COMMON FIXED POINT RESULTS IN S-FUZZY METRIC SPACES

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Abstract. In this paper we prove fixed point results in integral type mapping with rational expression under weak compatible condition.

Keywords: Self mapping, complete metric spaces, t-norm, common fixed point theorem.

1. Introduction

Concept of Fuzzy initiated by Zadeh [8] in 1965 with the publication of his paper fuzzy sets. The notion of fuzzy set is a turning point in the development of mathematics. Consequently the last three decades were very productive of mathematics. In this paper we establish general common fixed point theorems, which generalize the result of Singh and Chouhan [3, 4], Singh and Sharma [5, 6, 7] in S-fuzzy metric space.

In 2002, an analogue of a Banach contraction principle for integral type inequality Branciari [1] obtained a fixed point theorem for a single mapping. A lot of research works have been carried out on generalizing contractive conditions of integral type for different contractive mappings satisfying various known properties.

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Theorem 1.1 ([1]). *Let (X,d) be a complete metric space, $c \in (0,1)$ and let $f : X \to X$ be a mapping such that for each $x, y \in X$,*

$$\int_0^{d(fx,fy)} \varphi(t)dt \leq c \int_0^{d(x,y)} \varphi(t)dt.$$

Where $\varphi : [0,+\infty) \to [0,+\infty)$ is a Lebesgue - integrable mapping which is summable on each compact subset of $[0,+\infty)$, non negative and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t) > 0$, then f has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n\to\infty} f^n x = a$.

Theorem 1.2 ([2]). *Let (X,d) be a complete metric space and $f : X \to X$ such that*

$$\int_0^{d(fx,fy)} u(t)dt \leq \alpha \int_0^{d(x,fx)+d(y,fy)} u(t)dt + \beta \int_0^{d(x,y)} u(t)dt + \gamma \int_0^{\max\{d(x,fy),d(y,fy)\}} u(t)dt,$$

For each $x, y \in X$, with non-negative real α, β, γ such that $2\alpha + \beta + 2\gamma < 1$. Where $u : [0,+\infty) \to [0,+\infty)$ is a Lesbesgue integrable mapping which is summable, non negative and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} u(t)dt > 0$. Then f has a unique fixed point in X.

There is a gap in the proof of Theorem 1.2 In fact, the authors [2] used the inequality $\int_0^{a+b} u(t) \leq \int_0^a u(t)dt + \int_0^b u(t)dt$ for $0 \leq a < b$, which is not true in general. In this chapter we present in the presence of this inequality an extension of Theorem 1.2 using altering distance functions.

On taking the concept of Branciari [1] we establish common fixed point theorems in S fuzzy space for four integral type mappings.

2. Preliminaries

Definition 2.1. *The 3-tuple $(X, S, *)$ is said to be a S-Fuzzy Metric Space if X is an arbitrary set, $*$ is a continuous t-norm and S is a Fuzzy set on $X^3 \times (0,\infty)$, satisfying the following conditions:*

(*i*) $S(x,y,z,t) > 0$;

(*ii*) $S(x,y,z,t) = 1$ *if and only if* $x = y = z$;

(*iii*) $S(x,y,z,t) = S(y,z,x,t) = S(z,y,x,t)$ *(Symmetry);*

(*iv*) $S(x,y,z,r+s+t) \geq S(x,y,w,r) * S(x,w,z,s) * S(w,y,z,t)$ *(Tetrahedral inequality);*

(*v*) $S(x,y,z,.) : (0,\infty) \to [0,1]$ *is continuous for all* $x,y,z,w \in X$ *and* $r,s,t > 0$.

Geometrically $S(x, y, z, t)$ represents the Fuzzy Perimeter of the triangle whose vertices are the points x, y and z with respect to $t > 0$.

Definition 2.2. *Two mappings A and S on fuzzy metric space X are weakly compatible iff $S(ATu, TAu, y, t) \geq S(Au, Tu, y, t)$ for $u \in X$.*

Lemma 2.3. *If for all $x, y \in X$, $t > 0$ and for a number $q \in (0, 1)$,*

$$S(x, y, z, qt) \geq S(x, y, z, t),$$

then $x = y = z$.

