# REPRESENTATION OF GRAPHS USING $m$-POLAR FUZZY ENVIRONMENT

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**Abstract.** In this research article, we introduce the concepts of products in $m-$polar fuzzy graphs and investigate some of their interesting properties. We describe various properties of certain $m-$polar fuzzy graphs. We establish formulae of distance in complete $m-$polar fuzzy graphs and complete bipartite $m-$polar fuzzy graphs. We present an algorithm for computing the distance matrix, eccentricity of the vertices, radius and diameter in $m-$polar fuzzy graphs. We also discuss applications of $m-$polar fuzzy graphs in traveling and product manufacturing.
**Keywords:** $m-$polar fuzzy graphs, algorithm, eccentricity, central vertices, peripheral vertices, decision support systems.

## 1. Introduction

A fuzzy set [21] is an important mathematical structure to represent a collection of objects whose boundary is vague. Fuzzy models are becoming useful because of their aim in reducing the differences between the traditional numerical models used in engineering and sciences and the symbolic models used in expert systems. In 1994, Zhang [23] introduced the notion of bipolar fuzzy sets and relations. Bipolar fuzzy sets are extension of fuzzy sets whose membership degree ranges $[-1, 1]$. The membership degree $(0, 1]$ indicates that the object satisfies a certain property whereas the membership degree $[-1, 0)$ indicates that the object satisfies the counter property. Positive information represent what is considered to be possible and negative information represent what is granted to be impossible. Actually, a variety of decision making problems are based on two-sided bipolar thinking and judgements on a positive side and a negative side. Recently, Chen et al. [9] generalized the idea of bipolar fuzzy sets to $m-$polar fuzzy sets. In an $m-$polar fuzzy set, the membership value ranges over $[0, 1]^m$. In lots of real World problems, data are sometimes come from $n$ agents($n \geq 2$), that is, multipolar information exist which cannot be represented

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well by means of the existing graphs such as fuzzy graphs(correspond to single valued logic), bipolar fuzzy graphs(correspond to two valued logic), etc. Considering that graphic structures, $m-$polar fuzzy sets can be used to describe the relationship among several individuals. $m-$polar fuzzy sets have many applications in decision making problems when it is necessary to make judgements with a group of agreements. For example, weighted games, a country elects its leader or a group of friends wants to plan to visit a country.

Based on Zadeh's fuzzy relations [22] Kaufmann defined in [11] a fuzzy graph. The fuzzy relations between fuzzy sets were also considered by Rosenfeld [18] and he developed the structure of fuzzy graphs, obtaining analogs of several graph theoretical concepts. Later on, Bhattacharya [8] gave some remarks on fuzzy graphs, and some operations on fuzzy graphs were introduced by Mordeson and Peng [13]. Tom and Sunitha [20] introduced the concept of sum distance in fuzzy graphs and studied some of its properties. Akram *et al.* [1-7] introduced many new concepts, including bipolar fuzzy graphs, certain notions of $m$-polar fuzzy graphs, $m-$polar fuzzy competition graphs and $m-$polar fuzzy hypergraphs. In this research article, we introduce the concepts of products in $m-$polar fuzzy graphs and investigate some of their interesting properties. We establish formulae of distance in complete $m-$polar fuzzy graphs, complete bipartite $m-$polar fuzzy graphs and certain products of $m-$polar fuzzy graphs. We present an algorithm for computing the distance matrix, eccentricity of the vertices, radius and diameter in $m-$polar fuzzy graphs. We also discuss applications of $m-$polar fuzzy graphs in traveling and product manufacturing.

## 2. Representation of graphs using $m$-polar fuzzy environment

**Definition 2.1.** [6] Let $C$ be an $m-$polar fuzzy set on a non-empty crisp set $X$. An *$m-$polar fuzzy relation* is an $m-$polar fuzzy subset $D = (P_1 \circ D, P_2 \circ D, \ldots, P_m \circ D)$ of $X \times X$ such that $D(xy) \leq \inf\{C(x), C(y)\}$, for all $x, y \in X$, that is, for all $x, y \in X$ and for each $1 \leq i \leq m$, $P_i \circ D(xy) \leq \inf\{P_i \circ C(x), P_i \circ C(y)\}$, where $P_i \circ C(x)$ denotes the $i$th degree of membership of the vertex $x$ and $P_i \circ D(xy)$ denotes the $i$th degree of membership of the edge $xy$.

**Definition 2.2.** [6, 9] An *$m$-polar fuzzy graph* on a non-empty $X$ is a pair $G = (C, D)$ where, $C : X \to [0, 1]^m$ is an $m$-polar fuzzy set on the set of vertices $X$ and $D : X \times X \to [0, 1]^m$ is an $m$-polar relation such that $D(xy) \leq \inf\{C(x), C(y)\}$, for all $xy \in E$, and $D(xy) = \mathbf{0}$, for all $xy \in X \times X - E$ where, $\mathbf{0} = (0, 0, \ldots, 0)$ and $E \subseteq X \times X$ is the set of edges.

Throughout this paper, we use $G^*$ as a crisp graph and $G$ as an $m-$polar fuzzy graph.

**Definition 2.3.** An *$m$-polar fuzzy walk* in an $m$-polar fuzzy graph is an alternating sequence of vertices and edges $y_0$ ,$e_1$, $y_1$ ,$e_2$, $\ldots$ ,$e_{n-1}$, $y_{n-1}$ such that $P_i \circ C(y_{j-1}) > 0$ and $P_i \circ D(e_j) > 0$, for all $1 \leq j \leq n$, for at least one $i$.

**Definition 2.4.** An *$m$-polar fuzzy path* in an $m$-polar fuzzy graph is a sequence of distinct vertices $x_1$, $x_2$, … , $x_n$ such that $P_i \circ D(x_j x_{j+1}) > 0$, for all $1 \leq j \leq n-1$, for at least one $i$. It is denoted by $\widetilde{P}_n$. The graph of $\widetilde{P}_5$ is shown in Figure. 2.4.

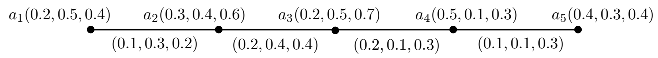



Figure 1: $\widetilde{P}_5$

If $x_1 = x_n$, the $m$-polar fuzzy path is known as an *$m$-polar fuzzy cycle*, denoted by $\widetilde{C}_n$.

**Definition 2.5.** The degree of a vertex $x$ in an $m$-polar fuzzy graph $G = (C, D)$ is denoted by the $m$-tuple,

$$deg(x) = (deg^{(1)}(x), deg^{(2)}(x), \dots, deg^{(m)}(x)),$$

that is, $deg(x) = (\sum_{xx_j \in E} P_1 \circ D(xx_j), \sum_{xx_j \in E} P_2 \circ D(xx_j), \dots, \sum_{xx_j \in E} P_m \circ D(xx_j))$.

If all vertices of $G$ have same degree, then $G$ is known as a *regular $m$-polar fuzzy graph*.

**Definition 2.6.** An $m$-polar fuzzy graph is known as a *complete $m$-polar fuzzy graph* if $P_i \circ D(xy) = P_i \circ C(x) \wedge P_i \circ C(y)$, for all $x, y \in X$, $1 \leq i \leq m$.

**Definition 2.7.** An $m$-polar fuzzy graph is known as *bipartite $m$-polar fuzzy graph* if the set of vertices $X$ can be written as the union of two disjoint sets $X_1$ and $X_2$ such that, for some $k$ and $j$,

1. $P_i \circ D(x_k x_j) = 0$, if $x_k, x_j \in X_1$ or $x_k, x_j \in X_2$, for all $1 \leq i \leq m$,
2. $P_i \circ D(x_k x_j) > 0$, if $x_k \in X_1$ and $x_j \in X_2$ or $x_k \in X_2$ and $x_j \in X_1$, for at least one $i$.

**Example 2.1.** Let $C$ be a $3-$polar fuzzy set on $X = \{a_1, a_2, a_3\} \cup \{b_1, b_2, b_3\}$ and $D$ be a $3-$polar fuzzy relation in $X$. The bipartite $3-$polar fuzzy graph $G$ is shown in Fig. 2.1.

**Definition 2.8.** An $m$-polar fuzzy graph is called *complete bipartite $m-$polar fuzzy graph* if the set of vertices $X$ can be written as the union of two disjoint sets $X_1$ and $X_2$ such that, for all $k$ and $j$,

1. $P_i \circ D(x_k x_j) = 0$, if $x_k, x_j \in X_1$ or $x_k, x_j \in X_2$, for all $1 \leq i \leq m$,
2. $P_i \circ D(x_k x_j) = P_i \circ C(x_k) \wedge P_i \circ C(x_j)$, if $x_k \in X_1$ and $x_j \in X_2$ or $x_k \in X_2$ and $x_j \in X_1$, for at least one $i$.

