \end{aligned}$$

and so

$$\|u_k\| \le \|y_k^\circ\| + \epsilon \text{ and } \|z_k\| \le \|y_k^\circ\| + \epsilon.$$

Therefore, $\|u\|_\infty, \|z\|_\infty \le \|(y_n^\circ)\|_\infty + \epsilon < \infty$. Hence,

$$\begin{aligned} \phi &\ne B\left(y_n^\circ, \epsilon\right) \cap \ell^\infty(W) \supseteq \{u\} \\ \phi &\ne B\left(y_n^\circ, \epsilon\right) \cap \left(\ell^\infty(W)\right)^c \supseteq \{z\}, \end{aligned}$$

and $(y_n^\circ) \in bd\left(\ell^\infty(W)\right)$.

(b) Let $(w_n) \in \ell^\infty(W)$ such that $\Phi\left(w_k, -y_k^\circ\right) = \sup\{\lambda \in \mathbb{R} : \lambda 1 \le w_k - y_k^\circ\} > 0$ for some $k \in S$. Then there exists $\lambda_\circ > 0$ such that $\lambda_\circ 1 \le w_k - y_k^\circ$. This means that $\lambda_\circ 1 + y_k^\circ \le w_k$. Since W is a downward set and $w_k \in W$, it follows that $\lambda_\circ 1 + y_k^\circ \in W$. Therefore, by (Proposition 3.1 in [4]) we have, $y_k^\circ \in \text{int}(W)$. This is a contradiction. □

Corollary 13. *Let W be a closed downward subset of X, $y_n^\circ \in bd(W)$ for all n. Then $\psi\left((w_n), (-y_n^\circ)\right) \le 0$, for all $(w_n) \in \ell^\infty(W)$.*

Proof. Since $y_n^\circ \in bd(W)$, for all n, by Theorem 12, $\Phi\left(w_n, -y_n^\circ\right) < 0$. Hence $\psi\left((w_n), (-y_n^\circ)\right) \le 0$. □

In the following two theorems we give some characterizations of the *downward set $\ell^\infty(W)$ in terms of the function ψ.*

Theorem 14. *Let W be a subset of X and ψ be the coupling function of* (2.1). *Then the following are equivalent:*

(1) *$\ell^\infty(W)$ is a downward set.*

(2) *For each $(x_n) \in \ell^\infty(X)\backslash\ell^\infty(W)$, there exist $\phi \ne S \subseteq \mathbb{N}$, $\Phi\left(w_k, -x_k\right) < 0, \forall k \in S$ and $(w_n) \in \ell^\infty(W)$.*

(3) *For each $(x_n) \in \ell^\infty(X)\backslash\ell^\infty(W)$, there exists $(L_n) \in \ell^\infty(X)$ and $\phi \ne S \subseteq \mathbb{N}$,*

$$\Phi\left(w_k, L_k\right) < 0 \le \Phi\left(x_k, L_k\right), \ \forall k \in S \text{ and } (w_n) \in \ell^\infty(W).$$

Proof. (1) $\Rightarrow$ (2) Let $\ell^\infty(W)$ be downward set and $(x_n) \in \ell^\infty(X)\backslash\ell^\infty(W)$. Suppose that for all $n \in \mathbb{N}$, $\Phi\left(w_n, -x_n\right) \ge 0$. Then by Proposition 8(2), $0 \le \left(\Phi\left(w_n, -x_n\right) 1\right) \le (w_n - x_n)$. Since W is downward set and $w_n \in W$, it follows that for all $n, x_n \in W$, which is a contradiction.

Hence $S = \{k, \Phi\left(w_k, -y_k^\circ\right) < 0\} \ne \phi$.

(2) $\Rightarrow$ (3). Assume that (2) holds and $(x_n) \in \ell^\infty(X)\backslash\ell^\infty(W)$ is arbitrary. Then, by hypothesis, there exists $\phi \ne S \subseteq \mathbb{N}$, such that $\Phi\left(w_k, -x_k\right) < 0, \forall k \in S$.

