# RIEMANN-LIOUVILLE FRACTIONAL SIMPSON'S INEQUALITIES THROUGH GENERALIZED $(m, h_1, h_2)$-PREINVEXITY

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**Abstract.** The authors first introduce the concept of generalized $(m, h_1, h_2)$-preinvex function. Second, a new Riemann-Liouville fractional integral identity involving first order differentiable function on an $m$-invex is found. By using this identity, we present new Riemann-Liouville fractional Simpson's inequalities through generalized $(m, h_1, h_2)$-preinvexity. These inequalities can be viewed as significant generalization of some previously known results.
**Keywords:** Simpson's inequality, generalized $(m, h_1, h_2)$-preinvex functions, Riemann-Liouville fractional integrals.

## 1. Introduction

The following inequality named Simpson's inequality plays a significant role in analysis.

**Theorem 1.1.** *Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on $(a, b)$ and $||f^{(4)}||_\infty := \sup_{x\in(a,b)} |f^{(4)}(x)| < \infty$. Then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\Big(\frac{a+b}{2}\Big)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\ &\le \frac{1}{2880}||f^{(4)}||_\infty(b-a)^4. \end{aligned}\tag{1.1}$$

The study of Simpson type integral inequalities involving various kinds of convex functions has been carried out by many researchers, including Chun and Qi [7] in the study of Simpson's inequalities for extended $s$-convex functions, Du et al. [11] in generalization of Simpson type inequality for extended

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$(s,m)$-convex functions, Hussain and Qaisar [13] in generalizations of Simpson's inequalities through preinvexity and Qaisar et al. [28] in generalizations of Simpson's type inequality via $(\alpha,m)$-convex functions. For more results and recent development on the Simpson's inequality, see [14, 18, 19, 20, 26, 27, 30, 36, 38, 45, 47] and the references therein.

In 2013, Sarikaya et al. [34] considered the following interesting Hermite-Hadamard-type inequalities containing Riemann-Liouville fractional integrals.

**Theorem 1.2** ([34]). *Let $f:[u,v]\to\mathbb{R}$ be a positive function with $0\le u<v$ and let $f\in L^1[u,v]$. Suppose $f$ is a convex function on $[u,v]$, then the following inequalities for fractional integrals hold:*

$$f\Big(\frac{u+v}{2}\Big)\le\frac{\Gamma(\alpha+1)}{2(v-u)^\alpha}[J^\alpha_{u^+}f(v)+J^\alpha_{v^-}f(u)]\le\frac{f(u)+f(v)}{2}, \tag{1.2}$$

*where the symbol $J^\alpha_{u^+}f$ and $J^\alpha_{v^-}f$ denote respectively the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha\in\mathbb{R}^+$ defined by*

$$J^\alpha_{u^+}f(x)=\frac{1}{\Gamma(\alpha)}\int_u^x(x-t)^{\alpha-1}f(t)\mathrm{d}t,\quad u<x$$

*and*

$$J^\alpha_{v^-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^v(t-x)^{\alpha-1}f(t)\mathrm{d}t,\quad x<v.$$

*Here, $\Gamma(\alpha)$ is the Gamma function and its definition is*

$$\Gamma(\alpha)=\int_0^\infty e^{-\mu}\mu^{\alpha-1}\mathrm{d}\mu.$$

We observe that, for $\alpha=1$, the inequality (1.2) can be reduced to the following termed Hermite-Hadamard inequality

$$f\left(\frac{u+v}{2}\right)\le\frac{1}{v-u}\int_u^v f(x)\mathrm{d}x\le\frac{f(u)+f(v)}{2}, \tag{1.3}$$

where $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is a convex mapping on the interval $I$ of real numbers and $u,v\in I$ with $u<v$.

For more recent results which generalize, refine, and extend this classic inequality (1.3), One can see contributions [9, 15, 17, 21, 25, 29, 31, 44, 48] and references therein.

Recently, Riemann-Liouville fractional Hermite-Hadamard inequalities and its extensively application have been attached more and more attentions, see [1, 2, 4, 5, 6, 8, 16, 23, 35, 42, 46], however, there are few works on the Simpson type fractional integrals. Thus, it is natural to offer to study Simpson's inequalities involving Riemann-Liouville fractional integral.

Stimulated by above works, in this paper we introduce a class of generalized $(m, h_1, h_2)$-preinvex functions and derive new Riemann-Liouville fractional Simpson's inequalities involving the class of functions whose first derivatives in absolute values are generalized $(m, h_1, h_2)$-preinvex functions. Some results can be regarded as generalization of recent results that appeared in Refs. [22, 32, 33, 40].

## 2. Preliminaries

**Definition 2.1** ([43])**.** *A set $S \subseteq \mathbb{R}^n$ is said to be invex set with respect to the mapping $\eta : S \times S \to \mathbb{R}^n$, if $x + t\eta(y, x) \in S$ for every $x, y \in S$ and $t \in [0, 1]$. The invex set $S$ is also termed an $\eta$-connected set.*

**Definition 2.2** ([10])**.** *A set $K \subseteq \mathbb{R}^n$ is said to be $m$-invex with respect to the mapping $\eta : K \times K \times (0, 1] \to \mathbb{R}^n$ for some fixed $m \in (0, 1]$, if $mx + t\eta(y, x, m) \in K$ holds for each $x, y \in K$ and any $t \in [0, 1]$.*

We next explore a new concept, to be referred to as the generalized $(m, h_1, h_2)$-preinvex functions and its variant forms.

**Definition 2.3.** *Let $K \subseteq \mathbb{R}$ be an open $m$-invex set with respect to $\eta : K \times K \times (0, 1] \to \mathbb{R}$. A function $f : K \to \mathbb{R}$, $h_1, h_2 : [0, 1] \to \mathbb{R}_0$, if*

$$f(mx + t\eta(y, x, m)) \leq mh_1(t)f(x) + h_2(t)f(y) \tag{2.1}$$

*is valid for all $x, y \in K$ and $t \in [0, 1]$, with $m \in (0, 1]$, then we say that $f(x)$ is a generalized $(m, h_1, h_2)$-preinvex function with respect to $\eta$. If the inequality (2.1) reverses, then $f$ is said to be $(m, h_1, h_2)$-preincave on $K$.*

**Remark 2.1.** In Definition 2.3, let $h_1(t) = (1 - t)^{-s}$, $h_2(t) = t^{-s}$ and $s \in [0, 1]$, if

$$f\Big(mx + t\eta(y, x, m)\Big) \leq m(1 - t)^{-s}f(x) + t^{-s}f(y) \tag{2.2}$$

is valid for all $x, y \in K$ and $t \in (0, 1)$, with $m \in (0, 1]$, then we say that $f(x)$ is a generalized $(m, s)$-Godunova-Levin-preinvex function with respect to $\eta$.

We now discuss some special cases of generalized $(m, h_1, h_2)$-preinvex function, which appears to be new ones.