Lemma 2.4. *Let X be a set, f, g be weakly compatible self maps of X. If f and g have a unique point of coincidence, $w = fx = gx$, then w is the unique common fixed point of f and g.*

3. Main results

Theorem 3.1. *Let A, B, E and F be self mappings of a complete S fuzzy metric space $(X, S, *)$ with continuous t-norm $*$ defined by $a * b * c = \min\{(a, b, c) : a, b, c \in [0, 1]\}$ satisfying the following conditions*

$$A(X) \subset F(X), B(X) \subset E(X). \tag{3.1}$$

Mappings E and $F$are continuous.

$\{A, E\}$ and $\{B, F\}$ are weakly compatible mappings

For all $x, y, z \in X, t > 0$, $k \in [0, 1]$

$$\int_0^{S(Ax,By,z,kt)} \varphi(t)dt \geq \int_0^J \varphi(t)dt. \tag{3.2}$$

Where

$$\begin{aligned} &J = \min\{S(Ax, Ex, z, t), S(By, Fy, z, t), S(Ex, Fy, z, t) \\ &\frac{S(Ax, Ex, z, 2t)S(By, Fy, z, 2t)}{S(Ex, Fy, z, t)}, \frac{S(Bx, Ex, z, 2t)}{S(Ex, Fy, z, t)} \\ &\frac{S(Bx, Fy, z, t) + S(By, Ex, z, t)}{2}\}, \\ &\text{for all } x, y, z \in X, \lim S(x, y, z, t) \to 1 \text{ as } t \to \infty. \end{aligned}$$

Then A, B, E and F have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. Since $A(x) \subseteq F(x)$, we can find a point x_1 in X such that $Ax_0 = Fx_1$. Also since $B(x) \subseteq E(x)$, we can choose a point x_2 with $Bx_1 = Ex_2$.

Using this argument repeatedly we can construct sequence $\{y_n\}$ in X such that $y_{2n-1} = Fx_{2n-1} = Bx_{2n-2}$ and $y_{2n} = Ex_{2n} = Bx_{2n-1}, n = 1, 2, 3.$

From (3.2) we have

$$\int_0^{S(y_{2n+1},y_{2n+2},z,kt)} \varphi(t)dt = \int_0^{S(Ax_{2n},Bx_{2n+1},z,kt)} \varphi(t)dt \geq \int_0^{J_1} \varphi(t)dt.$$

Where

$$J_1 = \min\{S(Ax_{2n}, Ex_{2n}, z, t), S(Bx_{2n+1}, Fx_{2n+1}, z, t)S(Ex_{2n}, Fx_{2n+1}, z, t)$$
$$\frac{S(Ax_{2n}, Ex_{2n}, z, t)S(Bx_{2n+1}, Fx_{2n+1}, z, 2t)}{S(Ex_{2n}, Fx_{2n+1}, z, t)},$$
$$\frac{S(Ax_{2n}, Ex_{2n}, z, 2t)}{S(Ex_{2n}, Fx_{2n+1}, z, t)},$$
$$\frac{S(Ax_{2n}, Fx_{2n+1}, z, t) + S(Bx_{2n+1}, Ex_{2n}, z, t)}{2}\}$$
$$\int_0^{J_2} \varphi(t)dt.$$

Where

$$J_2 = \min\{S(y_{2n}, y_{2n+1}, z, t), S(y_{2n+2}, y_{2n+1}, z, t), S(y_{2n}, y_{2n+1}, z, t),$$
$$\frac{S(y_{2n+1}, y_{2n}, z, 2t)S(y_{2n+2}, y_{2n+1}, z, 2t)}{S(y_{2n}, y_{2n+1}, z, t)}, \frac{S(y_{2n+1}, y_{2n}, z, 2t)}{S(y_{2n}, y_{2n+1}, z, t)},$$
$$\frac{S(y_{2n+1}, y_{2n+1}, z, t) + S(y_{2n+2}, y_{2n}, z, t)}{2}\}$$

$$\int_0^{S(y_{2n+1},y_{2n+2},z,kt)} \varphi(t)dt \geq \int_0^{S(y_{2n},y_{2n+1},z,t)} \varphi(t)dt.$$