Now, let $(L_n) = (-x_n) \in \ell^\infty(X)$. Using proposition 8 (2), we have for each $(w_n) \in \ell^\infty(W)$ and $k \in S$.

$$\Phi(w_k, L_k) = \Phi(w_k, -x_k) < 0 = \Phi(x_k, -x_k) = \Phi(x_k, L_k)$$

(3) $\Rightarrow$ (1). Suppose that $\ell^\infty(W)$ is not a downward set. Then there exists $(w_n^\circ) \in \ell^\infty(W)$ and $(x_n^\circ) \in \ell^\infty(X) \backslash \ell^\infty(W)$ with $(x_n^\circ) \leq (w_n^\circ)$. Using (3), there exists $(L_n) \in \ell^\infty(X)$ and $\phi \neq S \subseteq \mathbb{N}$, such that for all $k \in S$.

$$\Phi(w_k^\circ, L_k) < 0 \leq \Phi(x_k^\circ, L_k) \tag{2.4}$$

But ψ is increasing, we have $\psi_{(L_n)}((x_n^\circ)) \leq \psi_{(L_n)}((w_n^\circ))$. This mean

$$\Phi(x_n^\circ, L_n) \leq \Phi(w_n^\circ, L_n),$$

for all $n \in \mathbb{N}$ and this is a contradiction to (2.4). □

Theorem 15. *Let ψ be the function defined by* (2.1). *Then for a subset W of X the following are equivalent:*

(1) *$\ell^\infty(W)$ is a closed downward subset of $\ell^\infty(X)$.*

(2) *$\ell^\infty(W)$ is downward, and for each $(x_n) \in \ell^\infty(X)$ the set*

$$H = \{(\lambda_n) \in \ell^\infty(\mathbb{R}) : (x_n + \lambda_n 1) \in \ell^\infty(W)\}$$

is closed in $\ell^\infty(\mathbb{R})$.

(3) *For each $(x_n) \in \ell^\infty(X) \backslash \ell^\infty(W)$, there exists $(L_n) \in \ell^\infty(X)$ and $\phi \neq S \subseteq \mathbb{N}$, such that,*

$$\Phi(w_k, L_k) < 0 < \Phi(x_k, L_k),$$

for all $(w_n) \in \ell^\infty(W))$ and for all $k \in S$.

(4) *For each $(x_n) \in \ell^\infty(X) \backslash \ell^\infty(W)$, there exists $(L_n) \in \ell^\infty(X)$ and $\phi \neq S \subseteq \mathbb{N}$ such that,*

$$\sup_{(w_n) \in \ell^\infty(W)} \Phi(w_k, L_k) < \Phi(x_k, L_k).$$

Proof. (1) $\Rightarrow$ (2). Let $(x_n) \in \ell^\infty(X)$, $(\lambda_n^k) \in \ell^\infty(\mathbb{R}), (x_n + \lambda_n^k 1) \in \ell^\infty(W)$ $(k = 1, 2, ...)$ and $(\lambda_n^k) \longrightarrow (\lambda_n)$ in ℓ^∞ norm. Then,

$$\begin{aligned} \left\| (x_n + \lambda_n^k 1) - (x_n + \lambda_n 1) \right\|_\infty &= \left\| (\lambda_n^k - \lambda_n) 1 \right\|_\infty \\ &= \sup_n \left| \lambda_n^k - \lambda_n \right| \longrightarrow 0 \text{ as } k \longrightarrow +\infty. \end{aligned}$$

Since $(x_n + \lambda_n^k 1) \in \ell^\infty(W)$ and $\ell^\infty(W)$ is closed, it follows that $(x_n + \lambda_n 1) \in \ell^\infty(W)$. Hence, $(\lambda_n) \in H$ and H is a closed subset of $\ell^\infty(\mathbb{R})$.