**I)** If the mapping $\eta(y, x, m)$ with $m = 1$ degenerates into $\eta(y, x)$, $h_1(t) = h(1 - t)$ and $h_2(t) = h(t)$, then Definition 2.3, reduces to:

**Definition 2.4** ([21])**.** *Let $h : [0, 1] \to \mathbb{R}$ be a non-negative function and $h \neq 0$. The function $f$ on the invex set $X$ is said to be $h$-preinvex with respect to $\eta$, if*

$$f\big(x + t\eta(y, x)\big) \leq h(1 - t)f(x) + h(t)f(y) \tag{2.3}$$

*for each $x, y \in X$ and $t \in [0, 1]$ where $f(\cdot) > 0$.*

**II)** If the mapping $\eta(y,x,m)$ with $m=1$ degenerates into $\eta(y,x)$, $h_1(t)=(1-t)^s$ and $h_2(t)=t^s$, then Definition 2.3, reduces to:

**Definition 2.5** ([40]). *Let $S\subseteq\mathbb{R}^n$ be an invex set with respect to $\eta: S\times S\to\mathbb{R}^n$. A function $f: S\to\mathbb{R}_0=[0,\infty)$ is said to be s-preinvex with respect to $\eta$ and $s\in(0,1]$, if for every $x,y\in S$ and $t\in[0,1]$,*

$$f\big(x+t\eta(y,x)\big)\le(1-t)^s f(x)+t^s f(y). \tag{2.4}$$

**III)** If the mapping $\eta(y,x,m)$ with $m=1$ degenerates into $\eta(y,x)$, $h_1(t)=(1-t)^{-s}$ and $h_2(t)=t^{-s}$, then Definition 2.3, reduces to:

**Definition 2.6** ([24]). *A function $f: K\to R$ is said to be s-Godunova-Levin preinvex functions of second kind , if*

$$f\Big(x+t\eta(y,x)\Big)\le(1-t)^{-s}f(x)+t^{-s}f(y) \tag{2.5}$$

*for each $x,y\in K$, $t\in(0,1)$ and $s\in[0,1]$.*

**IV)** If the mapping $\eta(y,x,m)$ with $m=1$ degenerates into $\eta(y,x)$, $h_1(t)=1-t^\alpha$ and $h_2(t)=t^\alpha$, then Definition 2.3, reduces to:

**Definition 2.7** ([41]). *Let $S\subseteq\mathbb{R}^n$ be an invex set with respect to $\eta: S\times S\to\mathbb{R}^n$. A function $f: S\to\mathbb{R}$ is said to be $\alpha$-preinvex with respect to $\eta$ for $\alpha\in(0,1]$, if for every $x,y\in S$ and $t\in[0,1]$,*

$$f\big(x+t\eta(y,x)\big)\le(1-t^\alpha)f(x)+t^\alpha f(y). \tag{2.6}$$

**V)** If the mapping $\eta(y,x,m)$ with $m=1$ degenerates into $\eta(y,x)$, $h_1(t)=1-t$ and $h_2(t)=t$, then Definition 2.3, reduces to:

**Definition 2.8** ([43]). *A function $f$ defined on the invex set $K\subseteq\mathbb{R}^n$ is said to be preinvex respecting $\eta$, if*

$$f\big(x+t\eta(y,x)\big)\le(1-t)f(x)+tf(y),\forall x,y\in K,t\in[0,1]. \tag{2.7}$$

**VI)** If the mapping $\eta(y,x,m)=y-mx$ with $m=1$, $h_1(t)=h(1-t)$ and $h_2(t)=h(t)$, then Definition 2.3, reduces to:

**Definition 2.9** ([39]). *Let $h: J\subseteq\mathbb{R}\to\mathbb{R}$ be a positive function, $h\not\equiv 0$. We say that $f: I\subseteq\mathbb{R}\to\mathbb{R}$ is h-convex, if $f$ is non-negative and for all $x,y\in I$ and $t\in(0,1)$, one has*

$$f\big((1-t)x+ty\big)\le h(1-t)f(x)+h(t)f(y). \tag{2.8}$$

**VII)** If the mapping $\eta(y,x,m)=y-mx$ with $m=1$, $h_1(t)=(1-t)^s$ and $h_2(t)=t^s$, then Definition 2.3, reduces to:

**Definition 2.10** ([3]). *Let $s$ be a real number, $s \in (0,1]$. A function $f:[0,\infty) \to \mathbb{R}$ is said to be $s$-convex, if*

$$f\big((1-t)x+ty\big) \le (1-t)^s f(x) + t^s f(y). \tag{2.9}$$

*for all $x, y \in [0,\infty)$ and $t \in [0,1]$.*

**VIII)** If the mapping $\eta(y,x,m) = y - mx$ with $m = 1$, $h_1(t) = (1-t)^{-s}$ and $h_2(t) = t^{-s}$, then Definition 2.3, reduces to:

**Definition 2.11** ([12]). *We say that the function $f : K \subseteq X \to [0,\infty)$ is of $s$-Godunova-Levin type, with $s \in [0,1]$, if*

$$f\Big((1-t)x+ty\Big) \le (1-t)^{-s} f(x) + t^{-s} f(y) \tag{2.10}$$

*for each $x, y \in X$, $t \in (0,1)$.*

**IX)** If the mapping $\eta(y,x,m) = y - mx$ with $m = 1$, and $h_1(t) = h_2(t) = t(1-t)$, then Definition 2.3, reduces to:

**Definition 2.12** ([37]). *Let $f : K \subseteq \mathbb{R} \to \mathbb{R}$ be a nonnegative function, we say that $f : K \longrightarrow \mathbb{R}$ is a tgs-convex function on $K$ if the inequality*

$$f\Big((1-t)x+ty\Big) \le t(1-t)[f(x)+f(y)] \tag{2.11}$$

*holds for all $x, y \in K$ and $t \in (0,1)$. We say that $f$ is tgs-concave if $(-f)$ is tgs-convex.*

## 3. Main results

Let $f : K \to \mathbb{R}$ be a differentiable function, throughout this section we will take

$$\begin{aligned} &K_f(\alpha;\eta,m,a,b) \\ &:= \frac{1}{6}\left[f\big(ma\big) + 4f\left(ma + \frac{\eta(b,a,m)}{2}\right) + f\Big(ma + \eta(b,a,m)\Big)\right] \\ &- \frac{2^{\alpha-1}\Gamma(\alpha+1)}{\eta^\alpha(b,a,m)}\left[J^\alpha_{(ma)^+} f\left(ma + \frac{\eta(b,a,m)}{2}\right)\right. \\ &\left. + J^\alpha_{\big(ma+\eta(b,a,m)\big)^-} f\left(ma + \frac{\eta(b,a,m)}{2}\right)\right], \end{aligned} \tag{3.1}$$

where $K \subseteq \mathbb{R}$ be an open $m$-invex subset with respect to $\eta : K \times K \times (0,1] \to \mathbb{R}$ for some fixed $m \in (0,1]$, $a, b \in K$ with $a < b$, $\alpha > 0$ and $\Gamma$ is the Euler Gamma function.

Before presenting our main results, we claim the following integral identity.

**Lemma 3.1.** *Let $K \subseteq \mathbb{R}$ be an open $m$-invex subset with respect to $\eta : K \times K \times (0,1] \to \mathbb{R}$ for some fixed $m \in (0,1]$ and let $a, b \in K$, $a < b$ with $\eta(b,a,m) > 0$. Assume that $f : K \to \mathbb{R}$ is a first differentiable function, $f'$ is integrable on $[ma, ma+\eta(b,a,m)]$, then the following identity for Riemann-Louville fractional integral with $\alpha > 0$ and $x \in [ma, ma+\eta(b,a,m)]$ holds:*

$$
\begin{aligned}
K_f(\alpha;\eta,m,a,b) =& \frac{\eta(b,a,m)}{2}\int_0^1 \Bigg[\left(\frac{1}{3}-\frac{t^\alpha}{2}\right)f'\Big(ma+\frac{1-t}{2}\eta(b,a,m)\Big) \\
&+\left(\frac{t^\alpha}{2}-\frac{1}{3}\right)f'\Big(ma+\frac{1+t}{2}\eta(b,a,m)\Big)\Bigg]\mathrm{d}t.
\end{aligned}
\tag{3.2}
$$

**Proof.** By integration by parts, we have

$$
\begin{aligned}
&\int_0^1\left(\frac{1}{3}-\frac{t^\alpha}{2}\right)f'\Big(ma+\frac{1-t}{2}\eta(b,a,m)\Big)\mathrm{d}t \\
=&\frac{-2}{\eta(b,a,m)}\left[\left(\frac{1}{3}-\frac{t^\alpha}{2}\right)f\left(tma+\Big(1-t\Big)\Big(ma+\frac{\eta(b,a,m)}{2}\Big)\right)\right]\Bigg|_0^1 \\
&-\int_0^1(\alpha t^{\alpha-1})\frac{1}{\eta(b,a,m)}f\left(tma+\Big(1-t\Big)\Big(ma+\frac{\eta(b,a,m)}{2}\Big)\right)\mathrm{d}t \\
=&\frac{2}{\eta(b,a,m)}\left[\frac{1}{6}f(ma)+\frac{1}{3}f\Big(ma+\frac{\eta(b,a,m)}{2}\Big)\right] \\
&-\frac{\alpha 2^\alpha}{\eta^{\alpha+1}(b,a,m)}\int_{ma}^{ma+\frac{\eta(b,a,m)}{2}}\left(\Big(ma+\frac{\eta(b,a,m)}{2}\Big)-u\right)^{\alpha-1}f(u)\mathrm{d}u \\
=&\frac{2}{\eta(b,a,m)}\left[\frac{1}{6}f(ma)+\frac{1}{3}f\Big(ma+\frac{\eta(b,a,m)}{2}\Big)\right] \\
&-\frac{2^\alpha\Gamma(\alpha+1)}{\eta^{\alpha+1}(b,a,m)}J^\alpha_{(ma)^+}f\Big(ma+\frac{\eta(b,a,m)}{2}\Big).
\end{aligned}
\tag{3.3}
$$