Which implies in general

$$\int_0^{S(y_n,y_{n+1},z,kt)} \varphi(t)dt \geq \int_0^{S(y_{n-1},y_n,z,t)} \varphi(t)dt. \tag{3.3}$$

To prove that $\{y_n\}$ is a Cauchy sequence we shall prove

$$S(y_n, y_{n+m}, z, t) \geq 1 - \lambda \tag{3.4}$$

is true for all $n \geq n_0$ and every $m \in \mathbb{N}$.

Here we use induction method from (3.3) we have

$$\int_0^{S(y_n,y_{n+1},z,t)} \varphi(t)dt \geq \int_0^{S(y_{n-1},y_n,z,\frac{t}{k})} \varphi(t)dt$$
$$\geq \int_0^{S(y_{n-2},y_{n-1},z,t/k^2)\ldots S(y_0,y_1,z,t/k^n)} \varphi(t)dt \to 1 \text{ as } n \to \infty, \tag{3.5}$$

i.e. For $t > 0$, $\lambda \in (0, 1)$ we can choose $n_0 \in N$ such that $S(y_n, y_{n+1}, z, t) \geq 1-\lambda$. Thus (3.4) is true for m = 1 suppose (3.4) is true for m then we shall prove that it is also true for $m + 1$.

Using the definition of fuzzy metric space by (3.3) and (3.4) we have

$$\int_0^{S(y_n,y_{n+m+1},z,t)} \varphi(t)dt \geq \int_0^{\min\{S(y_n,y_{n+m},z,t/2),S(y_{n+m},y_{n+m+1},z,t/2)\}} \varphi(t)dt \geq 1-\lambda.$$

Hence (3.4) is true for $m + 1$. Thus $\{y_n\}$ is a Cauchy sequence. By completeness for $(X, S, *)$, $\{y_n\}$ converges to some point z in X. Thus $\{Ax_{2n}\}$, $\{Ex_{2n}\}$, $\{Bx_{2n-1}\}$ and $\{Fx_{2n-1}\}$ also converges to z_1. Now $Ax_{2n} \to z$ and E is continuous.

Hence $EAx_{2n} \to Ez_1$.

Thus for $t>0, \lambda\in(0, 1)$ there exist an $n_0\in N$ such that $S(EAx_{2n}, Ez_1, z, t/2) \geq 1 - \lambda$, for all $n > n_0$. Using given condition (weak compatible mappings) we have

$$\int_0^{S(AEx_{2n},EAx_{2n},z,t/2)} \varphi(t)dt \to 1$$

$$\int_0^{S(AEx_{2}n,Ez,z,t/2)} \varphi(t)dt \geq \int_0^{\min\{S(AEx_{2n},EAx_{2n},z,t/2),S(EAx_{2n},Ez,z,t/2)\}} \varphi(t)dt$$

$$\geq 1 - \lambda$$

(3.6) Hence $AEx_{2n} \to Ez_1$.

(3.7) Similarly $BFx_{2n-1} \to Fz_1$.

Using (3.2) we have

$$\int_0^{S(AEx_{2n},BFx_{2n-1},z,kt)} \varphi(t)dt \geq \int_0^{J_3} \varphi(t)dt.$$