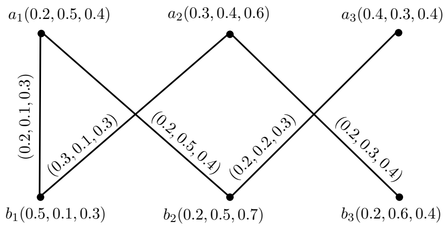
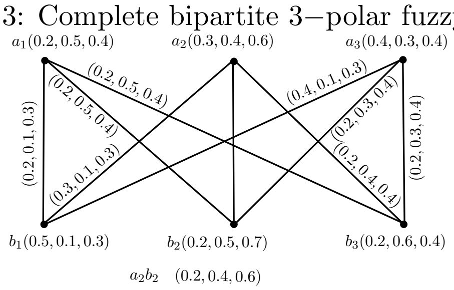



Figure 2: Bipartite 3−polar fuzzy graph

Figure 3: Complete bipartite 3−polar fuzzy graph




**Example 2.2.** An example of complete bipartite 3−polar fuzzy graph on the crisp graph $G^* = (X, E)$ where $X = \{a_1, a_2, a_3\} \cup \{b_1, b_2, b_3\}$, is shown in Fig. 3.

**Definition 2.9.** An $m$-polar fuzzy graph $H = (C^{'}, D^{'})$ is called an *m-polar fuzzy subgraph* of $m$-polar fuzzy graph $G = (C, D)$ if, $C^{'} \subseteq C$ and $D^{'} \subseteq D$.

**Definition 2.10.** Let $G = (C, D)$ be an $m$-polar fuzzy graph. The $P_i$*- strength* of an $m$-polar fuzzy path $x_1 - x_2 - \ldots - x_n$ is defined as, $S_{P_i}(x_1, x_n) = \inf\{P_i \circ D(x_k x_{k+1}) : 1 \leq k \leq n-1\}$. The *strength* of $m$-polar fuzzy path $x_1 - x_n$ is computed as, $S(x_1, x_n) = (S_{P_1}(x_1, x_n), S_{P_2}(x_1, x_n), \ldots, S_{P_m}(x_1, x_n))$. A strongest path between any two vertices is the path with supremum strength. The strength of the strongest path $x - y$ is defined as the $m$-tuple $P^{\infty}(x, y) = (P_1^{\infty}(x, y), P_2^{\infty}(x, y), \ldots, P_m^{\infty}(x, y))$, such that for all $x, y \in X$ and $1 \leq i \leq m$, $P_i^{\infty}(x, y) = \sup\{S_{P_i}(x, y),\ \ x - y$ is an $m$-polar fuzzy path in G$\}$. It is referred as *strength of connectedness* between $x$ and $y$.

**Example 2.3.** Consider a 3-polar fuzzy graph as shown in Fig. 2.3. The strength of the path $b - a - c$ is $(0.3 \wedge 0.1, 0.2 \wedge 0.3, 0.1 \wedge 0.0) = (0.1, 0.2, 0)$ and that of $b - d - c$ is $(0.2 \wedge 0.2, 0.3 \wedge 0.4, 0.1 \wedge 0.1) = (0.2, 0.3, 0.1)$. The strength of connectedness between the vertices $b$ and $c$ is $(0.2, 0.3, 0.1)$.

**Definition 2.11.** For any $m$-polar fuzzy path, $R$: $x_1 - x_2 - ... - x_n$, the $P_i$−length of $R$ is defined as the sum of $P_i \circ D$ values of the edges, that is, $L_i(R) = \sum\limits_{j=2}^{n} P_i \circ D(x_{j-1}x_j)$, for all $1 \leq i \leq m$. The *length* of $m$−polar fuzzy path $R$ is represented by the $m$-tuple $L(R) = (L_1(R), L_2(R), \ldots, L_m(R))$. For any two vertices $x, y$ of

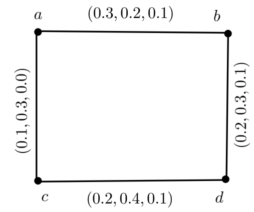



Figure 4: 3−polar fuzzy graph

$G$, let $\mathbb{R} = \{R_i,\ R_i$ is an $x-y$ $m$-polar fuzzy path, $i = 1, 2, 3, ...\}$, be the set of all $m$-polar fuzzy paths from $x$ to $y$. Then, $P_i-$distance of the path $x-y$, denoted by $d_i(x, y)$ and is defined as, $d_i(x, y) = \inf\{L_i(R_j) : R_j \in \mathbb{R}, j = 1, 2, 3, ...\}$, for all $1 \leq i \leq m$. The *distance* of $m-$polar fuzzy path $x - y$, denoted by $d(x, y)$ or $d_G(x, y)$, is defined as the $m$-tuple $d(x, y) = (d_1(x, y), d_2(x, y), \ldots, d_m(x, y))$ or $d_G(x, y) = (d_{1,G}(x, y), d_{2,G}(x, y), \ldots, d_{m,G}(x, y))$.

**Definition 2.12.** Let $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ be two $m-$polar fuzzy graphs on $X_1$ and $X_2$, respectively. The *Cartesian product* of $G_1$ and $G_2$ is denoted by $G_1 \Box G_2$ and defined as a pair $(C_1 \Box C_2, D_1 \Box D_2)$, such that for each $1 \leq i \leq m$,

1. $P_i \circ (C_1 \Box C_2)(x_1, x_2) = P_i \circ C_1(x_1) \wedge P_i \circ C_2(x_2)$, for all $(x_1, x_2) \in X_1 \times X_2$,
2. $P_i \circ (D_1 \Box D_2)((x_1, x_2)(x_1, y_2)) = P_i \circ C_1(x_1) \wedge P_i \circ D_2(x_2 y_2)$, for all $x_1 \in X_1$ and $x_2 y_2 \in E_2$,
3. $P_i \circ (D_1 \Box D_2)((x_1, x_2)(y_1, x_2)) = P_i \circ D_1(x_1 y_1) \wedge P_i \circ C_2(x_2)$, for all $x_2 \in X_2$ and $x_1 y_1 \in E_1$.

**Example 2.4.** The Cartesian product of two 3−polar fuzzy paths is shown in Fig. 2.4.

**Theorem 2.1.** *Let $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ be two $m$-polar fuzzy graphs. If $(x_1, x_2)$ and $(y_1, y_2)$ are vertices of the Cartesian product $G_1 \Box G_2$, then $d_{G_1 \Box G_2}((x_1, x_2), (y_1, y_2)) \leq d_{G_1}(x_1, y_1) + d_{G_2}(x_2, y_2)$.*

**Proof.** Assume that $d_{G_1}(x_1, y_1)$ and $d_{G_2}(x_2, y_2)$ are finite. Let $R_2, R_2, \ldots, R_m$ be the $m-$polar fuzzy paths in $G_1$ and $Q_1, Q_2, \ldots, Q_m$ are $m-$polar fuzzy paths in $G_2$ where, $R_i =\ x_{i1}, x_{i2}, \ldots, x_{in} = y_1$ such that,

$$d_{G_1}(x_1, y_1) = (L_1(R_1), L_2(R_2), \ldots, L_m(R_m)),$$

and $Q_i : x_2 = y_{i1}, y_{i2}, \ldots, y_{in'} = y_2$ be an $m-$polar fuzzy path in $G_2$ such that,

$$d_{G_2}(x_2, y_2) = (L_1(Q_1), L_2(Q_2), \ldots, L_m(Q_m)).$$

This establishes the following $m-$polar fuzzy paths in $G_1 \Box G_2$,

$$R_i \times \{y_{i1}\} = (x_{i1}, y_{i1}), (x_{i2}, y_{i1}), \ldots, (x_{in}, y_{i1})$$
$$\{x_{in}\} \times Q_i = (x_{in}, y_{i1}), (x_{in}, y_{i2}), \ldots, (x_{in}, y_{in'}), \qquad 1 \leq i \leq m$$

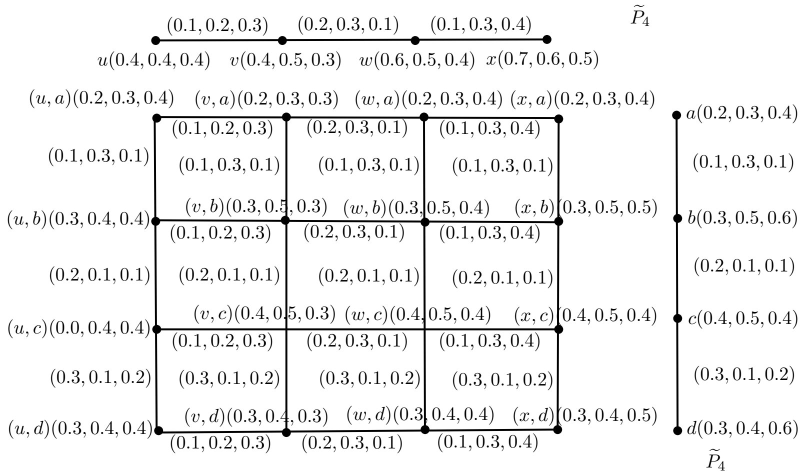



Figure 5: $\widetilde{P}_4 \square \widetilde{P}_4$.

whose join are the $m-$polar fuzzy paths of length $L_i(R_i \times \{x_2\}) + L_i(\{y_1\} \times Q_i)$, $1 \leq i \leq m$. It is clear that,

$$d_{i,G_1 \square G_2}((x_1,x_2),(y_1,y_2)) \leq L_i(R_i \times \{x_2\}) + L_i(\{y_1\} \times Q_i). \tag{2.1}$$