Similarly, we get

$$
\begin{aligned}
&\int_0^1\left(\frac{t^\alpha}{2}-\frac{1}{3}\right)f'\Big(ma+\frac{1+t}{2}\eta(b,a,m)\Big)\mathrm{d}t \\
=&\frac{2}{\eta(b,a,m)}\left[\left(\frac{t^\alpha}{2}-\frac{1}{3}\right)f\left(\Big(1-t\Big)\Big(ma+\frac{\eta(b,a,m)}{2}\Big)\right.\right. \\
&\left.\left.+t\Big(ma+\eta(b,a,m)\Big)\right)\right]\Bigg|_0^1
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
&-\int_0^1 (\alpha t^{\alpha-1})\frac{1}{\eta(b,a,m)}f\bigg(\Big(1-t\Big)\Big(ma+\frac{\eta(b,a,m)}{2}\Big)\\
&+t\Big(ma+\eta(b,a,m)\Big)\bigg)\mathrm{d}t\\
=&\frac{2}{\eta(b,a,m)}\left[\frac{1}{6}f\Big(ma+\eta(b,a,m)\Big)+\frac{1}{3}f\Big(ma+\frac{\eta(b,a,m)}{2}\Big)\right]\\
&-\frac{\alpha 2^{\alpha}}{\eta^{\alpha+1}(b,a,m)}\int_{ma+\frac{\eta(b,a,m)}{2}}^{ma+\eta(b,a,m)}\left(u-\Big(ma+\frac{\eta(b,a,m)}{2}\Big)\right)^{\alpha-1}f(u)\mathrm{d}u\\
=&\frac{2}{\eta(b,a,m)}\left[\frac{1}{6}f\Big(ma+\eta(b,a,m)\Big)+\frac{1}{3}f\Big(ma+\frac{\eta(b,a,m)}{2}\Big)\right]\\
&-\frac{2^{\alpha}\Gamma(\alpha+1)}{\eta^{\alpha+1}(b,a,m)}J^{\alpha}_{(ma+\eta(b,a,m))^-}f\Big(ma+\frac{\eta(b,a,m)}{2}\Big).
\end{aligned}
$$

From (3.3) and (3.4), we get (3.2). This completes the proof. □

**Remark 3.1.** If $\eta(b,a,m)=b-ma$ with $m=1$ in Lemma 3.1, then the identity (3.2) reduces to the following identity

$$
\begin{aligned}
&\frac{1}{6}\left[f(a)+4f\Big(\frac{a+b}{2}\Big)+f(b)\right]\\
&-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f\Big(\frac{a+b}{2}\Big)+J^{\alpha}_{b^-}f\Big(\frac{a+b}{2}\Big)\right]\\
&=\frac{b-a}{2}\int_0^1\left[\Big(\frac{1}{3}-\frac{t^{\alpha}}{2}\Big)f'\Big(\frac{1+t}{2}a+\frac{1-t}{2}b\Big)\right.\\
&\left.+\Big(\frac{t^{\alpha}}{2}-\frac{1}{3}\Big)f'\Big(\frac{1+t}{2}b+\frac{1-t}{2}a\Big)\right]\mathrm{d}t,
\end{aligned}
\tag{3.5}
$$

which is proved by Matłoka in [22]. Based on this identity, he established some interesting inequalities for $h$-convex functions via fractional integrals.

**Remark 3.2.** In Lemma 3.1, if the mapping $\eta(b,a,m)$ with $m=1$ degenerates into $\eta(b,a)$ and we choose $\alpha=1$ , then (3.2) becomes

$$
\begin{aligned}
&\frac{1}{6}\left[f(a)+4f\Big(a+\frac{\eta(b,a)}{2}\Big)+f\Big(a+\eta(b,a)\Big)\right]\\
&-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\\
&=\frac{\eta(b,a)}{4}\int_0^1\left[\Big(\frac{2}{3}-t\Big)f'\Big(a+\frac{1-t}{2}\eta(b,a)\Big)\right.\\
&\left.+\Big(t-\frac{2}{3}\Big)f'\Big(a+\frac{1+t}{2}\eta(b,a)\Big)\right]\mathrm{d}t,
\end{aligned}
\tag{3.6}
$$

which is proved by Wang et al. in [40]. Based on this identity, they established some interesting Simpson type inequalities for $s$-preinvex functions.

If $\eta(b,a)=b-a$ in (3.6), it follows that

$$\begin{aligned}&\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\\ =&\frac{b-a}{2}\int_0^1\left[\left(\frac{1}{3}-\frac{t}{2}\right)f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right.\\ &\left.+\left(\frac{t}{2}-\frac{1}{3}\right)f'\left(\frac{1+t}{2}b+\frac{1-t}{2}a\right)\right]\mathrm{d}t,\end{aligned}\tag{3.7}$$

which is proved by Sarikaya et al. in [33]. Based on this identity, they established some interesting Simpson type inequalities for $s$-convex functions.

We can achieve the following consequences by the above fractional integral identity.

**Theorem 3.1.** *Let $K\subseteq\mathbb{R}$ be an open $m$-invex subset with respect to $\eta:K\times K\times(0,1]\to\mathbb{R}$ for some fixed $m\in(0,1]$ and let $a,b\in K$, $a<b$ with $\eta(b,a,m)>0$. Suppose that $f:K\to\mathbb{R}$ is a first differentiable function, $h_1,h_2:[0,1]\to\mathbb{R}_0$, $\left|f'\right|$ is generalized $(m,h_1,h_2)$-preinvex on $[ma,ma+\eta(b,a,m)]$, then the following inequality for Riemann-Louville fractional integral with $\alpha>0$ and $x\in[ma,ma+\eta(b,a,m)]$ holds:*

$$\begin{aligned}&\left|K_f(\alpha;\eta,m,a,b)\right|\\ &\le\frac{\eta(b,a,m)}{3}\left(m|f'(a)|\int_0^1h_1(t)\mathrm{d}t+|f'(b)|\int_0^1h_2(t)\mathrm{d}t\right).\end{aligned}\tag{3.8}$$

**Proof.** From Lemma 3.1, $|f'|$ is generalized $(m,h_1,h_2)$-preinvex, we get

$$\begin{aligned}&\left|K_f(\alpha;\eta,m,a,b)\right|\\ &\le\frac{\eta(b,a,m)}{2}\int_0^1\left|\frac{1}{3}-\frac{t^\alpha}{2}\right|\left[\left|f'\left(ma+\frac{1+t}{2}\eta(b,a,m)\right)\right|\right.\\ &\left.+\left|f'\left(ma+\frac{1-t}{2}\eta(b,a,m)\right)\right|\right]\mathrm{d}t\\ &\le\frac{\eta(b,a,m)}{2}\int_0^1\left|\frac{1}{3}-\frac{t^\alpha}{2}\right|\left[mh_1\left(\frac{1+t}{2}\right)\left|f'(a)\right|+h_2\left(\frac{1+t}{2}\right)\left|f'(b)\right|\right.\\ &\left.+mh_1\left(\frac{1-t}{2}\right)\left|f'(a)\right|+h_2\left(\frac{1-t}{2}\right)\left|f'(b)\right|\right]\mathrm{d}t\\ &=\frac{\eta(b,a,m)}{2}\left\{\int_0^1\left|\frac{1}{3}-\frac{t^\alpha}{2}\right|\left[m\left(h_1\left(\frac{1-t}{2}\right)+h_1\left(\frac{1+t}{2}\right)\right)\left|f'(a)\right|\right]\mathrm{d}t\right.\\ &\left.+\int_0^1\left|\frac{1}{3}-\frac{t^\alpha}{2}\right|\left[\left(h_2\left(\frac{1-t}{2}\right)+h_2\left(\frac{1+t}{2}\right)\right)\left|f'(b)\right|\mathrm{d}t\right]\right\}\\ &\le\frac{\eta(b,a,m)}{3}\left[m|f'(a)|\int_0^1h_1(t)\mathrm{d}t+|f'(b)|\int_0^1h_2(t)\mathrm{d}t\right],\end{aligned}\tag{3.9}$$

where we used the fact that $|\frac{t^\alpha}{2}-\frac{1}{3}|\leq\frac{1}{3}$ for all $t\in[0,1]$. The proof is completed. □