Where

$$J_3 =$$

$$= \min\{S(AEx_{2n}, E^2x_{2n}, z, t), S(BFx_{2n-1}, F^2x_{2n-1}, z, t), S(E^2x_{2n}, F^2x_{2n-1}, z, t),$$

$$= \frac{S(AEx_{2n}, E^2x_{2n}, z, 2t)S(BFx_{2n-1}, F^2x_{2n-1}, z, 2t)}{S(E^2x_{2n}, F^2x_{2n-1}, z, t)}, \frac{S(AEx_{2n}, E^2x_{2n}, z, 2t)}{S(E^2x_{2n}, F^2x_{2n-1}, z, t)},$$

$$\frac{S(AEx_{2n}, F^2x_{2n-1}, z, t) + S(BFx_{2n-1}, E^2x_{2n}, z, t)}{2}\}.$$

Taking limit as $n \to \infty$ and using (3.6) and (3.7) we get

$$\int_0^{S(Ez_1Fz_1,z,kt)} \varphi(t)dt \geq \int_0^{S(Ez_1,Fz_1,z,t)} \varphi(t)dt. \tag{3.8}$$

Which implies that

$$Ez_1 = Fz_1. \tag{3.9}$$

Now

$$\int_0^{S(Az_1, BFx_{2n-1}, z, kt)} \varphi(t)dt \geq \int_0^{J_4} \varphi(t)dt. \tag{3.10}$$

Where

$$\begin{aligned}J_4 = \min\{&S(Az_1, Ez_1, z, t), S(BFx_{2n-1}, F^2x_{2n-1}, z, t), S(Ez_1, F^2x_{2n-1}, t),\\ &\frac{S(Az_1, Ez_1, z, 2t)S(BFx_{2n-1}, F^2x_{2n-1}, z, 2t)}{S(Ez_1, F^2x_{2n-1}, z, t)}, \frac{S(AEz_1, Ez_1, z, 2t)}{S(Ez_1, F^2x_{2n-1}, z, t)},\\ &\frac{S(AEz_1, F^2x_{2n-1}, z, t) + S(BFx_{2n-1}, Ez_1, z, t)}{2}\}.\end{aligned}$$

Taking the limit as $n \to \infty$ and using (3.7) and (3.9) we get

$$\int_0^{S(Az, Fz_1, z, kt)} \varphi(t)dt \geq \int_0^{S(Az, Fz_1, z, t)} \varphi(t)dt.$$

Which implies that

$$Az_1 = Fz_1 \tag{3.11}$$

$$\int_0^{S(Az_1, Bz_1, z, kt)} \varphi(t)dt \geq \int_0^{J_5} \varphi(t)dt.$$

Where

$$\begin{aligned}J_5 = \min\{&S(Az_1, Ez_1, z, t), S(Bz_1, Fz_1, z, t), S(Ez_1, Fz_1, z, t),\\ &\frac{S(Az_1, Ez_1, z, 2t)S(Bz_1, Fz_1, z, 2t)}{S(Ez_1, Fz_1, t)}, \frac{S(AEz_1, Ez_1, z, 2t)}{S(Ez_1, Fz_1, z, t)},\\ &\frac{S(AEz_1, Fz_1, z, t) + S(Bz_1, Ez_1, z, t)}{2}\}.\end{aligned}$$

$$\int_0^{S(Az_1, Bz_1, z, kt)} \varphi(t)dt \geq \int_0^{J_6} \varphi(t)dt.$$

Where

$$\begin{aligned}J_6 \min\{&S(Fz_1, Ez_1, z, t), S(Bz_1, Az_1, z, t), S(Ez_1, Fz_1, z, t),\\ = &\frac{S(Az_1, Ez_1, z, 2t)S(Bz_1, Fz_1, z, 2t)}{S(Ez_1, Fz_1, z, t)}, \frac{S(AEz_1, Ez_1, z, 2t)}{S(Ez_1, Fz_1, z, t)},\\ &\frac{S(AEz_1, Fz_1, z, t) + S(Bz_1, Ez_1, a, t)}{2}\}.\end{aligned}$$