For each $1 \leq i \leq m$,

$$\begin{aligned}
&L_i(R_i \times \{x_2\}) \\
&= P_i \circ D_1(x_{i1}x_{i2}) \wedge P_i \circ C_2(x_2) + P_i \circ D_1(x_{i2}x_{i3}) \\
&\wedge P_i \circ C_2(x_2) + \ldots + P_i \circ D_1(x_{\overline{in-1}}x_{in}) \wedge P_i \circ C_2(x_2), \\
&\leq P_i \circ D_1(x_{i1}x_{i2}) + P_i \circ D_1(x_{i2}x_{i3}) + \ldots + P_i \circ D_1(x_{\overline{in-1}}x_{in}), \\
&= L_i(R_i)
\end{aligned}$$

$$L_i(R_i \times \{x_2\}) \leq L_i(R_i) = d_{i,G_1}(x_1,y_1), \tag{2.2}$$

$$\Rightarrow L_i(R_i \times \{x_2\}) \leq d_{i,G_1}(x_1,y_1). \tag{2.3}$$

By using similar argument, we can prove that

$$L_i(\{y_1\} \times Q_i) \leq L_i(Q_i) = d_{i,G_2}(x_2,y_2). \tag{2.4}$$

From Equation. (2.1), (2.2) and (2.4), we conclude that

$$\begin{aligned}
&d_{i,G_1 \square G_2}((x_1,x_2),(y_1,y_2)) \leq d_{i,G_1}(x_1,y_1) + d_{i,G_2}(x_2,y_2), \quad 1 \leq i \leq m. \\
&\Rightarrow d_{G_1 \square G_2}((x_1,x_2),(y_1,y_2)) \leq d_{G_1}(x_1,y_1) + d_{G_2}(x_2,y_2).
\end{aligned}$$
□

**Definition 2.13.** Let $G_1^* \bullet G_2^* \bullet \cdots \bullet G_k^*$ be any product of the graphs $G_1^*$, $G_2^*, \cdots$, $G_k^*$ where, $\bullet$ represents any product, Cartesian product, direct product, strong product or lexicographic product. The mapping $f_{G_i} : G_1^* \bullet G_2^* \bullet \cdots \bullet G_k^* \to G_i^*$, defined by

$$f_{G_i}(x_1, x_2, \cdots, x_k) = x_i, \qquad x_i \in V_i, 1 \leq i \leq k,$$

is called the projection of $G_i^*$ onto $G_1^* \bullet G_2^* \bullet \cdots \bullet G_k^*$.

**Theorem 2.2.** *Let $S$ be an $m-$polar fuzzy path in $G_1\Box G_2$ and for all $1 \leq i, j \leq m$, $P_i \circ C_1 \geq P_j \circ D_2$ and $P_i \circ C_2 \geq P_j \circ D_1$ then, $L(S) = L(f_{G_1}(S)) + L(f_{G_2}(S))$.*

**Proof.** Let $P$ : $x_1, x_1, \ldots, x_n$ be an $m-$polar fuzzy path in $G_1$ and $Q$ : $y_1, y_2, \ldots, y_{n'}$ be an $m-$polar fuzzy path in $G_2$. Let $S$ be a path in $G_1\Box G_2$ which is established as follows,

$$S = (x_1, y_1), (x_2, y_1), \ldots, (x_n, y_1), (x_n, y_1), (x_n, y_2), \ldots, (x_n, y_{n'}),$$

Clearly, $f_{G_1}(S) = P$, $f_{G_2}(S) = Q$ and $L(S) = (L_1(S), L_2(S), \ldots, L_m(S))$. It follows that,

$$\begin{aligned} L_i(S) &= P_i \circ D_1(x_1x_2) \wedge P_i \circ C_2(y_1) + P_i \circ D_1(x_2x_3) \wedge P_i \circ C_2(y_1) \\ &\quad + \ldots + P_i \circ D_1(x_{n-1}x_n) \wedge P_i \circ C_2(y_1) + P_i \circ C_1(x_n) \wedge P_i \circ D_2(y_1y_2) \\ &\quad + \ldots + P_i \circ C_1(x_n) \wedge P_i \circ D_2(y_{n'-1}y(n')), \\ &= P_i \circ D_1(x_1x_2) + P_i \circ D_1(x_2x_3) + \ldots + P_i \circ D_1(x_{n-1}x_n) \\ &\quad + P_i \circ D_2(y_1y_2) + \ldots + P_i \circ D_2(y_{n'-1}y_{n'}), \\ &= L(P) + L(Q) = L(f_{G_1}(S)) + L(f_{G_2}(S)). \end{aligned}$$ □

**Lemma 2.1.** *Let $G_1$ and $G_2$ be two $m$-polar fuzzy graphs and $(x_1, y_1)$ and $(x_2, y_2)$ are vertices of a Cartesian product $G_1\Box G_2$. If for all $1 \leq i, j \leq m$, $P_i \circ C_1 \geq P_j \circ D_2$ and $P_i \circ C_2 \geq P_j \circ D_1$, then, $d_{G_1\Box G_2}((x_1, x_2), (y_1, y_2)) = d_{G_1}(x_1, y_1) + d_{G_2}(x_2, y_2)$.*

**Proof.** By Theorem. 2.1,

$$d_{G_1\Box G_2}((x_1, x_2), (y_1, y_2)) \leq d_{G_1}(x_1, y_1) + d_{G_2}(x_2, y_2). \tag{2.5}$$

Conversely, let $S_1, S_2, \ldots, S_m$ be the shortest $m-$polar fuzzy paths between vertices $(x_1, x_2)$ and $(y_1, y_2)$ such that $d_{G_1\Box G_2}((x_1, x_2), (y_1, y_2)) = (L_1(S_1), L_2(S_2), \ldots, L_m(S_m))$. The projections $f_{G_1}(S_i)$ and $f_{G_2}(S_i)$, $1 \leq i \leq m$, are the $m-$polar fuzzy paths between the vertices $x_1$ and $y_1$ in $G_1$ and $x_2$ and $y_2$ in $G_2$. Consider,

$$\begin{aligned} & d_{G_1}(x_1, y_1) + d_{G_2}(x_2, y_2) \\ & \leq (L_1(f_{G_1}(S_1)), L_2(f_{G_1}(S_2)), \ldots, L_m(f_{G_1}(S_m))) \\ & + (L_1(f_{G_2}(S_1)), L_2(f_{G_2}(S_2)), \ldots, L_m(f_{G_2}(S_m))), \\ & = (L_1(S_1), L_2(S_2), \ldots, L_m(S_m)) = d_{G_1\Box G_2}((x_1, x_2), (y_1, y_2)). \end{aligned} \tag{2.6}$$

By combining Eqs. (2.5) and (2.6), required result is obtained. □

**Example 2.5.** Consider the Cartesian product of two $m-$polar fuzzy paths in Fig. 2.4. It can be easily seen that , $d_{\widetilde{P}_3\Box\widetilde{P}_3}((u, a), (v, b)) = (0.2, 0.5, 0.4) = d_{\widetilde{P}_3}(u, v) + d_{\widetilde{P}_3}(a, b)$, $d_{\widetilde{P}_3\Box\widetilde{P}_3}((u, a), (u, c)) = (0.3, 0.4, 0.2) = d_{\widetilde{P}_3}(u, u) + d_{\widetilde{P}_3}(a, c)$. Similarly, for the other vertices.

**Theorem 2.3.** *Let $G_1$ and $G_2$ be two $m$-polar fuzzy graphs. If $x_1 \in X_1$, $x_2 \in X_2$, and for all $1 \leq i, j \leq m$, $P_i \circ C_1 \geq P_j \circ D_2$ and $P_i \circ C_2 \geq P_j \circ D_1$, then, $deg_{G_1 \Box G_2}((x_1, x_2) = deg_{G_1}(x_1) + deg_{G_2}(x_2)$.*

**Definition 2.14.** The *direct product* of two $m$-polar fuzzy graphs $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ is denoted by $G_1 \times G_2$ and defined as a pair $(C_1 \times C_2, D_1 \times D_2)$, such that for each $1 \leq i \leq m$,

1. $P_i \circ (C_1 \times C_2)(x_1, x_2) = P_i \circ C_1(x_1) \wedge P_i \circ C_2(x_2)$, for all $(x_1, x_2) \in X_1 \times X_2$,
2. $P_i \circ (D_1 \times D_2)((x_1, x_2)(y_1, y_2)) = P_i \circ D_1(x_1 y_1) \wedge P_i \circ D_2(x_2 y_2)$, for all $x_1 y_1 \in E_1$ and $x_2 y_2 \in E_2$.