**Corollary 3.1.** *Under the assumptions of Theorem* 3.1 *with* $h_1(t) = h(1-t)$, $h_2(t) = h(t)$, *if the mapping* $\eta(b, a, m)$ *with* $m = 1$ *degenerates into* $\eta(b, a)$, *then the inequality* (3.8) *becomes the following fractional inequality for an* $h$*-preinvex function*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(a+\frac{\eta(b,a)}{2}\right)+f\Big(a+\eta(b,a)\Big)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\eta^\alpha(b,a)}\left[J^\alpha_{a^+}f\left(a+\frac{\eta(b,a)}{2}\right)+J^\alpha_{\left(a+\eta(b,a)\right)^-}f\left(a+\frac{\eta(b,a)}{2}\right)\right]\right|\\&\leq\frac{\eta(b,a)}{3}\Big(|f'(a)|+|f'(b)|\Big)\int_0^1h(t)\mathrm{d}t,\end{aligned}\tag{3.10}$$

*specially for* $\eta(b, a) = b - a$, *we achieve a inequality for* $h$*-convex function*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{a^+}f\left(\frac{a+b}{2}\right)+J^\alpha_{b^-}f\left(\frac{a+b}{2}\right)\right]\right|\\&\leq\frac{b-a}{3}\Big(|f'(a)|+|f'(b)|\Big)\int_0^1h(t)\mathrm{d}t,\end{aligned}\tag{3.11}$$

*which is the same as the inequality established in* [22, Theorem6].

**Corollary 3.2.** *In Theorem* 3.1, *when* $\eta(b, a, m)$ *with* $m = 1$ *reduces to* $\eta(b, a)$, *if* $s \in (0, 1], h_1(t) = (1-t)^s$, $h_2(t) = t^s$, *then the inequality* (3.8) *becomes the following fractional inequality for* $s$*-preinvex function*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(a+\frac{\eta(b,a)}{2}\right)+f\Big(a+\eta(b,a)\Big)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\eta^\alpha(b,a)}\left[J^\alpha_{a^+}f\left(a+\frac{\eta(b,a)}{2}\right)J^\alpha_{\left(a+\eta(b,a)\right)^-}f\left(a+\frac{\eta(b,a)}{2}\right)\right]\right|\\&\leq\frac{\eta(b,a)}{3(s+1)}\Big(|f'(a)|+|f'(b)|\Big),\end{aligned}\tag{3.12}$$

*specially for* $\eta(b, a) = b - a$ *and* $\alpha = 1$, *then from the proof of Theorem* 3.1 *it follows that the following inequality for* $s$*-convex function holds*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^bf(x)\mathrm{d}x\right|\\&\leq(b-a)\frac{(s-4)6^{s+1}+2\times5^{s+2}-2\times3^{s+2}+2}{6^{s+2}(s+1)(s+2)}\Big(|f'(a)|+|f'(b)|\Big),\end{aligned}\tag{3.13}$$

*which is one of the inequalities given in* [33, Theorem7].

**Corollary 3.3.** *In Theorem* 3.1, *if* $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ *and* $s \in [0,1)$, *we get a fractional inequality for generalized* $(m,s)$*-Godunova-Levin-preinvex function*

$$\big|K_f(\alpha;\eta,m,a,b)\big| \le \frac{\eta(b,a,m)}{3(1-s)}\big[m|f'(a)| + |f'(b)|\big], \tag{3.14}$$

*specially for* $\alpha = 1$ *and* $\eta(b,a,m) = b - ma$ *with* $m = 1$, *then from the proof of Theorem* 3.1 *it follows that the following inequality for* $s$*-Godunova-Levin functions holds*

$$\begin{aligned} &\left|\frac{1}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\ &\le (b-a)\frac{(-s-4)6^{1-s} + 2\times 5^{2-s} - 2\times 3^{2-s} + 2}{6^{2-s}(1-s)(2-s)}\big[|f'(a)| + |f'(b)|\big]. \end{aligned} \tag{3.15}$$

**Corollary 3.4.** *In Theorem* 3.1, *when* $\eta(b,a,m)$ *with* $m = 1$ *degenerates into* $\eta(b,a)$, *if* $h_1(t) = 1 - t^\alpha$, $h_2(t) = t^\alpha$, *for* $\alpha \in (0,1]$, *then the inequality* (3.8) *becomes the following fractional inequality for* $\alpha$*-preinvex function*

$$\begin{aligned} &\Bigg|\frac{1}{6}\left[f(a) + 4f\left(a + \frac{\eta(b,a)}{2}\right) + f\Big(a + \eta(b,a)\Big)\right] \\ &- \frac{2^{\alpha-1}\Gamma(\alpha+1)}{\eta^\alpha(b,a)}\left[J^\alpha_{a^+} f\left(a + \frac{\eta(b,a)}{2}\right) J^\alpha_{\left(a+\eta(b,a)\right)^-} f\left(a + \frac{\eta(b,a)}{2}\right)\right]\Bigg| \\ &\le \frac{\eta(b,a)}{3(\alpha+1)}\Big(\alpha\big|f'(a)\big| + \big|f'(b)\big|\Big), \end{aligned} \tag{3.16}$$

*specially for* $\eta(b,a) = b - a$ *and* $\alpha = 1$, *then from the proof of Theorem* 3.1 *it follows that the following inequality holds*

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\ &\le \frac{5(b-a)}{72}\Big(\big|f'(a)\big| + \big|f'(b)\big|\Big). \end{aligned} \tag{3.17}$$

*This is one of the inequalities given in [32, Theorem 5].*

**Corollary 3.5.** *In Theorem* 3.1, *if* $\alpha = 1, h_1(t) = h_2(t) = t(1-t)$, *and* $\eta(b,a,m) = b - ma$ *with* $m = 1$, *then from the proof of Theorem* 3.1 *it follows that the following inequality for tgs-convex function holds*

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\ &\le \frac{211(b-a)}{7776}\Big(\big|f'(a)\big| + \big|f'(b)\big|\Big). \end{aligned} \tag{3.18}$$

**Theorem 3.2.** *Let $K \subseteq \mathbb{R}$ be an open $m$-invex subset with respect to $\eta : K\times K\times (0,1] \to \mathbb{R}$ for some fixed $m \in (0,1]$ and let $a, b \in K$, $a < b$ with $\eta(b,a,m) > 0$. Assume that $f : K \to \mathbb{R}$ is a first differentiable function, $h_1, h_2 : [0,1] \to \mathbb{R}_0$, $\left|f'\right|^q$ is $(m, h_1, h_2)$-preinvex on $[ma, ma+\eta(b,a,m)]$, $\frac{1}{p}+\frac{1}{q}=1$ and $q > 1$, then the following inequality for Riemann-Louville fractional integral with $\alpha > 0$ and $x \in [ma, ma+\eta(b,a,m)]$ holds*

$$
\begin{aligned}
&\left|K_f(\alpha;\eta,m,a,b)\right| \\
&\le \frac{\eta(b,a,m)}{6}\Bigg\{\left[\left(m|f'(a)|^q\int_0^1 h_1\Big(\frac{1+t}{2}\Big)\mathrm{d}t+|f'(b)|^q\int_0^1 h_2\Big(\frac{1+t}{2}\Big)\mathrm{d}t\right)\right]^{\frac{1}{q}} \\
&+\left[\left(m|f'(a)|^q\int_0^1 h_1\Big(\frac{1-t}{2}\Big)\mathrm{d}t+|f'(b)|^q\int_0^1 h_2\Big(\frac{1-t}{2}\Big)\mathrm{d}t\right)\right]^{\frac{1}{q}}\Bigg\}.
\end{aligned}
\tag{3.19}
$$