(2) $\Rightarrow$ (3). Let $(x_n) \in \ell^\infty(X) \backslash \ell^\infty(W)$ be arbitrary. We claim that there exists $(\lambda_n^\circ) > (0)$ such that $(-\lambda_n^\circ) \notin H$. Indeed, if $(-\lambda_n) \in H$ for all $(\lambda_n) > (0, 0, ..., 0, ...)$. Then due to the closedness of H, we have $(0, 0, ..., 0..) \in H$. This implies $(x_n) = (x_n + 0 \cdot 1) \in \ell^\infty(W)$. This is a contradiction.

Now, let $(L_n) = (\lambda_n^\circ 1 - x_n) \in \ell^\infty(X)$. We show that, $\exists \phi \neq S \subseteq \mathbb{N}$ such that $\Phi(w_k, L_k) < 0$, for all $k \in S$ and for all $(w_n) \in \ell^\infty(W)$. Assume that there exists $(w_n^\circ) \in \ell^\infty(W)$ such that $\psi((w_n^\circ), (L_n)) \geq (0)$. Then by proposition 8 (2), for all n,

$$0 \leq \Phi(w_n^\circ, L_n)1 \leq w_n^\circ + L_n$$

and so $w_n^\circ \geq -L_n = x_n - \lambda_n^\circ 1$. Since $\ell^\infty(W)$ is downward and $(w_n^\circ) \in \ell^\infty(W)$, it follows that $(x_n - \lambda_n^\circ 1) \in \ell^\infty(W)$, and consequently $-\lambda_n \in H$. This is a contradiction. Hence, $\exists S \neq \phi$,

$$\Phi(w_k, L_k) < 0 \text{ for all } (w_n) \in \ell^\infty(W)), \text{ for all } k \in S.$$

On the other hand, for all $k \in S$

$$\begin{aligned}
\Phi(x_k, L_k) &= \sup\{\lambda \in \mathbb{R} : \lambda 1 \leq x_k + L_k\} \\
&= \sup\{\lambda \in \mathbb{R} : \lambda 1 \leq x_k + \lambda_k^\circ 1 - x_k = \lambda_k^\circ 1\} \\
&= \sup\{\lambda \in \mathbb{R} : (\lambda - \lambda_k^\circ)1 \leq 0\}.
\end{aligned}$$

Let $\lambda - \lambda_k^\circ = \alpha_k$. Then $\lambda = \lambda_k^\circ + \alpha_k$. Hence

$$\begin{aligned}
\Phi(w_k, L_k) &= \sup\{\alpha_k + \lambda_k^\circ \in \mathbb{R} : \alpha_k 1 \leq 0\} \\
&= \sup\{\alpha_k \in \mathbb{R} : \alpha_k 1 \leq 0\} + \lambda_k^\circ \\
&= \lambda_k^\circ > 0.
\end{aligned}$$

(3) $\Rightarrow$ (4). By (3) for each $(x_n) \in \ell^\infty(X) \backslash \ell^\infty(W)$, there exists $(L_n) \in \ell^\infty(X)$ and $\phi \neq S \subseteq \mathbb{N}$

$$\Phi(w_k, L_k) < 0 < \Phi(x_k, L_k),$$

for all $(w_n) \in \ell^\infty(W))$. Then

$$\sup_{(w_n) \in \ell^\infty(W)} \Phi(w_k, L_k) < \Phi(x_k, L_k), \text{ for all } k \in S.$$