**Example 2.6.** The direct product of two 3−polar fuzzy paths is shown in Fig. 2.6.

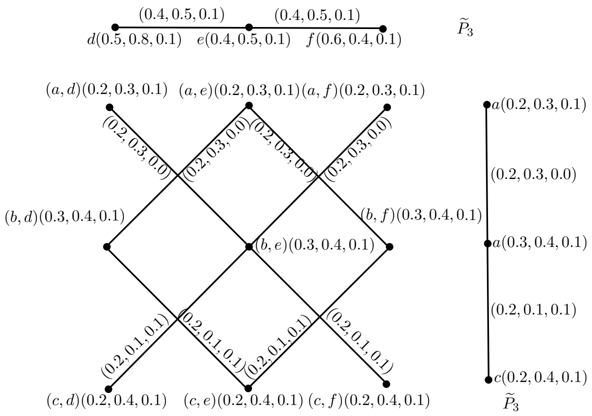



Figure 6: $\widetilde{P}_3 \times \widetilde{P}_3$

**Theorem 2.4.** *The direct product $G_1 \times G_2$ of two $m$-polar fuzzy graphs $G_1$ and $G_2$ is an $m$-polar fuzzy graph.*

**Theorem 2.5.** *Let $G_1$ and $G_2$ be two $m$-polar fuzzy graphs. If $x_1, y_1 \in X_1$, $x_2, y_2 \in X_2$, and for each $1 \leq i, j \leq m$, $P_i \circ D_1 \leq P_j \circ D_2$. Let $R_1, R_2, \ldots, R_m$ be the shortest $m$-polar paths between $(x_1, x_2)$ and $(y_1, y_2)$ such that, $d_{G_1 \times G_2}((x_1, x_2), (y_1, y_2)) = (L_1(R_1, L_2(R_2), \ldots, L_m(R_m)))$ then*

$$d_{G_1 \times G_2}((x_1, x_2), (y_1, y_2)) = (L_1(f_{G_1}(R_1)), L_2(f_{G_1}(R_2)), \ldots, L_m(f_{G_1}(R_m))).$$

**Proof.** Assume that for each $1 \leq i \leq m$, $R_i : (x_1, x_2) = (x_1^{(i1)}, x_2^{(i1)}), (x_1^{(i2)}, x_2^{(i2)}), \ldots, (x_1^{(in)}, x_2^{(in)}) = (y_1, y_2)$ are the shortest $m$-polar fuzzy paths between $(x_1, x_2)$ and $(y_1, y_2)$ and,

$$d_{G_1 \times G_2}((x_1, x_2), (y_1, y_2)) = (L_1(R_1), L_2(R_2), \ldots, L_m(R_m)). \tag{2.7}$$

If $E$ is the set of edges in $G_1 \times G_2$ then for each $1 \leq i \leq m$,

$$\begin{aligned}
&L_i(R_i)\\
&= \sum\nolimits_{(x_1^{(ij)},x_2^{(ij)})(x_1^{(\overline{ij+1})},x_2^{(\overline{ij+1})})\in E} P_i \circ (D_1 \times D_2)(x_1^{(ij)}, x_2^{(ij)})(x_1^{(\overline{ij+1})}, x_2^{(\overline{ij+1})}),\\
(2.8)\quad &= \sum\nolimits_{x_1^{(ij)}x_1^{(\overline{ij+1})}\in E_1, x_2^{(ij)}x_2^{(\overline{ij+1})}\in E_2} P_i \circ D_1(x_1^{(ij)}x_1^{(\overline{ij+1})}) \wedge P_i \circ D_2(x_2^{(ij)}x_2^{(\overline{ij+1})}),\\
&= \sum\nolimits_{x_1^{(j)}x_1^{(j+1)}\in E_1} P_i \circ D_1(x_1^{(ij)}x_1^{(\overline{ij+1})}),\\
&= L_i(f_{G_1}(R_i)).
\end{aligned}$$

From Equation. (2.7) and (2.8),

$$d_{G_1 \times G_2}((x_1, x_2), (y_1, y_2)) = (L_1(f_{G_1}(R_1)), L_2(f_{G_1}(R_2)), \ldots, L_m(f_{G_1}(R_m))). \square$$

**Remark 2.1.** If $P_i \circ D_2 \geq P_i \circ D_1$, and $R_i$, $1 \leq i \leq m$ are the shortest $m$-polar fuzzy paths between the vertices $(x_1, x_2)$ and $(y_1, y_2)$ of $G_1 \times G_2$ then,

$$d_{G_1 \times G_2}((x_1, x_2), (y_1, y_2)) = (L_1(f_{G_2}(R_1)), L_2(f_{G_2}(R_2)), \ldots, L_m(f_{G_2}(R_m))).$$

**Example 2.7.** In Fig. 2.6, the shortest 3−polar fuzzy path between the vertices $(a, d)$ and $(a, f)$ is $S : (a, d) - (b, e) - (a, f)$. $f_{\widetilde{P}_3}(S) = a - b - a = P$. It can be easily seen that $d_{\widetilde{P}_3 \times \widetilde{P}_3}((a, d), (a, f)) = (0.4, 0.6, 0.0) = L(P)$. For the vertices $(a, d)$ and $(c, f)$, the shortest 3−polar fuzzy path is $S : (a, d) - (b, e) - (c, f)$. Therefore, $d_{\widetilde{P}_3 \times \widetilde{P}_3}((a, d), (c, f)) = (0.4, 0.4, 0.1)$. The projection of $S$ in $\widetilde{P}_3$ is $a - b - c$ whose length is equal to $d_{\widetilde{P}_3 \times \widetilde{P}_3}((a, d), (c, f))$.

**Theorem 2.6.** *Let $G_1$ and $G_2$ be two $m$-polar fuzzy graphs such that $P_i \circ D_1 \leq P_j \circ D_2$, for all $1 \leq i, j \leq m$. For any two vertices $(x_1, x_2)$ and $(y_1, y_2)$ of the direct product $G_1 \times G_2$, let $k$ be a smallest positive integer such that $G_1^*$ has a $x_1, y_1$-walk of length $k$ and $G_2^*$ has a $x_2, y_2$-walk of length $k$. The $i$th distance between $(x_1, x_2)$ and $(y_1, y_2)$ is the smallest $P_i$-length of any $m$-polar fuzzy walk between $x_1$ and $y_1$ whose length in the crisp graph $G_1^*$ is $k$.*

**Example 2.8.** The 2−polar fuzzy graph in Fig. 2.8 is the direct product of $\widetilde{P}_3$ and $\widetilde{C}_3$. Take the vertices $(a, d)$ and $(c, e)$. The smallest $a - c$ and $d - e$ walks are of length 2. Therefore, the distance between $(a, d)$ and $(c, e)$ must be the smallest length of a 2−polar fuzzy walk in $\widetilde{P}_3$ whose length in $P_3$ is 2. Such walk in $\widetilde{P}_3 \times \widetilde{C}_3$ is $(a, d) - (b, f) - (c, e)$. Hence, $d_{\widetilde{P}_3 \times \widetilde{C}_3}((a, d), (c, e)) = (0.4, 0.4)$.

**Theorem 2.7.** *Let $G_1$ and $G_2$ be two $m$-polar fuzzy graphs. If $x_1 \in X_1$ and $x_2 \in X_2$, and for all $1 \leq i, j \leq m, P_i \circ D_1 \geq P_j \circ D_2$, then, $deg_{G_1 \times G_2}((x_1, x_2)) =$ (number of vertices adjacent to $x_2$)$deg_{G_1}(x_1)$. If $P_i \circ D_2 \geq P_i \circ D_1$, $deg_{G_1 \times G_2}((x_1, x_2))$ $=$ (number of vertices adjacent to $x_1$)$deg_{G_2}(x_2)$.*

**Definition 2.15.** The *strong product* of two $m$-polar fuzzy graphs $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$, denoted by $G_1 \boxtimes G_2$ and defined as a pair $(C_1 \boxtimes C_2, D_1 \boxtimes D_2)$, such that for each $1 \leq i \leq m$,

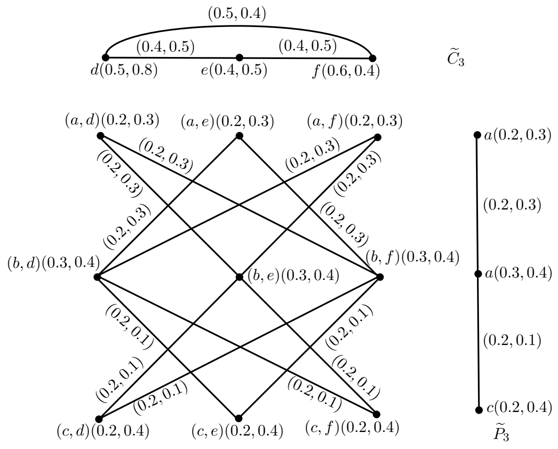



Figure 7: $\widetilde{P}_3 \times \widetilde{C_3}$

1. $P_i \circ (C_1 \boxtimes C_2)(x_1, x_2) = P_i \circ C_1(x_1) \wedge P_i \circ C_2(x_2)$, for all $(x_1, x_2) \in X_1 \times X_2$,
2. $P_i \circ (D_1 \boxtimes D_2)((x_1, x_2)(x_1, y_2)) = P_i \circ C_1(x_1) \wedge P_i \circ D_2(x_2 y_2)$, for all $x_1 \in X_1$ and $x_2 y_2 \in E_2$,
3. $P_i \circ (D_1 \boxtimes D_2)((x_1, x_2)(y_1, x_2)) = P_i \circ D_1(x_1 y_1) \wedge P_i \circ C_2(x_2)$, for all $x_2 \in X_2$ and $x_1 y_1 \in E_1$,
4. $P_i \circ (D_1 \boxtimes D_2)((x_1, x_2)(y_1, y_2)) = P_i \circ D_1(x_1 y_1) \wedge P_i \circ D_2(x_2 y_2)$, for all $x_1 y_1 \in E_1$ and $x_2 y_2 \in E_2$.