Which implies that

$$Az_1 = Bz_1. \tag{3.12}$$

Using (3.9) and (3.12) we get

$$Az_1 = Bz_1 = Ez_1 = Fz_1 \tag{3.13}$$

Now $\int_0^{S(Ax_{2n},Bz_1,z,kt)} \varphi(t)dt \geq \int_0^{J_7} \varphi(t)dt$. Where

$$\begin{aligned} J_7 \min\{&S(Ax_{2n}, Ex_{2n}, z, t), S(Bz_1, Fz_1, z, t), S(Ex_{2n}, Fz_1, z, t),\\ =& \frac{S(Ax_{2n}x_{2n}, z, 2t)S(Bz_1, Fz_1, z, 2t)}{S(Ex_{2n}, Fz_1, z, t)}, \frac{S(Ax_{2n}, Ex_{2n}, z, 2t)}{S(Ex_{2n}, Fz_1, z, t)},\\ &\frac{S(Ax_{2n}, Fz_1, z, t) + S(Bz, Ex_{2n}, z, t)}{2}\} \geq \int_0^{J_8} \varphi(t)d. \end{aligned}$$

Where

$$\begin{aligned} J_8 = \min\{&S(Ex_{2n}, Fz_1, z, t), S(Ax_{2n}, Ex_{2n}, z, t), S(Bz_1, Fz_1, z, t),\\ &\frac{S(Ax_{2n}, Ex_{2n}, z, 2t)S(Bz_1, Fz_1, z, 2t)}{S(Ex_{2n}, Fz_1, z, t)}\\ &\frac{S(Ax_{2n}, Ex_{2n}, z, 2t)}{S(Ex_{2n}, Fz_1, z, t)}, \frac{S(Ax_(2n,)Fz_1, z, t) + S(Bz_1, Ex_{2n}, z, t)}{2}\} \end{aligned}$$

Taking limit as $n \to \infty$ and using (3.13) we get

$$\int_0^{S(z_1,Bz_1,z,kt)} \varphi(t)dt \geq \int_0^{S(z_1,Bz_1,z,t)} \varphi(t)dt.$$

Which implies that

$$z_1 = Bz_1. \tag{3.14}$$

Thus z_1 is a common fixed point of A, B, E and F. For the uniqueness let w be another common fixed point of said mappings. Then from (3.2)

$$\int_0^{S(Az_1,Bw,z,kt)} \varphi(t)dt \geq \int_0^{J_9} \varphi(t)dt.$$

Where

$$\begin{aligned} J_9 = \min\{&S(Az_1, Ez_1, z, t), S(Bw, Fw, z, t), S(Ez, Fw, z, t),\\ &\frac{S(Az_1, Ez_1, z, 2t)S(Bw, Fw, z, 2t)}{S(Ez, Fw, t)}, \frac{S(Az_1, Ez_1, z, 2t)}{S(Ez, Fw, t)},\\ &\frac{S(Az_1, Fw, z, t) + S(Bw, Ez_1, z, t)}{2}\} \end{aligned}$$

i.e. $\int_0^{(z_1,w,z,kt)} \varphi(t)dt \geq \int_0^{S(z_1,w,z,t)} \varphi(t)dt$.

Hence $z_1 = w$. This completes the proof. □

Theorem 3.2. *Let* p, q, f *and* g *be self mappings of complete* S *fuzzy metric space* $(X, S, *)$ *with continuous* t*-norm* $*$ *defined by* $a * b * c = \min\{(a, b, c) : a, b, c \in [0, 1]\}$ *satisfying the following conditions*

(3.15) *Pairs* $\{p, f\}$ *and* $\{q, g\}$ *be oscillatory weakly commutative.*

(3.16) *For all* $x, y, z \in X, t > 0, h \in [0, 1]$

$$\int_0^{S(px,qy,z,ht)} \varphi(t)dt \geq \int_0^{J} \varphi(t)dt.$$

Where $J = \min\{S(fx, gy, z, t), S(fx, px, z, t), S(qy, gy, z, t), [S(px, gy, z, t)+ S(fx, px, z, t)]/S(fx, px, z, t), [S(qy, fx, z, t)+S(qy, gy, z, t)]/S(qy, gy, z, t)\}$*. For all* $x, y, z \in X$, $\lim S(x, y, z, t) \to 1$ *as* $t \to \infty$*. And* $\emptyset \in \Phi$*, for all* $0 < t < 1$*, then there exists a unique point* $w \in X$ *such that* $pw = fw = w$ *and a unique point* $z_1 \in X$ *such that* $qz_1 = gz_1 = z_1$*. Moreover,* $z_1 = w$*, so that there is a unique common fixed point of* p, f, q *and* g*.*