(4) $\Rightarrow$ (1). Suppose that $\ell^\infty(W)$ is not a downward set. Then there exists $(w_n^\circ) \in \ell^\infty(W)$ and $(x_n^\circ) \in \ell^\infty(X) \backslash \ell^\infty(W)$ with $(x_n^\circ) \leq (w_n^\circ)$. By hypothesis, there exists $(L_n) \in \ell^\infty(X)$ and $\phi \neq S \subseteq \mathbb{N}$,

$$\sup_{(w_n) \in \ell^\infty(W)} \Phi(w_k, L_k) < \Phi(x_k^\circ, L_k),$$

for all $k \in S$. Since $\psi(., (L_n)) = \psi_{(L_n)}(.)$ is increasing, it follows that

$$\psi((x_n^\circ), (L_n)) \leq \psi((w_n^\circ), (L_n))$$

Hence, for all $k \in S$

$$\Phi(x_k^\circ, L_k) \leq \sup_{(w_n) \in \ell^\infty(W)} \Phi(w_k, L_k) < \Phi(x_k^\circ, L_k).$$

This is a contradiction. Hence, $\ell^\infty(W)$ is a downward set.

Finally, assume that $\ell^\infty(W)$ is not closed. Then there exists a sequence $\{w_n^m\}_{m\geq 1} \subset \ell^\infty(W)$ and $(x_n^\circ) \in \ell^\infty(X)\backslash\ell^\infty(W)$ such that

$$\|w_n^m - x_n^\circ\|_\infty \longrightarrow 0 \text{ as } m \longrightarrow +\infty.$$

Since $(x_n^\circ) \in \ell^\infty(X)\backslash\ell^\infty(W)$, by hypothesis, there exists $(L_n) \in \ell^\infty(X)$ and $\phi \neq S \subseteq \mathbb{N}$, such that

$$\sup_{(w_n)\in\ell^\infty(W)} \Phi(w_k, L_k) < \Phi(x_k^\circ, L_k),$$

for all $k \in S, \forall (w_n) \in \ell^\infty(W)$. Hence

$$\Phi((w_k^m), (L_k)) \leq \sup_{(w_n)\in\ell^\infty(W)} \Phi((w_k), (L_k)),$$

for all $m, \forall k \in S$. By continuity of $\psi_{L_n}(., (L_n)) = (\Phi_{L_n}(., L_n))$ it follows that

$$\Phi((x_k^\circ), (L_k)) \leq \sup_{(w_n)\in\ell^\infty(W)} \Phi((w_k), (L_k)),$$

for all $k \in S$. This is a contradiction. □

3. Best approximation of $\ell^\infty(W)$ in $\ell^\infty(X)$

A subset W in a Banach space X is said to be proximinal if there corresponds to each $x \in X$ at least one $w \in W$ such that $\|x - w\| = \text{dist}(x, W) = \inf_{z\in W} \|(x - z\|$. A necessary condition for proximinality of a subset W of a normed linear space X is closeness (see, [2]). The set (possibly empty) of best approximations to x from W is defined by: $P_W(x) = \{w \in W : \|x - w\| = d(x, W)\}$.

In this section we prove that if W is a closed downward set in X, then $\ell^\infty(W)$ is proximinal in $\ell^\infty(X)$ and the set $P_{\ell^\infty(W)}((x_n))$ of all of points of best approximation of the point $x = (x_n) \in \ell^\infty(X)$ in $\ell^\infty(W)$ has minimal element.

Theorem 16. *Let W be a closed downward subset of X. Then $\ell^\infty(W)$ is proximinal in $\ell^\infty(X)$.*

Proof. Let $(x_n^\circ) \in \ell^\infty(X)\backslash\ell^\infty(W)$ be arbitrary and

$$\begin{aligned} d((x_n^\circ), \ell^\infty(W)) &= \inf_{(w_n)\in\ell^\infty(W)} \|(x_n^\circ) - (w_n)\|_\infty \\ &= \inf_{(w_n)\in\ell^\infty(W)} \sup_n \|x_n^\circ - w_n\| = r > 0. \end{aligned}$$