**Example 2.9.** Fig. 2.9 is an example of strong product of two 3−polar fuzzy paths $\widetilde{P}_3$ and $\widetilde{P}_3$.

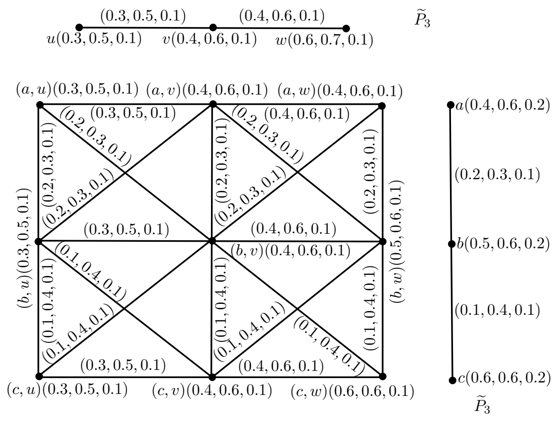



Figure 8: $\widetilde{P}_3 \boxtimes \widetilde{P}_3$

**Theorem 2.8.** *The strong product $G_1 \boxtimes G_2$ of two $m$-polar fuzzy graphs $G_1$ and $G_2$ is an $m$-polar fuzzy graph.*

**Theorem 2.9.** *Let $G_1$ and $G_2$ be two $m$-polar fuzzy graphs such that for all $1 \leq i, j \leq m$, $P_i \circ C_1 \geq P_j \circ D_2$, $P_i \circ C_2 \geq P_j \circ D_1$ and $P_i \circ D_1 \leq P_j \circ D_2$ , the following conditions are satisfied,*

1. *If $x_2 = y_2$, $d_{G_1 \boxtimes G_2}((x_1, x_2), (y_1, y_2)) = d_{G_1}(x_1, y_1)$.*
2. *If $x_1{=}y_1$ and $x_2{\neq}y_2$ or $x_1{\neq}y_1$ and $x_2{\neq}y_2$ then, $d_{i,G_1 \boxtimes G_2}((x_1, x_2), (y_1, y_2)) = L_i(W) \wedge L_i(T)$ where,*

   *a) $W$ is an $m$-polar fuzzy walk of smallest length in $G_1 \times G_2$ from $(x_1, x_2)$ to $(y_1, y_2)$ whose length in crisp direct product is the positive integer $k$ such that $k = d_{G_1^*(x_1,y_1)} \vee d_{G_2^*(x_2,y_2)}$.*

   *b) $L_i(T)$ is the smallest $P_i$-length of any $m-$polar fuzzy walk $T$, from $x_1$ to $y_1$, in $G_1$ such that the length of $T^*$ is greater than $k$.*

**Example 2.10.** Consider the strong product in Fig. 2.9,

1. $d_{\widetilde{P}_3 \boxtimes \widetilde{P}_3}((a, u), (c, u)) = (0.3, 0.7, 0.2) = d_{\widetilde{P}_3}(a, c)$.
2. $d_{\widetilde{P}_3 \boxtimes \widetilde{P}_3}((a, u), (a, w)) = (0.4, 0.6, 0.2) = L(W), W : a - b - a$.
3. $d_{\widetilde{P}_3 \boxtimes \widetilde{P}_3}((a, u), (c, w)) = (0.3, 0.7, 0.2) = L(W)$, here $W = (a, u) - (b, v) - (c, w)$. It clear that $k = L(W^*) = d_{P_3}(a, c) \vee d_{P_3}(u, w)$.
4. $d_{\widetilde{P}_3 \boxtimes \widetilde{P}_3}((a, u), (b, w)) = (0.4, 0.9, 0.2) = (L_1(f_{\widetilde{P}_3}(W_1)), L_2(W), L_3(W))$ where, $W = (a, u) - (b, v) - (b, w)$, $d_{P_3}(a, b) \vee d_{P_3}(u, w) = 2 = L(W^*)$ and $W_1 = (a, u) - (b, v) - (c, w) - (b, w)$. Clearly, $f_{\widetilde{P}_3}(W_1) = a - b - c - b$ whose crisp length is 3 which is greater than $k$ and $L_1(f_{\widetilde{P}_3}(W_1)) < L_(W)$ .

**Theorem 2.10.** *Let $G_1$ and $G_2$ be two $m$-polar fuzzy graphs. If $x_1 \in X_1$ and $x_2 \in X_2$, and for each $1 \leq i, j \leq m$, $P_i \circ C_1 \geq P_j \circ D_2$, $P_i \circ C_2 \geq P_j \circ D_1$ and $P_i \circ D_1 \leq P_j \circ D_2$. then, $deg_{G_1 \boxtimes G_2}((x_1, x_2) = deg_{G_1}(x_1) + deg_{G_2}(x_2) + r_2 deg_{G_1}(x_1)$, where $r_2$ is the number of vertices adjacent to $x_2$.*

**Definition 2.16.** The *lexicographic product* of two $m$-polar fuzzy graphs $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$, denoted by $G_1 \circ G_2$, is defined as a pair $(C_1 \circ C_2, D_1 \circ D_2)$, such that for each $1 \leq i \leq m$,

1. $P_i \circ (C_1 \circ C_2)(x_1, x_2) = P_i \circ C_1(x_1) \wedge P_i \circ C_2(x_2)$, for all $(x_1, x_2) \in X_1 \times X_2$,
2. $P_i \circ (D_1 \circ D_2)((x, x_2)(x, y_2)) = P_i \circ C_1(x) \wedge P_i \circ D_2(x_2 y_2)$ for a ll $x \in X_1$ and $x_2 y_2 \in E_2$,
3. $P_i \circ (D_1 \circ D_2)((x_1, x_2)(y_1, y_2)) = P_i \circ D_1(x_1 y_1) \wedge P_i \circ D_2(x_2 y_2)$, for all $x_1 y_1 \in E_1$ and $x_2 y_2 \in E_2$.

**Example 2.11.** The lexicographic product of $\widetilde{P}_3$ and $\widetilde{P}_3$ is given in Fig. 2.11.

**Theorem 2.11.** *The lexicographic product $G_1 \circ G_2$ of two $m$-polar fuzzy graphs $G_1$ and $G_2$ is an $m$-polar fuzzy graph.*

**Theorem 2.12.** *Let $G_1$ and $G_2$ be two $m$-polar fuzzy graphs such that for each $1 \leq i \leq m$, $P_i \circ C_1 \geq P_i \circ D_2$, $P_i \circ C_2 \geq P_i \circ D_2$ and $P_i \circ D_1 \geq P_i \circ D_2$, the following conditions are satisfied.*

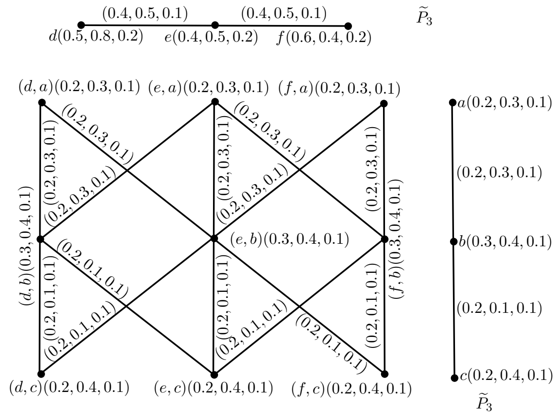


Figure 9: $\widetilde{P}_3 \circ \widetilde{P}_3$

1. *If* $d_{G_1^*}(x_1, y_1) = k$*, where* $k$ *is even(or odd) and* $d_{G_2^*}(x_2, y_2)$ *is also even(or odd) then,* $d_{i,G_1 \circ G_2}((x_1, x_2), (y_1, x_2)) = L_i(W)$*, where* $W$ *is an* $m$*-polar fuzzy walk of smallest* $P_i$*-length in* $G_2$ *such that* $W^*$ *is a walk of length* $k$ *in* $G_2^*$*.*

2. *If* $d_{G_1^*}(x_1, y_1) = k$*, where* $k$ *is even(or odd) and* $d_{G_2^*}(x_2, y_2)$ *is odd(or even) then,* $d_{i,G_1 \circ G_2}((x_1, x_2), (y_1, x_2)) = L_i(W)$*, where* $W$ *is an* $m$*-polar fuzzy walk of smallest* $P_i$*-length in* $G_2$ *such that* $W^*$ *is a walk of length* $k + 1$ *in* $G_2^*$*.*

**Example 2.12.** In Figure. 2.11,

1. $d_{\widetilde{P}_3 \circ \widetilde{P}_3}((d, a), (f, c)) = (0.4, 0.4, 0.2) = L(W)$, $W = a - b - c$. It clear that $L(W^*) = d_{P_3}(d, f)$ because both $d_{P_3}(d, f)$ and $d_{P_3}(a, c)$ are even.