Proof. Let the pairs $\{p, f\}$ and $\{q, g\}$ be owc, so there are points $x, y, z \in X$ such that $px = fx$ and $qy = gy$. We claim that $px = qy$. If not, by inequality (3.16)

$$\int_0^{S(px,qy,z,ht)} \varphi(t)dt \geq \int_0^{J} \varphi(t)dt$$

Where $J = \min\{S(fx, gy, z, t), S(fx, px, z, t), S(qy, gy, z, t), [S(px, gy, z, t)+ S(fx, px, z, t)]/S(fx, px, z, t), [S(qy, fx, z, t) +S(qy, gy, z, t)]/S(qy, gy, z, t)\}$

$$\int_0^{S(px,qy,z,ht)} \phi(t)dt \geq \int_0^{J_1} \phi(t)dt,$$

where $J_1 = \min\{S(px, qyz, t), S(px, px, z, t), S(qy, qy, z, t), [S(px, qy, z, t)+ S(px, px, z, t)].S(px, px, z, t), [S(qy, px, z, t) + S(qy, qy, z, t)]/S(qy, qy, z, t)\}$

$$\int_0^{S(px,qy,z,ht)} \phi(t)dt \geq \int_0^{J_2} \phi(t)dt,$$

where $J_2 = S(px, qy, z, t)$. Therefore $px = qy$, i.e. $px = fx = qy = gy$. Suppose that there is another point z such that $pz = fz$ then by (3.15) we have $pz = fz = qy = gy$, so $px = pz$ and $w = px = fx$ is the unique point of coincidence of p and f. By Lemma 2.4 w is the only common fixed point of p and f. Similarly there is a unique point $z_1 \in X$ such that $z_1 = qz_1 = gz_1$. Assume that $w \neq zz_1$. We have

$$\int_0^{S(w,z_1,z,ht)} \phi(t)dt = \int_0^{S(pw,fz_1,z,ht)} \phi(t)dt \geq \int_0^{S(w,z_1,z,ht)} \phi(t)dt.$$

$$J_3 = \min\{S(fw, gz_1, z, t), S(fw, pw, z, t), S(qz_1, gz_1, z, t), [S(pw, gz_1, z, t) + S(fw, pw, z, t)]/S(fw, pw, z, t), [S(qz_1, fw, z, t) + S(qz_1, gz_1, z, t)]/S(qz_1, gz_1, z, t)\}$$

$$\int_0^{S(w,z_1,z,ht)} \phi(t)dt \geq \int_0^{S(w,z_1,z,ht)} \phi(t)dt.$$

Therefore we have $z_1 = w$. Hence z_1 is a common fixed point of p, f, q and g. The uniqueness of the fixed point holds from (3.15). □

Theorem 3.3. *Let complete S fuzzy metric space $(X, S, *)$ with continuous t-norm $*$ defined by $a*b*c = \min\{(a, b, c) : a, b, c \in [0, 1]\}$ satisfying the following conditions*

(i) R and T be continuous mappings of X;

(ii) if there exists continuous mappings A of X into $R(X) \cap T(X)$ which weakly compatible with R and T and

$$\int_0^{S(Ax,Ay,z,ht)} \varphi(t)dt \geq \int_0^{J} \varphi(t)dt \tag{3.17}$$

where

$$J = \min\{S(Ty, Ay, z, t), S(Rx, Ax, z, t), S(Rx, Ty, z, t), \frac{S(Rx, Ty, z, t)}{S(Ax, Ty, z, t)}, \frac{S(Ty, Ay, z, t)}{S(Rx, Ax, z, t)}, \frac{S(Rx, Ax, z, t)}{S(Ty, Ay, z, t)}\},$$

for all $x, y, z \in X$, $t > 0$, and $0 < h < 1$. Then R, T and A have a unique common fixed point.