This implies for all $\epsilon > 0$, there exists $(w_{n\epsilon}) \in \ell^\infty(W)$ such that

$$\|(x_n^\circ) - (w_{n\epsilon})\|_\infty < r + \epsilon.$$

Consequently using (1.2) we get

$$\begin{aligned} B((x_n^\circ), r+\epsilon) &= \left\{ \begin{array}{c} (w_{n\epsilon}) \in \ell^\infty(X) : ||x_n^\circ - w_{n\epsilon}|| \le \sup_n ||x_n^\circ - w_{n\epsilon}|| \\ = ||(x_n^\circ) - (w_{n\epsilon})||_\infty \le r+\epsilon \end{array} \right\} \\ &= \{(w_{n\epsilon}) \in \ell^\infty(X) : x_n^\circ - (r+\epsilon)1 \le w_{n\epsilon} \le x_n^\circ + (r+\epsilon)1\}. \end{aligned}$$

If $(w_n^\circ) = (x_n^\circ - r1)$, then

$$||(x_n^\circ) - (w_n^\circ)||_\infty = \sup_n ||x_n^\circ - w_n^\circ|| = \sup_n ||r|| = r.$$

Hence $(w_n^\circ - \epsilon 1) = (x_n^\circ - r1 - \epsilon 1) \le (w_{n\epsilon})$. Since W is closed downward set and $(w_{n\epsilon}) \in \ell^\infty(W)$, it follows that $(w_n^\circ - \epsilon 1) \in \ell^\infty(W)$, for all $\epsilon > 0$ and $w_n^\circ \in W$. So $(w_n^\circ) \in P_{\ell^\infty(W)}((x_n^\circ))$. □

Remark 17. We prove that for each $(x_n^\circ) \in \ell^\infty(X) \backslash \ell^\infty(W)$, the set $P_{\ell^\infty(W)}((x_n^\circ))$ contains $(w_n^\circ) = (x_n^\circ - r1)$ with $r = d((x_n^\circ), \ell^\infty(W))$. If $(x_n^\circ) \in \ell^\infty(W)$, then $(w_n^\circ) = (x_n^\circ)$ and $P_{\ell^\infty(W)}((x_n^\circ)) = \{(w_n^\circ)\}$.

Theorem 18. *Let W be a closed downward subset of X and $(x_n^\circ) \in \ell^\infty(X)$.*

Then there exists the least element $(w_n^\circ) = \min P_{\ell^\infty(W)}((x_n^\circ))$ of the set $P_{\ell^\infty(W)}((x_n^\circ))$, namely, $(w_n^\circ) = (x_n^\circ - r1)$, where $r = d((x_n^\circ), \ell^\infty(W))$.

Proof. If $(x_n^\circ) \in \ell^\infty(W)$, then the result holds. Assume that $(x_n^\circ) \notin \ell^\infty(W)$ and $(w_n^\circ) = (x_n^\circ - r1)$. Then by (Remark 17), we have

$$(w_n^\circ) = (x_n^\circ - r1) \in P_{\ell^\infty(W)}((x_n^\circ)).$$

Since applying (1.2) and the equality $\|(x_n^\circ) - (w_n)\|_\infty = r$, we get

$$\begin{aligned} B((x_n^\circ), r) &= \{(x_n) \in \ell^\infty(X) : \|(x_n) - (x_n^\circ)\|_\infty \le r\} \\ &= \left\{(x_n) \in \ell^\infty(X) : \sup_n \|x_n - x_n^\circ\| \le r\right\}. \end{aligned}$$

Consequently for all n,

$$\|x_n - x_n^\circ\| \le \|(x_n) - (x_n^\circ)\|_\infty = \sup_n \|x_n - x_n^\circ\| \le r,$$

and using (1.1) we have

$$-r1 \le x_n - x_n^\circ \le r1 \Rightarrow x_n^\circ - r1 \le x_n \le x_n^\circ + r1.$$

Hence, $w_n^\circ = x_n^\circ - r1 \le x_n$, and so $(w_n^\circ) \le (x_n)$ for all $(x_n) \in B((x_n^\circ), r)$, and this implies (w_n°) is the least element of the closed ball $B((x_n^\circ), r)$.