2. $d_{\widetilde{P}_3 \circ \widetilde{P}_3}((d, a), (f, b)) = (0.6, 0.5, 0.2) = L(W)$, $W = a - b - c - b$. Here $L(W^*) = d_{P_3}(d, f) + 1$ because $d_{P_3}(d, f)$ is even and $d_{P_3}(a, b)$ is odd.

**Theorem 2.13.** *Let* $G_1$ *and* $G_2$ *be two* $m$*-polar fuzzy graphs. If* $x_1 \in X_1$ *and* $x_2 \in X_2$*, and for each* $1 \leq i, j \leq m$*,* $P_i \circ C_1 \geq P_j \circ D_2$*,* $P_i \circ C_2 \geq P_j \circ D_1$ *and* $P_i \circ D_1 \leq P_j \circ D_2$ *then,* $deg_{G_1 \circ G_2}((x_1, x_2) = deg_{G_2}(x_2) + r_2 deg_{G_1}(x_1)$*, where* $r_2$ *is the number of vertices adjacent to* $x_2$*.*

We now define the concept of metric in $m-$polar fuzzy graphs.

**Theorem 2.14.** *For an* $m$*-polar fuzzy graph* $G = (C, D)$*,* $d = (d_1, d_2, \ldots, d_m) : X \times X \to [0, 1]^m$ *defines a metric on* $X$*, with the following conditions:*

(1) $d(x, y) \geq \mathbf{0}$,
(2) $d(x, y) = \mathbf{0} \Leftrightarrow x = y$,
(3) $d(x, y) = d(y, x)$,
(4) $d(x, z) \leq d(x, y) + d(y, z)$, *for all* $x, y, z \in X$.

**Proof.** It is very clear from the definition of $d_i$, for each $1 \leq i \leq m$ $d_i(x,y) \geq 0$, for all $x, y \in X \Rightarrow d(x,y) \geq \mathbf{0}$. If $x = y$ then, $d(x,y) = d(x,x) = (d_1(x,x), d_2(x,x), \dots, d_m(x,x)) = (0,0,\dots,0)$. The inverse of any $x-y$ $m-$polar fuzzy path is a $y-x$ $m-$polar fuzzy path and vice versa of same distance. Let $P_1, P_2, \dots, P_m$ be $x-y$ $m-$polar fuzzy paths and $Q_1, Q_2, \dots, Q_m$ be $y-z$ $m-$polar fuzzy paths such that

$$d(x,y) = (d_1(x,y), d_2(x,y), \dots, d_m(x,y)) = (L_1(P_1), L_2(P_2), \dots, L_m(P_m)),$$
$$d(y,z) = (d_1(y,z), d_2(y,z), \dots, d_m(y,z)) = (L_1(Q_1), L_2(Q_2), \dots, L_m(Q_m)).$$

$P_1$ followed by $Q_1$, $P_2$ followed by $Q_2$ and so on $P_m$ followed by $Q_m$ are $x-z$ $m-$polar fuzzy walks, each of which contains only one $m-$polar fuzzy path whose length cannot exceed $d_1(x,y) + d_2(y,z)$, $d_2(x,y) + d_2(y,z)$ and so on $d_m(x,y) + d_m(y,z)$, respectively. Thus, we can write,

$$\begin{aligned}(d_1(x,z), d_2(x,z), \dots, d_m(x,z)) &\leq (d_1(x,y) + d_1(y,z), d_2(x,y) + d_2(y,z),\\ &\qquad \dots, d_m(x,y) + d_m(y,z))\\ (d_1(y,z), d_2(y,z), \dots, d_m(y,z)) &\leq (d_1(x,y), d_2(x,y), \dots, d_m(x,y))\\ &\qquad + (d_1(y,z), d_2(y,z), \dots, d_m(y,z)),\\ d(x,z) &\leq d(x,y) + d(y,z).\end{aligned}$$ □

**Definition 2.17.** Let $G = (C, D)$ be an $m$-polar fuzzy graph. The eccentricity of a vertex $x$ is denoted by the $m$-tuple $e(x) = (e_1(x), e_2(x), \dots, e_m(x))$ and defined as the distance to a vertex farthest from $x$, i.e.,

$$e_i(x) = \max\{d_i(x,y) : y \in X\}, \qquad 1 \leq i \leq m,$$
$$e(x) = \max\{d(x,y) : y \in X\}.$$

**Definition 2.18.** The radius of an $m$-polar fuzzy graph is the minimum of all the eccentricities of the vertices, i.e., $r(G) = \min\{e(x) : x \in X\}$.

**Definition 2.19.** The diameter of $m$-polar fuzzy graph is, denoted by $diam(G)$, defined as the maximum of all of the eccentricities of the vertices, i.e., $diam(G) = \max\{e(y) : y \in X\}$.

**Definition 2.20.** A vertex $y$ at a distance $e(x)$ from $x$ is called *eccentric vertex* of $x$.

**Definition 2.21.** A vertex $x$ is called a *central vertex* if $e(x) = r(G)$.

**Definition 2.22.** The $m-$polar fuzzy subgraph induced by the central vertices is known as *center* of the $m$-polar fuzzy graph. If the center of $G$ is $G$ itself then $G$ is called a *self-centered $m$-polar fuzzy graph.*

**Definition 2.23.** A vertex $y$ is called a *peripheral vertex* if $e(y) = diam(G)$.

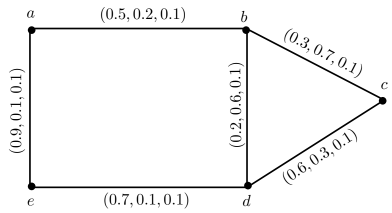



Figure 10: 3−polar fuzzy graph

**Example 2.13.** Consider an 3−polar fuzzy graph as shown in Fig. 2.13. Here, $d$ is a central vertex and $c$ is a peripheral vertex

**Theorem 2.15.** *For any m−polar fuzzy graph G, radius and diameter satisfy the inequality,* $r(G) \leq diam(G) \leq 2r(G)$

**Proof.** By Definition 2.18 and 2.19, we have $r_i(G) \leq diam_i(G), 1 \leq i \leq m, r(G) = (r_1(G), r_2(G), \ldots, r_m(G)), \Rightarrow r(G) \leq (diam_1(G), diam_2(G), \ldots, diam_m(G)), \Rightarrow r(G) \leq diam(G)$. Let $u$, $v$ be central vertices of $G$, respectively then, $e(u) = r(G)$ and $e(v) = r(G)$. Let $x$ be a peripheral vertex of $G$. Since $d$ defines a metric, for some vertices $y_1, y_2, \ldots, y_m \in X$,

$$\begin{aligned} diam(G) &= (diam_1(G), diam_2(G), \ldots, diam_m(G)), \\ &= (d_1(x, y_1), (d_2(x, y_2), \ldots, d_m(x, y_m)), \\ &\leq (d_1(x, u) + d_1(u, y_1), d_2(x, u) + d_2(u, y_2), \ldots, d_m(x, u) + d_m(u, y_m)), \\ &\quad \text{for some } u, v \in X \\ &\leq (2r_1(G), 2r_2(G), \ldots, 2r_m(G)), \\ &= 2r(G). \end{aligned}$$ □

The following theorem gives an absolute difference between the eccentricities of any two adjacent vertices.

**Theorem 2.16.** *For any two adjacent vertices x and y in an m−polar fuzzy graph G,* $|e(x) - e(y)| \leq \mathbf{1}$.

By assuming any two arbitrary vertices $x$ and $y$ in Theorem 2.16, we obtain the following result.

**Theorem 2.17.** *For any two vertices x and y in an m-polar fuzzy graph G,* $|e(x) - e(y)| \leq d(x, y)$.

**Theorem 2.18.** *For any two adjacent vertices x and y in an m-polar fuzzy graph G,* $|d(x, z) - d(y, z)| \leq \mathbf{1}$*, for every vertex z in G.*

We now generalize Theorem. 2.18 for any two vertices $x$ and $y$ in the following theorem.