**Proof.** From Lemma 3.1 and the Hölder inequality, we have

$$
\begin{aligned}
&\left|K_f(\alpha;\eta,m,a,b)\right| \\
&\le \frac{\eta(b,a,m)}{2}\Bigg\{\left(\int_0^1\left|\frac{t^\alpha}{2}-\frac{1}{3}\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \\
&\cdot\left(\int_0^1\left|f'\Big(ma+\frac{1+t}{2}\eta(b,a,m)\Big)\right|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&+\left(\int_0^1\left|\frac{1}{3}-\frac{t^\alpha}{2}\right|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\Big(ma+\frac{1-t}{2}\eta(b,a,m)\Big)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\Bigg\}.
\end{aligned}
\tag{3.20}
$$

Because $|f'|^q$ is generalized $(m, h_1, h_2)$-preinvex, we get

$$
\begin{aligned}
&\int_0^1\left|f'\Big(ma+\frac{1+t}{2}\eta(b,a,m)\Big)\right|^q\mathrm{d}t \\
&\le m\left|f'(a)\right|^q\int_0^1 h_1\Big(\frac{1+t}{2}\Big)\mathrm{d}t+\left|f'(b)\right|^q\int_0^1 h_2\Big(\frac{1+t}{2}\Big)\mathrm{d}t
\end{aligned}
\tag{3.21}
$$

and

$$
\begin{aligned}
&\int_0^1\left|f'\Big(ma+\frac{1-t}{2}\eta(b,a,m)\Big)\right|^q\mathrm{d}t \\
&\le m\left|f'(a)\right|^q\int_0^1 h_1\Big(\frac{1-t}{2}\Big)\mathrm{d}t+\left|f'(b)\right|^q\int_0^1 h_2\Big(\frac{1-t}{2}\Big)\mathrm{d}t.
\end{aligned}
\tag{3.22}
$$

Using the fact that $\left|\frac{t^\alpha}{2}-\frac{1}{3}\right| \le \frac{1}{3}$ for all $t \in [0,1]$ and using the last two inequalities in (3.20) we obtain (3.19). This completes the proof of the theorem. □

**Corollary 3.6.** *Under the circumstance of Theorem 3.2 with $h_1(t) = h(1-t)$, $h_2(t) = h(t)$, if the mapping $\eta(b,a,m)$ with $m = 1$ degenerates into $\eta(b,a)$, then* *the inequality* (3.19) *becomes the following inequality for an h-preinvex function*

$$\tag{3.23}\begin{aligned}&\Bigg|\frac{1}{6}\Bigg[f(a)+4f\Bigg(a+\frac{\eta(b,a)}{2}\Bigg)+f\Bigg(a+\eta(b,a)\Bigg)\Bigg]\\&-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\Bigg[J_{a^+}^{\alpha}f\Bigg(\frac{a+b}{2}\Bigg)+J_{b^-}^{\alpha}f\Bigg(\frac{a+b}{2}\Bigg)\Bigg]\Bigg|\\&\leq\frac{\eta(b,a)}{6}\Bigg\{\Bigg[|f'(a)|^q\int_0^1 h\Big(\frac{1-t}{2}\Big)\mathrm{d}t+|f'(b)|^q\int_0^1 h\Big(\frac{1+t}{2}\Big)\mathrm{d}t\Bigg)\Bigg]^{\frac{1}{q}}\\&+\Bigg[|f'(a)|^q\int_0^1 h\Big(\frac{1+t}{2}\Big)\mathrm{d}t+|f'(b)|^q\int_0^1 h\Big(\frac{1-t}{2}\Big)\mathrm{d}t\Bigg)\Bigg]^{\frac{1}{q}}\Bigg\},\end{aligned}$$

*specially for* $\alpha=1,\eta(b,a)=b-a$, *then from the proof of Theorem* 3.2 *it follows that the following inequality for h-convex function holds*

$$\tag{3.24}\begin{aligned}&\Bigg|\frac{1}{6}\Bigg[f(a)+4f\Big(\frac{a+b}{2}\Big)+f(b)\Bigg]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\Bigg|\\&\leq\frac{b-a}{12}\Big(\frac{1+2^{p+1}}{3(p+1)}\Big)^{\frac{1}{p}}\Bigg\{\Bigg[|f'(a)|^q\int_0^1 h\Big(\frac{1-t}{2}\Big)\mathrm{d}t\\&+|f'(b)|^q\int_0^1 h\Big(\frac{1+t}{2}\Big)\mathrm{d}t\Big)\Bigg]^{\frac{1}{q}}\\&+\Bigg[|f'(a)|^q\int_0^1 h\Big(\frac{1+t}{2}\Big)\mathrm{d}t+|f'(b)|^q\int_0^1 h\Big(\frac{1-t}{2}\Big)\mathrm{d}t\Big)\Bigg]^{\frac{1}{q}}\Bigg\}.\end{aligned}$$

**Corollary 3.7.** *In Theorem* 3.2, *when* $\eta(b,a,m)$ *with* $m=1$ *reduces to* $\eta(b,a)$, *if* $s\in(0,1],h_1(t)=(1-t)^s$ *and* $h_2(t)=t^s$, *then the inequality* (3.8) *becomes the following inequality for s-preinvex function*

$$\tag{3.25}\begin{aligned}&\Bigg|\frac{1}{6}\Bigg[f(a)+4f\Bigg(a+\frac{\eta(b,a)}{2}\Bigg)+f\Bigg(a+\eta(b,a)\Bigg)\Bigg]\\&-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\eta^{\alpha}(b,a)}\Bigg[J_{a^+}^{\alpha}f\Bigg(a+\frac{\eta(b,a)}{2}\Bigg)+J_{(a+\eta(b,a))^-}^{\alpha}f\Bigg(a+\frac{\eta(b,a)}{2}\Bigg)\Bigg]\Bigg|\\&\leq\frac{\eta(b,a)}{6}\Bigg(\frac{2}{s+1}\Bigg)^{\frac{1}{q}}\Bigg\{\Bigg[|f'(a)|^q\frac{1}{2^{s+1}}+|f'(b)|^q\Bigg(1-\frac{1}{2^{s+1}}\Bigg)\Bigg]^{\frac{1}{q}}\\&+\Bigg[|f'(a)|^q\Bigg(1-\frac{1}{2^{s+1}}\Bigg)+|f'(b)|^q\frac{1}{2^{s+1}}\Bigg]^{\frac{1}{q}}\Bigg\},\end{aligned}$$

*specially for* $\alpha=1$, *then from the proof of Theorem* 3.2 *it follows that the following inequality holds*

$$\tag{3.26}\Bigg|\frac{1}{6}\Bigg[f(a)+4f\Bigg(a+\frac{\eta(b,a)}{2}\Bigg)+f\Bigg(a+\eta(b,a)\Bigg)\Bigg]-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\Bigg|$$

$$\leq \frac{\eta(b,a)}{36}\Big(\frac{q-1}{2q-1}\Big)^{1-\frac{1}{q}}\Big[\frac{3}{2^s(s+1)}\Big]^{\frac{1}{q}}\Big(2^{\frac{2q-1}{q-1}}+1\Big)^{1-\frac{1}{q}}$$
$$\times\left\{\Big[|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q\Big]^{\frac{1}{q}}+\Big[(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q\Big]^{\frac{1}{q}}\right\},$$