Proof. We define a sequence $\{x_n\}$ such that $Ax_{2n} = Sx_{2n-1}$ and $Ax_{2n-1} = Tx_{2n}$, $n = 1, 2, \ldots$. We shall prove that $\{Ax_n\}$ is a Cauchy sequence. For this suppose $x = x_{2n}$ and $y = x_{2n+1}$ in (3.17), we write

$$\int_0^{S(Ax_{2n},Ax_{2n+1},z,ht)} \varphi(t)dt \geq \int_0^{J_1} \varphi(t)dt$$

where

$$J_1 = \min\{S(Tx_{2n+1}, Ax_{2n+1}, z, t), S(Rx_{2n}, Ax_{2n}, z, t), S(Rx_2, Tx_{2n+1}, z, t), \frac{S(Rx_{2n}, Tx_{2n+1}, z, t)}{S(Ax_{2n}, Tx_{2n+1}, z, t)}, \frac{S(Tx_{2n+1}, Ax_{2n+1}, z, t)}{S(Rx_{2n}, Ax_{2n}, z, t)}, \frac{S(Rx_{2n}, Ax_{2n}, z, t)}{S(Tx_{2n+1}, Ax_{2n+1}, z, t)}$$

$$\int_0^{S(Ax_{2n},Ax_{2n+1},z,ht)} \varphi(t)dt \geq \int_0^{J_2} \varphi(t)dt$$

where

$$J_2 = \min\{S(Ax_{2n}, Ax_{2n+1}, z, t), S(Ax_{2n+1}, Ax_{2n}, z, t), S(Ax_{2n+1}, Ax_{2n}, z, t), \frac{S(Ax_{2n+1}, Tx_{2n}, z, t)}{S(Ax_{2n}, A_{2n}, z, t)}, \frac{S(Ax_{2n}, Ax_{2n+1}, z, t)}{S(Ax_{2n+1}, Ax_{2n}, z, t)}, \frac{S(Ax_{2n+1}, Ax_{2n}, z, t)}{S(Ax_{2n}, Ax_{2n+1}, z, t)}.$$

Now

$$\begin{aligned} J_2 &= \min\{S(Ax_{2n}, Ax_{2n+1}, z, t), S(Ax_{2n+1}, Ax_{2n}, z, t), \\ &\quad S(Ax_{2n+1}, Ax_{2n}, z, t), 1, 1, 1\} \\ &\geq \min\{S(Ax_{2n-1}, Ax_{2n}, z, \frac{t}{h}), S(Ax_{2n}, Ax_{2n-1}, z, \frac{t}{h})\}. \end{aligned}$$

Thus

$$\int_0^{S(Ax_{2n}, Ax_{2n+1}, z, ht)} \varphi(t)dt \geq \int_0^{S(Ax_{2n-1}, Ax_{2n}, z, \frac{t}{h})} \varphi(t)dt.$$

By induction

$$\int_0^{S(Ax_{2k}, Ax_{2m+1}, z, ht)} \varphi(t)dt \geq \int_0^{S(Ax_{2m}, Ax_{2k-1}, z, \frac{t}{h})} \varphi(t)dt.$$

For every k and m in N, Further if $2m+1 > 2k$, then

$$\begin{aligned} \int_0^{S(Ax_{2k}, Ax_{2m+1}, z, ht)} \varphi(t)dt &\geq \int_0^{S(Ax_{2k-1}, Ax_{2m}, z, \frac{t}{h})} \varphi(t)dt \\ \geq \dots \geq \int_0^{S(Ax_0, Ax_{2m+1-2k}, z, \frac{t}{h^{2m+1}})} \varphi(t)dt. \end{aligned}$$

If $2k > 2m+1$, then

$$\begin{aligned} \int_0^{S(Ax_{2k}, Ax_{2m+1}, z, ht)} \varphi(t)dt &\geq \int_0^{S(Ax_{2k-1}, Ax_{2m}, z, \frac{t}{h})} \varphi(t)dt \\ \geq \dots \geq \int_0^{S(Ax_{2k-(2m+1)}, Ax_0, z, \frac{t}{h^{2m+1}})} \varphi(t)dt. \end{aligned}$$