**Theorem 2.19.** *For any two vertices $x$ and $y$ in an $m-$polar fuzzy graph $G$, $|d(x,z) - d(y,z)| \leq d(x,y)$.*

**Theorem 2.20.** *If $G$ is a self centered $m-$polar fuzzy graph, then each vertex of $G$ is an eccentric vertex.*

**Proof.** Let $y$ be an eccentric vertex of $x$ then, $e(x) = d(x,y)$. Since $G$ is self centered, $e(x) = e(y) = r(G)$, $e(y) = d(x,y)$. It shows that $x$ is an eccentric vertex of $y$ . Since $x$ was taken to be arbitrary, theorem is true for all vertices. □

**Remark 2.2.** The converse of Theorem. 2.20 is not true in general as it can be seen in the following example.

**Example 2.14.** Consider a 2−polar fuzzy graph as shown in the Fig. 2.14. $e(x) = (0.9, 0.5)$, $e(y) = (0.9, 0.5)$, $e(z) = (0.8, 0.5)$ $e(w) = (0.8, 0.5)$, $r(G) =$

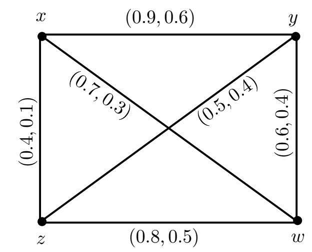
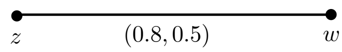



Figure 11: 2−polar fuzzy graph

$(0.8, 0.5)$, $diam(G) = (0.9, 0.5)$. All vertices of $G$ are eccentric but $G$ is not self-centered 2−polar fuzzy graph because the center of $G$ is shown as,




**Theorem 2.21.** *Let $G$ be a self centered m-polar fuzzy graph. For every two vertices $x, y \in G$ if $x \in Y^*$, then $y \in X^*$, where $X^*$ is the set of all eccentric vertices of $x$ any $Y^*$ is the set of all eccentric vertices of $y$.*

**Proof.** Since $x$ is a eccentric vertex of $y$, $e(x) = d(x,y) \Rightarrow x \in Y^*$. It is given that $G$ is a self centered $m-$polar fuzzy graph, so $e(y) = e(x) = d(y,x)$, i.e., $y$ is an eccentric vertex of $x$. Hence $y \in X^*$. □

**Remark 2.3.** The converse of Theorem. 2.21 is not true in general. From example 2.14, $x \in Y^*$, $y \in X^*$, $z \in W^*$ and $w \in Z^*$ but $G$ is not self-centered.

**Theorem 2.22.** *Let $G$ be an m-polar fuzzy graph then all peripheral vertices are eccentric vertices.*

**Proof.** Let $x$ be a peripheral vertex and $y$ be its eccentric vertex then, $diam(G) = e(x) = d(x,y) = d(y,x)$. It is only possible if $diam(G) = d(y,x) = e(y)$. It shows that $x$ is an eccentric vertex of $y$. Hence the proof. □

**Remark 2.4.** The condition in Theorem. 2.22 is not sufficient. In example 2.14, $z$ and $w$ are eccentric vertices of each other but these are not peripheral vertices.

**Theorem 2.23.** *Let $G$ be a complete $m-$polar fuzzy graph such that $P_i \circ C(x_1) \leq P_i \circ C(x_2) \leq P_i \circ C(x_3) \leq ... \leq P_i \circ C(x_n)$, for each $1 \leq i \leq m$, then the distance between any two vertices $x_l$ and $x_j$ is $(P_1 \circ D(x_l x_j) \wedge 2P_1 \circ C(x_1), P_m \circ D(x_l x_j) \wedge 2P_2 \circ C(x_1), \dots, P_m \circ D(x_l x_j) \wedge 2P_m \circ C(x_1))$.*

**Proof.** Let $x_l$ and $x_j$ be any two vertices of $G$ then for some $x_k^{(1)}, x_k^{(2)}, \dots, x_k^{(m)} \in X$, the distance between $x_l$ and $x_j$ can be defined as,

$$\begin{aligned} d(x_l, x_j) &= (d_1(x_l, x_j), d_2(x_l, x_j), \dots, d_m(x_l, x_j)), \\ &= (\inf\{P_1 \circ D(x_l x_j), P_1 \circ D(x_l x_k^{(1)}) + P_1 \circ D(x_k^{(1)} x_j)\}, \\ &\quad \inf\{P_2 \circ D(x_l x_j), P_2 \circ D(x_l x_k^{(2)}) + P_2 \circ D(x_k^{(2)} x_j)\}, \\ &\quad \dots, \inf\{P_m \circ D(x_l x_j), P_m \circ D(x_l x_k^{(m)}) + P_m \circ D(x_k^{(m)} x_j)\}. \end{aligned} \tag{2.9}$$

Since $G$ is a complete $m$-polar fuzzy graph, for each $1 \leq i \leq m$, $P_i \circ D(x_l x_k^{(i)}) = P_i \circ C(x_l) \wedge P_i \circ C(x_k^{(i)})$, Since, for each $1 \leq i \leq m$, $P_i \circ C(x_1) \leq P_i \circ C(x_2) \leq P_i \circ C(x_3) \leq \dots \leq P_i \circ C(x_n)$. Therefore, for $x_k^{(i)} = x_1$, $P_i \circ D(x_l x_k^{(i)}) = P_i \circ C(x_k^{(i)}) = P_i \circ C(x_1)$. Similarly, $P_i \circ D(x_k^{(i)} x_j) = P_i \circ C(x_k^{(i)}) = P_i \circ C(x_1)$. Equation. (2.10) takes the form as, $d(x_l, x_j) = (P_1 \circ D(x_l x_j) \wedge 2P_1 \circ C(x_1), P_2 \circ D(x_l x_j) \wedge 2P_2 \circ C(x_1), \dots, P_m \circ D(x_l x_j) \wedge 2P_m \circ C(x_1))$. □

**Theorem 2.24.** *Let $G$ be a complete bipartite $m-$polar fuzzy graph where, $X_1 = \{x_1, x_2, \dots, x_n\}$ and $X_2 = \{y_1, y_2, \dots, y_{n'}\}$, such that for each $1 \leq i \leq m$, $P_i \circ C(x_1) \leq P_i \circ C(x_2) \leq P_i \circ C(x_3) \leq ... \leq P_i \circ C(x_n)$, and $P_i \circ C(y_1) \leq P_i \circ C(y_2) \leq P_i \circ C(y_3) \leq ... \leq P_i \circ C(y_{n'})$. If $P_i \circ C(x_1) \leq P_i \circ C(y_1)$ and $P_i \circ C(y_1) \leq P_i \circ C(x_j)$ , for each $2 \leq j \leq n$, then,*

$$d_i(u, v) = \begin{cases} 2P_i \circ C(x_1), & u, v \in X_2 \\ P_i \circ C(x_1) + P_i \circ C(y_1), & u, v \in X_1, u = x_1 \text{ or } v = x_1 \\ 2P_i \circ C(y_1), & u, v \in X_1, u \neq x_1, v \neq x_1 \\ P_1 \circ D(uv) \wedge (2P_1 \circ C(x_1) + P_1 \circ C(y_1)), & u \in X_1 \text{ and } v \in X_2. \end{cases}$$

*If $P_i \circ C(y_1) \leq P_i \circ C(x_1)$ and $P_i \circ C(x_1) \leq P_i \circ C(y_j)$ , for each $2 \leq j \leq n^{'}$, then*

$$d_i(u, v) = \begin{cases} 2P_i \circ C(y_1), & u, v \in X_1 \\ P_i \circ C(y_1) + P_i \circ C(x_1), & u, v \in X_2, u = y_1 \text{ or } v = y_1 \\ 2P_i \circ C(x_1), & u, v \in X_2, u \neq y_1, v \neq y_1 \\ P_1 \circ D(uv) \wedge (2P_1 \circ C(y_1) + P_1 \circ C(x_1)), & u \in X_1 \text{ and } v \in X_2. \end{cases}$$

**Proof.** Consider the case $P_i \circ C(x_1) \leq P_i \circ C(y_1)$ and $P_i \circ C(y_1) \leq P_i \circ C(x_j)$, for each $2 \leq j \leq n$. Let $u, v \in X_1$ be any two vertices of $G$ then for some $y_k^{(1)}, y_k^{(2)}, \ldots, y_k^{(m)} \in X$, the distance between $u$ and $v$ can be defined as:

$$\begin{aligned} d(u,v) &= (d_1(u,v), d_2(u,v), \ldots, d_m(u,v)), \\ &= (P_1 \circ D(uy_k^{(1)}) + P_1 \circ D(y_k^{(1)}v), P_2 \circ D(uy_k^{(2)}) + P_2 \circ D(y_k^{(2)}v), \\ &\quad \ldots, P_m \circ D(uy_k^{(m)}) + P_m \circ D(y_k^{(m)}v)). \end{aligned} \tag{2.10}$$

Since, $P_i \circ D(uy_k^{(i)}) = P_i \circ C(u) \wedge P_i \circ C(y_k^{(i)})$. If $u = x_1$, then for $y_k^{(i)} = y_1$, $P_i \circ D(uy_k^{(i)}) = P_i \circ C(u) = P_i \circ C(x_1)$. Similarly, $P_i \circ D(y_k^{(i)}v) = P_i \circ C(y_k^{(i)}) = P_i \circ C(y_1)$. Equation. (2.10) takes the form as, $d(u,v) = (P_1 \circ C(x_1) + P_1 \circ C(y_1), P_2 \circ C(x_1) + P_2 \circ C(y_1), \ldots, P_m \circ C(x_1) + P_m \circ C(y_1))$. If $u, v \neq x_1$, then for $y_k^{(i)} = y_1$, $P_i \circ D(uy_k^{(i)}) = P_i \circ C(y_k^{(i)}) = P_i \circ C(y_1)$. Similarly, $P_i \circ D(y_k^{(i)}v) = P_i \circ C(y_k^{(i)}) = P_i \circ C(y_1)$. Equation. (2.10) takes the form as, $d(u,v) = (2P_1 \circ C(y_1), 2P_2 \circ C(y_1), \ldots, 2P_m \circ C(y_1))$. If $u, v \in X_2$, there exist some $x_k^{(1)}, x_k^{(2)}, \ldots, x_k^{(m)} \in X_1$ such that,

$$\begin{aligned} d(u,v) &= (d_1(u,v), d_2(u,v), \ldots, d_m(u,v)), \\ &= (P_1 \circ D(ux_k^{(1)}) + P_1 \circ D(x_k^{(1)}v), P_2 \circ D(ux_k^{(2)}) + P_2 \circ D(x_k^{(2)}v), \\ &\quad \ldots, P_m \circ D(ux_k^{(m)}) + P_m \circ D(x_k^{(m)}v). \end{aligned} \tag{2.11}$$