*which is the inequalities given in [40, Theorem 3.2].*

*Specially for $\eta(b,a)=b-a$, then from the proof of Theorem 3.2 it follows that the following inequality for $s$-convex function holds*

$$\begin{aligned}&\left|\frac{1}{6}\Big[f(a)+4f\Big(\frac{a+b}{2}\Big)+f(b)\Big]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\ &\leq\frac{(b-a)}{12}\Big(\frac{1+2^{p+1}}{3(p+1)}\Big)^{\frac{1}{p}}\left\{\Big(\frac{(2^{s+1}-1)|f'(a)|^q+|f'(b)|^q}{2^s(s+1)}\Big)^{\frac{1}{q}}\right.\\ &\left.+\Big(\frac{|f'(a)|^q+(2^{s+1}-1)|f'(b)|^q}{2^s(s+1)}\Big)^{\frac{1}{q}}\right\}.\end{aligned}\tag{3.27}$$

*This is one of the inequalities given in [33, Theorem 9].*

**Corollary 3.8.** *In Theorem* 3.2, *if $h_1(t)=(1-t)^{-s}$, $h_2(t)=t^{-s}$ and $s\in[0,1)$, we get a fractional inequality for generalized $(m,s)$-Godunova-Levin-preinvex function*

$$\begin{aligned}&\big|K_f(\alpha;\eta,m,a,b)\big|\\ &\leq\frac{\eta(b,a,m)}{6}\Big(\frac{2}{1-s}\Big)^{\frac{1}{q}}\left\{\Big[m|f'(a)|^q\Big(1-\frac{1}{2^{1-s}}\Big)+|f'(b)|^q\frac{1}{2^{1-s}}\Big]^{\frac{1}{q}}\right.\\ &\left.+\Big[m|f'(a)|^q\frac{1}{2^{1-s}}+|f'(b)|^q\Big(1-\frac{1}{2^{1-s}}\Big)\Big]^{\frac{1}{q}}\right\},\end{aligned}\tag{3.28}$$

*specially for $\alpha=1$, $\eta(b,a,m)$ with $m=1$ reduces to $\eta(b,a)$, then from the proof of Theorem* 3.2 *it follows that the following inequality holds*

$$\begin{aligned}&\left|\frac{1}{6}\Big[f(a)+4f\Big(a+\frac{\eta(b,a)}{2}\Big)+f\Big(a+\eta(b,a)\Big)\Big]-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\right|\\ &\leq\frac{\eta(b,a)}{36}\Big(\frac{q-1}{2q-1}\Big)^{1-\frac{1}{q}}\Big[\frac{3}{2^{-s}(1-s)}\Big]^{\frac{1}{q}}\Big(2^{\frac{2q-1}{q-1}}+1\Big)^{1-\frac{1}{q}}\\ &\times\left\{\Big[(2^{1-s}-1)|f'(a)|^q+|f'(b)|^q\Big]^{\frac{1}{q}}+\Big[|f'(a)|^q+(2^{1-s}-1)|f'(b)|^q\Big]^{\frac{1}{q}}\right\},\end{aligned}\tag{3.29}$$

*furthermore if $\eta(b,a)=b-a$, then from the proof of Theorem 3.2 it follows that the following inequality for $s$-Godunova-Levin function holds*

$$\left|\frac{1}{6}\Big[f(a)+4f\Big(\frac{a+b}{2}\Big)+f(b)\Big]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\tag{3.30}$$

$$\leq \frac{b-a}{12}\Big(\frac{1+2^{p+1}}{3(p+1)}\Big)^{\frac{1}{p}}\left\{\Big(\frac{(2^{1-s}-1)|f'(a)|^q+|f'(b)|^q}{2^{-s}(1-s)}\Big)^{\frac{1}{q}}\right.$$
$$\left.+\Big(\frac{(2^{1-s}-1)|f'(b)|^q+|f'(a)|^q}{2^{-s}(1-s)}\Big)^{\frac{1}{q}}\right\}.$$

**Corollary 3.9.** *In Theorem 3.2, when $\eta(b,a,m)$ with $m=1$ reduces to $\eta(b,a)$, if $h_1(t)=1-t^\alpha$, $h_2(t)=t^\alpha$, for $\alpha\in(0,1]$, then the inequality (3.8) becomes the following fractional inequality for $\alpha$-preinvex function*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(a+\frac{\eta(b,a)}{2}\right)+f\Big(a+\eta(b,a)\Big)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\eta^\alpha(b,a)}\left[J^\alpha_{a^+}f\left(a+\frac{\eta(b,a)}{2}\right)+J^\alpha_{\left(a+\eta(b,a)\right)^-}f\left(a+\frac{\eta(b,a)}{2}\right)\right]\right|\\&\leq\frac{\eta(b,a)}{6}\Big(\frac{1}{2^\alpha(\alpha+1)}\Big)^{\frac{1}{q}}\left\{\left[\Big(2^\alpha(\alpha-1)+1\Big)|f'(a)|^q+(2^{\alpha+1}-1)|f'(b)|^q\right]^{\frac{1}{q}}\right.\\&\left.+\left[\Big(2^\alpha(\alpha+1)-1\Big)|f'(a)|^q+|f'(b)|^q\right]^{\frac{1}{q}}\right\}.\end{aligned}\tag{3.31}$$

*Furthermore if $\eta(b,a)=b-a$, with $\alpha=1$, then from the proof of Theorem 3.2 it follows that the following inequality holds*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\&\leq\frac{b-a}{12}\Big(\frac{1+2^{p+1}}{3(p+1)}\Big)^{\frac{1}{p}}\left[\left(\frac{1}{4}|f'(a)|^q+\frac{3}{4}|f'(b)|^q\right)^{\frac{1}{q}}\right.\\&\left.+\left(\frac{3}{4}|f'(a)|^q+\frac{1}{4}|f'(b)|^q\right)^{\frac{1}{q}}\right],\end{aligned}\tag{3.32}$$

*which is given by Sarikaya et al. in [32, Theorem 6].*

**Corollary 3.10.** *In Theorem 3.2, if $h_1(t)=1-t$, $h_2(t)=t$, $\eta(b,a,m)=b-ma$ with $m=1$, then the following fractional inequality holds*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right.\\&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{a^+}f\left(\frac{a+b}{2}\right)+J^\alpha_{b^-}f\left(\frac{a+b}{2}\right)\right]\right|\\&\leq\frac{b-a}{6}\left[\left(\frac{1}{4}|f'(a)|^q+\frac{3}{4}|f'(b)|^q\right)^{\frac{1}{q}}+\left(\frac{3}{4}|f'(a)|^q+\frac{1}{4}|f'(b)|^q\right)^{\frac{1}{q}}\right].\end{aligned}\tag{3.33}$$

**Corollary 3.11.** *In Theorem 3.2, if $h_1(t) = h_2(t) = t(1-t)$, and $\eta(b,a,m) = b - ma$ with $m = 1$, we obtain a fractional inequality for tgs-convex function*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right.\\ &\left.- \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f\left(\frac{a+b}{2}\right) + J^{\alpha}_{b^-}f\left(\frac{a+b}{2}\right)\right]\right|\\ &\leq \frac{b-a}{3\sqrt[q]{6}}\Big(|f'(a)|^q + |f'(b)|^q\Big)^{\frac{1}{q}},\end{aligned} \tag{3.34}$$

*furthermore if $\alpha = 1$, then from the proof of Theorem 3.2 it follows that the following inequality holds*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\ &\leq \frac{b-a}{6}\left(\frac{1}{6}\right)^{\frac{1}{q}}\Big(\frac{1+2^{p+1}}{3(p+1)}\Big)^{\frac{1}{p}}\Big(|f'(a)|^q + |f'(b)|^q\Big)^{\frac{1}{q}}.\end{aligned} \tag{3.35}$$