By simple induction of above two equation we have

$$\int_0^{S(Ax_n, Ax_{n+p}, z, ht)} \varphi(t)dt \geq \int_0^{S(Ax_0, Ax_p, z, \frac{t}{h^n})} \varphi(t)dt.$$

For $n = 2k, p = 2m+1$ or $n = 2k+1$, $p = 2m+1$

$$S(Ax_n, Ax_{n+p}, z, ht) \geq S(Ax_0, Ax_1, z, \frac{t}{2h^n}) * S(Ax_1, Ax_p, z, \frac{t}{h^n}). \tag{3.18}$$

If $n = 2k, p = 2m$ or $n = 2k+1$, $p = 2m$. For every positive integer p and n in N, by nothing that

$$S(Ax_0, Ax_p, z, \frac{t}{h^n}) \to 1 \text{ as } n \to \infty.$$

Thus $\{Ax_n\}$ is a Cauchy sequence. Since the space X is complete there exists $z_1 \in X$, such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Rx_{2n-1} = \lim_{n\to\infty} Tx_{2n} = z_1$. It follows that $Az_1 = Rz_1 = Tz_1$ and by weakly compatibility therefore

$$\int_0^{S(Az_1,AA_{z_1},z,ht)} \varphi(t)dt \geq \int_0^{J_3} \varphi(t)dt$$

where

$$\begin{aligned} J_3 = \min\{&S(TAz_1, AAz_1, z, t), S(Rz_1, Az_1, z, t), S(Az_1, TAz_1, z, t), \\ &\frac{S(Rz_1, TAz_1, z, t)}{S(Az_1, TA_1, z, t)}, \frac{S(TAz_1, AAz_{z1}, z, t)}{S(Rz_1, Az_1, z, t)}, \frac{S(Rz_1, Az_1, z, t)}{S(TAz_1, AAz_1, z, t)}. \end{aligned} \tag{3.19}$$

$$\begin{aligned} &\int_0^{S(Az_1,AAz_1,z,ht)} \varphi(t)dt \geq \int_0^{S(Rz_1,TAz_1,z,t)} \varphi(t)dt \\ &\geq \int_0^{S(Rz_1,ATz_1,z,t)} \varphi(t)dt \geq \int_0^{S(Az_1,A^2,z_1,z,t)} \varphi(t)dt \\ &\geq \ldots \geq \int_0^{S(Az_1,A^2z_1,z,\frac{t}{h^n})} \varphi(t)dt. \end{aligned}$$

Since $\lim_{n\to\infty} S(Az_1, A^2z_1, z, \frac{t}{h^n}) = 1 \Rightarrow Az_1 = A^2z_1$.

Thus z_1 is common fixed point of A, R and T. For uniqueness, let $w(w \neq z_1)$ be another common fixed point of R, T and A. By (3.17), we write

$$\int_0^{S(Az_1,Aw,z,ht)} \varphi(t)dt \geq \int_0^{J_4} \varphi(t)dt$$

where

$$\begin{aligned} J_4 = \min\{&S(Tw, Aw, z, t), S(Rz_1, Az_1, z, t), S(Rz_1, Tw, z, t), \\ &\frac{S(Rz_1, Tw, z, t)}{S(Az_1, Tw, z, t)}, \frac{S(Tz_1, Aw, z, t)}{S(Rz_1, Az_1, z, t)}, \frac{S(Rz_1, Az_1, z, t)}{S(Tw, Aw, z, t)} \end{aligned}$$

$$\int_0^{S(Az_1,Aw,z,ht)} \varphi(t)dt \geq \int_0^{S(z_1,w,z,t)} \varphi(t)dt.$$

This implies that

$$\int_0^{S(z_1,w,z,ht)} \varphi(t)dt \geq \int_0^{S(z_1,w,z,t)} \varphi(t)dt.$$

Therefore by Lemma 2.4, we write $z_1 = w$. This complete the proof of Theorem (3.3). □

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