Now, let $(w_n) \in P_{\ell^\infty(W)}(x_n^\circ)$ be arbitrary. Then, $\|(x_n^\circ) - (w_n)\| = r$ and so $(w_n) \in B((x_n^\circ), r)$. Therefore, $(w_n) \ge (w_n^\circ)$. Hence, (w_n°) is the least element of the set $P_{\ell^\infty(W)}(x_n^\circ)$. □

Corollary 19. *Let W be a closed downward subset of X, $(x_n^\circ) \in \ell^\infty(X)$ and $(w_n^\circ) = \min P_{\ell^\infty(W)}(x_n^\circ)$. Then, $(w_n^\circ) \leq (x_n^\circ)$.*

Proof. Since $(w_n^\circ) = \min P_{\ell^\infty(W)}(x_n^\circ)$. Then by Theorem 18, we get $(w_n^\circ) = (x_n^\circ - r1) \leq (x_n^\circ)$. □

Corollary 20. *Let W be a closed downward subset of X and $(x_n) \in \ell^\infty(X)$ be arbitrary. Then $d((x_n), \ell^\infty(W)) = \min\{\lambda \geq 0, (x_n - \lambda 1) \in \ell^\infty(W)\}$.*

Proof. Let $A = \{\lambda \geq 0, (x_n - \lambda 1) \in \ell^\infty(W)\}$. If $(x_n) \in \ell^\infty(W)$, then $(x_n - 0.1) = (x_n) \in \ell^\infty(W)$, and so $\min(A) = 0 = d((x_n), \ell^\infty(W))$. Suppose that $(x_n) \notin \ell^\infty(W)$. Then $r = d((x_n), \ell^\infty(W)) > 0$. Let $\lambda > 0$ be arbitrary such that $(x_n - \lambda 1) \in \ell^\infty(W)$. Thus, we have

$$\lambda = \|(\lambda 1)\|_\infty = \|(x_n - x_n - \lambda 1)\|_\infty = \sup_n \|x_n - (x_n - \lambda 1)\| \geq d((x_n), \ell^\infty(W)) = r.$$

Since by (Theorem 18), $(x_n - r1) \in \ell^\infty(W)$, it follows that $r \in A$. Hence $\min(A) = r$. □

References

[1] D. Fang, X. Luo, C. Li, *Non linear Simultaneous approximation in complete Lattice Banach spaces*, Taiw. J. Math., 12, 9 (2008), 2373-2385.

[2] F. Girosi, T. Poggio, *Networks and the Best Approximation Property*, Biological Cybernetics, 63 (1990), 169-176.

[3] J. E. Martinez-Legaz, A. M. Rubinov and I. Singer, *Downward sets and their separation and approximation properties*, J. Global Optim. 23(2002), 111-137.

[4] S. Modarres, M. Dehghani, *New results for best approximation on Banach lattices*, Nonlinear Analysis, 70 (2009), 3342-3347.

[5] H. Mohebi, A. M. Rubinov, *Best approximation by downward sets with application*, Anal. Theory Appl. 22, 1 (2006), 20-40.

[6] A. M. Rubinov, I. Singer, *Best approximation by normal and conormal sets*, J. Approximation Theory, 107(2000), 212-243.

[7] A. M. Rubinov, *Abstract Convexity and Global Optimization*, Kluwer Academic Publishers, Boston, 2000.

[8] H. H. Schaefer, *Banach lattices and positive operators*, Springer-Verlag, Berlin Heidelberg New York, 1974.

[9] B. Z. Vulikh, *Introduction to the theory of partially ordered vector spaces*, Wolters-Noordhoff, Groningen, 1967.

Accepted: 26.04.2017