**Theorem 3.3.** *Let $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K\times K\times(0,1] \to \mathbb{R}$ for some fixed $m \in (0,1]$ and let $a, b \in K$, $a < b$ with $\eta(b,a,m) > 0$. Suppose that $f : K \to \mathbb{R}$ is a first differentiable function, $h_1, h_2 : [0,1] \to \mathbb{R}_0$, $|f'|^q$ is $(m, h_1, h_2)$-preinvex on $[ma, ma + \eta(b,a,m)]$ and $q \geq 1$, then the following inequality for Riemann-Louville fractional integral with $\alpha > 0$ and $x \in [ma, ma + \eta(b,a,m)]$ holds*

$$\begin{aligned}&\big|K_f(\alpha;\eta,m,a,b)\big|\\ &\leq \frac{\eta(b,a,m)}{6}\left[\frac{1}{\alpha+1}\left(\Big(\frac{2}{3}\Big)^{\frac{1}{\alpha}}2\alpha + \frac{1}{2} - \alpha\right)\right]^{1-\frac{1}{q}}\\ &\times\left\{\left[\left(m|f'(a)|^q\int_0^1 h_1\Big(\frac{1+t}{2}\Big)\mathrm{d}t + |f'(b)|^q\int_0^1 h_2\Big(\frac{1+t}{2}\Big)\mathrm{d}t\right)\right]^{\frac{1}{q}}\right.\\ &\left.+\left[\left(m|f'(a)|^q\int_0^1 h_1\Big(\frac{1-t}{2}\Big)\mathrm{d}t + |f'(b)|^q\int_0^1 h_2\Big(\frac{1-t}{2}\Big)\mathrm{d}t\right)\right]^{\frac{1}{q}}\right\}.\end{aligned} \tag{3.36}$$

**Proof.** From Lemma 3.1 and power mean inequality, we have

$$\begin{aligned}&\big|K_f(\alpha;\eta,m,a,b)\big|\\ &\leq \frac{\eta(b,a,m)}{2}\left(\int_0^1\left|\frac{t^{\alpha}}{2} - \frac{1}{3}\right|\mathrm{d}t\right)^{1-\frac{1}{q}}\\ &\times\left[\left(\int_0^1\left|\frac{t^{\alpha}}{2} - \frac{1}{3}\right|\left|f'\left(ma + \frac{1+t}{2}\eta(b,a,m)\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_0^1\left|\frac{1}{3} - \frac{t^{\alpha}}{2}\right|\left|f'\left(ma + \frac{1-t}{2}\eta(b,a,m)\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right].\end{aligned} \tag{3.37}$$

By the generalized $(m, h_1, h_2)$-preinvex of $|f'|^q$ and using the fact that $\left|\frac{t^\alpha}{2} - \frac{1}{3}\right| = \left|\frac{1}{3} - \frac{t^\alpha}{2}\right| \le \frac{1}{3}$ for all $t \in [0,1]$, we get

$$\begin{aligned}&\int_0^1 \left|\frac{t^\alpha}{2} - \frac{1}{3}\right| \left|f'\left(ma + \frac{1+t}{2}\eta(b,a,m)\right)\right|^q \mathrm{d}t \\ &\le \frac{1}{3}\left(m\left|f'(a)\right|^q \int_0^1 h_1\left(\frac{1+t}{2}\right)\mathrm{d}t + \left|f'(b)\right|^q \int_0^1 h_2\left(\frac{1+t}{2}\right)\mathrm{d}t\right)\end{aligned} \tag{3.38}$$

and

$$\begin{aligned}&\int_0^1 \left|\frac{1}{3} - \frac{t^\alpha}{2}\right| \left|f'\left(ma + \frac{1-t}{2}\eta(b,a,m)\right)\right|^q \mathrm{d}t \\ &\le \frac{1}{3}\left(m\left|f'(a)\right|^q \int_0^1 h_1\left(\frac{1-t}{2}\right)\mathrm{d}t + \left|f'(b)\right|^q \int_0^1 h_2\left(\frac{1-t}{2}\right)\mathrm{d}t\right).\end{aligned} \tag{3.39}$$

By simple computation,

$$\left(\int_0^1 \left|\frac{t^\alpha}{2} - \frac{1}{3}\right|\mathrm{d}t\right)^{1-\frac{1}{q}} = \left[\frac{1}{3(\alpha+1)}\left(\left(\frac{2}{3}\right)^{\frac{1}{\alpha}} 2\alpha + \frac{1}{2} - \alpha\right)\right]^{1-\frac{1}{q}}. \tag{3.40}$$

Using the last three inequalities in (3.37) we obtain (3.36). This completes the proof of the theorem. □

**Corollary 3.12.** *Under the assumptions of Theorem* 3.3 *with* $h_1(t) = h(1-t)$, $h_2(t) = h(t)$, *if the mapping* $\eta(b,a,m)$ *with* $m = 1$ *degenerates into* $\eta(b,a)$, *then the inequality* (3.36) *becomes the following inequality for an* $h$*-preinvex function*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a) + 4f\left(a + \frac{\eta(b,a)}{2}\right) + f\left(a + \eta(b,a)\right)\right]\right. \\ &\left.- \frac{2^{\alpha-1}\Gamma(\alpha+1)}{\eta^\alpha(b,a)}\left[J_{a^+}^\alpha f\left(a + \frac{\eta(b,a)}{2}\right) + J_{\left(a+\eta(b,a,)\right)^-}^\alpha f\left(a + \frac{\eta(b,a)}{2}\right)\right]\right| \\ &\le \frac{\eta(b,a)}{6}\left[\frac{1}{\alpha+1}\left(\left(\frac{2}{3}\right)^{\frac{1}{\alpha}} 2\alpha + \frac{1}{2} - \alpha\right)\right]^{1-\frac{1}{q}} \\ &\times \left\{\left[|f'(a)|^q \int_0^1 h\left(\frac{1-t}{2}\right)\mathrm{d}t + |f'(b)|^q \int_0^1 h\left(\frac{1+t}{2}\right)\mathrm{d}t\right]^{\frac{1}{q}}\right. \\ &\left.+ \left[|f'(a)|^q \int_0^1 h\left(\frac{1+t}{2}\right)\mathrm{d}t + |f'(b)|^q \int_0^1 h\left(\frac{1-t}{2}\right)\mathrm{d}t\right]^{\frac{1}{q}}\right\},\end{aligned} \tag{3.41}$$

*specially for* $\eta(b,a) = b - a$ *and* $\alpha = 1$, *we achieve a inequality for* $h$*-convex function*

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \tag{3.42}$$

$$\le \frac{b-a}{6}\Big(\frac{5}{12}\Big)^{1-\frac{1}{q}}\Bigg\{\Big[|f'(a)|^q\int_0^1 h\Big(\frac{1-t}{2}\Big)\mathrm{d}t+|f'(b)|^q\int_0^1 h\Big(\frac{1+t}{2}\Big)\mathrm{d}t\Big)\Big]^{\frac{1}{q}}$$
$$+\Big[|f'(a)|^q\int_0^1 h\Big(\frac{1+t}{2}\Big)\mathrm{d}t+|f'(b)|^q\int_0^1 h\Big(\frac{1-t}{2}\Big)\mathrm{d}t\Big]^{\frac{1}{q}}\Bigg\}.$$

**Corollary 3.13.** *In Theorem* 3.3, *when* $\eta(b,a,m)$ *with* $m=1$ *reduces to* $\eta(b,a)$, *if* $s\in(0,1], h_1(t)=(1-t)^s$, $h_2(t)=t^s$, *then the inequality* (3.36) *becomes the following fractional inequality for s-preinvex function*

$$\begin{aligned}
&\Bigg|\frac{1}{6}\Big[f(a)+4f\Big(a+\frac{\eta(b,a)}{2}\Big)+f\Big(a+\eta(b,a)\Big)\Big]\\
&-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\eta^{\alpha}(b,a)}\Big[J^{\alpha}_{a^+}f\Big(a+\frac{\eta(b,a)}{2}\Big)+J^{\alpha}_{\left(a+\eta(b,a)\right)^-}f\Big(a+\frac{\eta(b,a)}{2}\Big)\Big]\Bigg|\\
&\le\frac{\eta(b,a)}{6}\Big[\frac{1}{\alpha+1}\Big(\Big(\frac{2}{3}\Big)^{\frac{1}{\alpha}}2\alpha+\frac{1}{2}-\alpha\Big)\Big]^{1-\frac{1}{q}}\Big(\frac{2}{s+1}\Big)^{\frac{1}{q}}\\
&\times\Bigg\{\Big[|f'(a)|^q\Big(1-\frac{1}{2^{s+1}}\Big)+|f'(b)|^q\frac{1}{2^{s+1}}\Big]^{\frac{1}{q}}\\
&+\Big[|f'(a)|^q\frac{1}{2^{s+1}}+|f'(b)|^q\Big(1-\frac{1}{2^{s+1}}\Big)\Big]^{\frac{1}{q}}\Bigg\},
\end{aligned}\tag{3.43}$$

*specially for* $\eta(b,a)=b-a$, $\alpha=1$, *then from the proof of Theorem 3.3 it follows that the following inequality for s-convex function holds*

$$\begin{aligned}
&\Bigg|\frac{1}{6}\Big[f(a)+4f\Big(\frac{a+b}{2}\Big)+f(b)\Big]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\Bigg|\\
&\le\frac{(b-a)}{2}\Big(\frac{5}{36}\Big)^{1-\frac{1}{q}}\Bigg\{\Bigg(\frac{2\times5^{s+2}+(s-4)\times6^{s+1}-(2s+7)3^{s+1}}{3\times6^{s+1}(s+1)(s+2)}|f'(a)|^q\\
&+\frac{(2s+1)3^{s+1}+2}{3\times6^{s+1}(s+1)(s+2)}|f'(b)|^q\Bigg)^{\frac{1}{q}}+\Bigg(\frac{(2s+1)3^{s+1}+2}{3\times6^{s+1}(s+1)(s+2)}|f'(a)|^q\\
&+\frac{2\times5^{s+2}+(s-4)\times6^{s+1}-(2s+7)3^{s+1}}{3\times6^{s+1}(s+1)(s+2)}|f'(b)|^q\Bigg)^{\frac{1}{q}}\Bigg\},
\end{aligned}\tag{3.44}$$

*which is one of the inequalities proved in [33, Theorem 10].*

**Corollary 3.14.** *In Theorem* 3.3, *if* $h_1(t)=(1-t)^{-s}$, $h_2(t)=t^{-s}$ *and* $s\in[0,1)$, *we get a fractional inequality for generalized* $(m,s)$*-Godunova-Levin-preinvex* *function*

$$
\begin{aligned}
&\big|K_f(\alpha;\eta,m,a,b)\big|\\
\leq&\frac{\eta(b,a,m)}{6}\left[\frac{1}{\alpha+1}\bigg(\Big(\frac{2}{3}\Big)^{\frac{1}{\alpha}}2\alpha+\frac{1}{2}-\alpha\bigg)\right]^{1-\frac{1}{q}}\Big(\frac{2}{1-s}\Big)^{\frac{1}{q}}\\
&\times\left\{\left[m|f'(a)|^q\Big(1-\frac{1}{2^{1-s}}\Big)+|f'(b)|^q\frac{1}{2^{1-s}}\right]^{\frac{1}{q}}\right.\\
&\left.+\left[m|f'(a)|^q\frac{1}{2^{1-s}}+|f'(b)|^q\Big(1-\frac{1}{2^{1-s}}\Big)\right]^{\frac{1}{q}}\right\},
\end{aligned}
\tag{3.45}
$$

*specially for $\alpha=1$ and $\eta(b,a,m)=b-ma$ with $m=1$, then from the proof of Theorem* 3.3 *it follows that the following inequality for s-Godunova-Levin functions holds*

$$
\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\Big(\frac{a+b}{2}\Big)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\
&\leq\frac{(b-a)}{2}\Big(\frac{5}{36}\Big)^{1-\frac{1}{q}}\left\{\bigg(\frac{2\times5^{2-s}+(-s-4)\times6^{1-s}-(7-2s)3^{1-s}}{3\times6^{1-s}(1-s)(2-s)}|f'(b)|^q\right.\\
&+\frac{(1-2s)3^{1-s}+2}{3\times6^{1-s}(1-s)(2-s)}|f'(a)|^q\bigg)^{\frac{1}{q}}+\bigg(\frac{(1-2s)3^{1-s}+2}{3\times6^{1-s}(1-s)(2-s)}|f'(b)|^q\\
&\left.+\frac{(2\times5^{2-s}+(-s-4)\times6^{1-s}-(7-2s)3^{1-s}}{3\times6^{1-s}(1-s)(2-s)}|f'(a)|^q\bigg)^{\frac{1}{q}}\right\}.
\end{aligned}
\tag{3.46}
$$

**Corollary 3.15.** *In Theorem* 3.3, *when $\eta(b,a,m)$ with $m=1$ reduces to $\eta(b,a)$, if $h_1(t)=1-t^\alpha$, $h_2(t)=t^\alpha$, for $\alpha\in(0,1]$, then the inequality* (3.36) *becomes the following fractional inequality for $\alpha$-preinvex function*

$$
\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\Big(a+\frac{\eta(b,a)}{2}\Big)+f\Big(a+\eta(b,a)\Big)\right]\right.\\
&\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\eta^\alpha(b,a)}\left[J^\alpha_{a+}f\Big(a+\frac{\eta(b,a)}{2}\Big)J^\alpha_{(a+\eta(b,a))^-}f\Big(a+\frac{\eta(b,a)}{2}\Big)\right]\right|\\
&\leq\frac{\eta(b,a)}{6(\alpha+1)}\left[\bigg(\Big(\frac{2}{3}\Big)^{\frac{1}{\alpha}}2\alpha+\frac{1}{2}-\alpha\bigg)\right]^{1-\frac{1}{q}}\Big(\frac{1}{2^\alpha}\Big)^{\frac{1}{q}}\\
&\times\left\{\left[\Big(2^\alpha(\alpha-1)+1\Big)|f'(a)|^q+(2^{\alpha+1}-1)|f'(b)|^q\right]^{\frac{1}{q}}\right.\\
&\left.+\left[\Big(2^\alpha(\alpha+1)-1\Big)|f'(a)|^q+|f'(b)|^q\right]^{\frac{1}{q}}\right\},
\end{aligned}
\tag{3.47}
$$

*specially for $\eta(b,a) = b-a$, $\alpha = 1$, then from the proof of Theorem 3.3 it follows that the following inequality holds*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\ \leq&\frac{5(b-a)}{72}\left\{\left(\frac{29|f'(a)|^q+61|f'(b)|^q}{90}\right)^{\frac{1}{q}}+\left(\frac{61|f'(a)|^q+29|f'(b)|^q}{90}\right)^{\frac{1}{q}}\right\},\end{aligned} \tag{3.48}$$

*which is confirmed by Sarikaya et al. in [32, Theorem 7].*

**Corollary 3.16.** *If $\alpha = 1, h_1(t) = h_2(t) = t(1-t)$, and $\eta(b,a,m) = b - ma$ with $m = 1$, then from the proof of Theorem* 3.3 *it follows that the following inequality for tgs-convex function holds*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\\ \leq&\frac{5(b-a)}{36}\left(\frac{211}{1080}\right)^{\frac{1}{q}}\left(|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}}.\end{aligned} \tag{3.49}$$

**Corollary 3.17.** *In Theorem* 3.3*, if $h_1(t) = 1-t$, $h_2(t) = t$, and $\eta(b,a,m) = b - ma$ with $m = 1$, then the following fractional inequality holds*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right.\\ &\left.-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{a^+}f\left(\frac{a+b}{2}\right)+J^\alpha_{b^-}f\left(\frac{a+b}{2}\right)\right]\right|\\ &\leq\frac{b-a}{6}\left[\frac{1}{\alpha+1}\left(\left(\frac{2}{3}\right)^{\frac{1}{\alpha}}2\alpha+\frac{1}{2}-\alpha\right)\right]^{1-\frac{1}{q}}\\ &\times\left\{\left[\left(\frac{1}{4}|f'(a)|^q+\frac{3}{4}|f'(b)|^q\right)^{\frac{1}{q}}+\left(\frac{3}{4}|f'(a)|^q+\frac{1}{4}|f'(b)|^q\right)^{\frac{1}{q}}\right]\right\}.\end{aligned} \tag{3.50}$$

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