Since $P_i \circ C(x_1) \leq P_i \circ C(y_1)$, for $x_k^{(i)} = x_1$, $P_i \circ D(ux_k^{(i)}) = P_i \circ C(x_k^{(i)}) = P_i \circ C(x_1)$. Similarly, $P_i \circ D(x_k^{(i)}v) = P_i \circ C(x_k^{(i)}) = P_i \circ C(x_1)$. Equation. (2.11) takes the form as, $d(x_l, x_j) = (2P_1 \circ C(x_1), 2P_2 \circ C(x_1), \ldots, 2P_m \circ C(x_1))$. If $u \in X_1$ and $v \in X_2$,

$$\begin{aligned} d(u,v) &= (d_1(u,v), d_2(u,v), \ldots, d_m(u,v)), \\ &= (P_1 \circ D(uv) \wedge (P_1 \circ D(uy_1) + P_1 \circ D(y_1x_1) + P_1 \circ D(x_1v)), \\ &\quad P_2 \circ D(uv) \wedge (P_2 \circ D(uy_1) + P_2 \circ D(y_2x_2) + P_2 \circ D(x_1v))\}, \\ &\quad \ldots, P_m \circ D(uv) \wedge (P_m \circ D(uy_1) + P_1 \circ D(y_1x_1) + P_m \circ D(x_1v))\}). \end{aligned} \tag{2.12}$$

Equation (2.12) becomes, $d(u,v) = (P_1 \circ D(uv) \wedge (2P_1 \circ C(x_1) + P_1 \circ C(y_1)), P_2 \circ D(uv) \wedge (2P_2 \circ C(x_1) + P_2 \circ C(y_1)), \ldots, P_m \circ D(uv) \wedge (2P_m \circ C(x_1) + P_1 \circ C(y_m)))$
Other cases can be proved similarly. □

We now present an algorithm for computing distance, eccentricity of vertices, radius and diameter of any $m-$polar fuzzy graphs.

**Description and Time Complexity:** First the algorithm takes number of vertices $v$ and $pth$, $1 \leq p \leq m$, adjacency matrix of membership values $a(i,j)$ as input. Starting from a vertex $r$, lines $1-5$ find a vertex $i$ adjacent to $r$ such that $ri$ has the least weight. Lines $6-14$ calculate the distances of vertex $r$ to

**Algorithm**

1. Input: Enter the number of vertices $v$ and the $i$th adjacency matrix of membership values $a(i, j)$ row-wise.
2. Output: The distance between all the vertices, eccentricity of vertices, radius and diameter.
3. **do** $r$ from 1 to $v$
4. distance $= \infty$
5. find a vertex $i$ adjacent to $r$ with minimum weight
6. Take $u(n) = i, \text{sum}(r, i) = a(r, i)$ and $\text{sum}(r, k) = \infty$
7. **do** $k$ from 1 to $v$
8. **if**$(\text{sum}(r, u(n)) + a(u(n), k) < \text{sum}(r, k))$
9. $\text{sum}(r, u(n)) + a(u(n), k) \leftarrow \text{sum}(r, k)$
10. $u(n) = k$
11. **else**
12. $\text{sum}(r, k) \leftarrow \text{sum}(r, k)$
13. **end if**
14. $\min\{\text{distance}(r, k), \text{sum}(r, k)\} \leftarrow \text{distance}(r, k)$
15. **end do**
16. print of all the distances
17. **end do**
18. find eccentricities, radius and diameter.

all other vertices. The distances are printed in line 15. Line 17 calculates the eccentricities, radius and diameter. Repeat this algorithm $m$ times to find the distances in $m-$polar fuzzy graph.

The running time complexity of lines $1-6, 15-17$ is $v$ and lines $7-14$ is $v \times v$. Therefore, the net time complexity of the algorithm is $O(v^2)$, where $v$ is the number of vertices of an $m-$polar fuzzy graph.

## Applications of $m$-polar fuzzy graphs

We describe a pair of example applications of $m-$polar fuzzy graphs in decision support system.

**A. $m-$polar fuzzy graphs in product manufacturing**

A product can increase the profit of a company if it is sold in multiple areas. Before manufacturing a product, engineers and manufacturers test several important things in a product. Usually, graphical models are used in such decision making problems. $m$-polar fuzzy graphs are mostly used in decision making problems when it is necessary to gather a group of agreements. Suppose a multinational enterprize(MNE) has to decide to manufacture a product among four products $P_1$, $P_2$, $P_3$ and $P_4$ to market it in different countries. Every company consider the following points before manufacturing a product.

- Does the product follow the mass market demands?

- Is the product fast or time consuming to manufacture?
- Is the product sold at a high or low cost?
- Does the product appeal the people at global level?

We gather the above four points in a set as, $X = \{$Demand, Time, Cost, Appealing$\}$. Let the set of products is $P = \{P_1, P_2, P_3, P_4\}$. This phenomenon can be represented by a 4-polar fuzzy graph, taking $P$ as the set of vertices. The membership value of each product represents the degree of demand, time consumption, sale price and attraction to people at a global level. Let $C(P_1) = (0.5, 0.8, 0.6, 0.4)$, $C(P_2) = (0.8, 0.2, 0.5, 0.9)$, $C(P_3) = (0.4, 0.6, 0.7, 0.4)$, $C(P_4) = (0.6, 0.8, 0.6, 0.4)$. That is, the degrees of $P_1$ corresponding to demand, time consumption, sale price and attraction to people are $0.5, 0.8, 0.6$ and $0.4$, respectively and similarly for other products. The edge between two products represents the degree of using common power equipments, materials, engineer employs and agencies involved for both of the products. Let $D(P_1P_2) = (0.5, 0.5, 0.7, 0.4)$, $D(P_1P_3) = (0.1, 0.3, 0.2, 0.4)$, $D(P_1P_4) = (0.2, 0.3, 0.1, 0.1)$, $D(P_2P_3) = (0.3, 0.4, 0.5, 0.4)$, $D(P_2P_4) = (0.6, 0.2, 0.5, 0.4)$. This means that $P_1$ and $P_2$ use 50% common equipments, 50% same materials, 70% common trained engineers and 40% same agencies. It can be easily verified that it is as 4-polar fuzzy graph as shown in Fig. 2. By observation it is easy to see that the production of $P_2$ has more

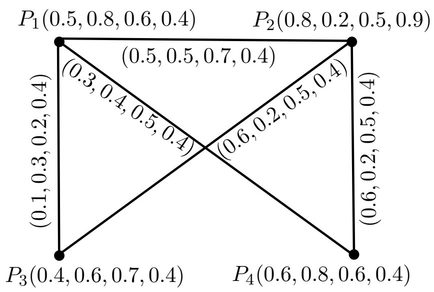



Figure 12: $4-$polar fuzzy graph

demand and attraction, minimum time consumption and its price is such that it is in the range of all classes of people. Also it shares a lot of common things with other products. Consequently, it will give more profit to company as compared to other products.

**B. Shortest path problem in $m-$polar fuzzy graphs**

Graph are used as a common source to model the communication networks such as transportation, to find the shortest paths between any two points of the network. $m-$polar fuzzy graphs can be used to find the shortest paths when it is required to consider a group of consequences. An agency wants to deliver a secret envelope from Astana Kazakhstan to British Columbia through delivery. The delivery car can only travel 4000 kilometers before refilling the tank. The agency requires to deliver the envelope with minimum cost of fuel and safety.

From Astana to Columbia, there are seventeen fuel stations namely, Moscow Russia, London England, Barcelona Spain, Algiers Algeria, San Jaun, Boston, Houston Taxas, Helena Montana, Beijing China, Delhi India, Tokyo, Singapore, Australia, Honolulu, Alaska and California. The agency demands to consider the following things.

- Refill the tank at a station having minimum fuel cost, minimum time consumption and less danger of robbery.
- Take into account the distance travelled(in kilometers) between two fuel stations.
- How much fuel is used from one station to another?
- Use the path in which there is less danger of robbery.

We represent the fuel stations by vertices. The membership value of each vertex represents the degree of fuel cost, time consumption and danger. The edge between two stations represent the degree of distance travelled, fuel used and danger. This situation can b represented by 3-polar fuzzy graph in Fig. 2. Since, in calculating the distance there is no use of membership value of vertices, we do not write these values in the 3-polar fuzzy graph. It is necessary to travel

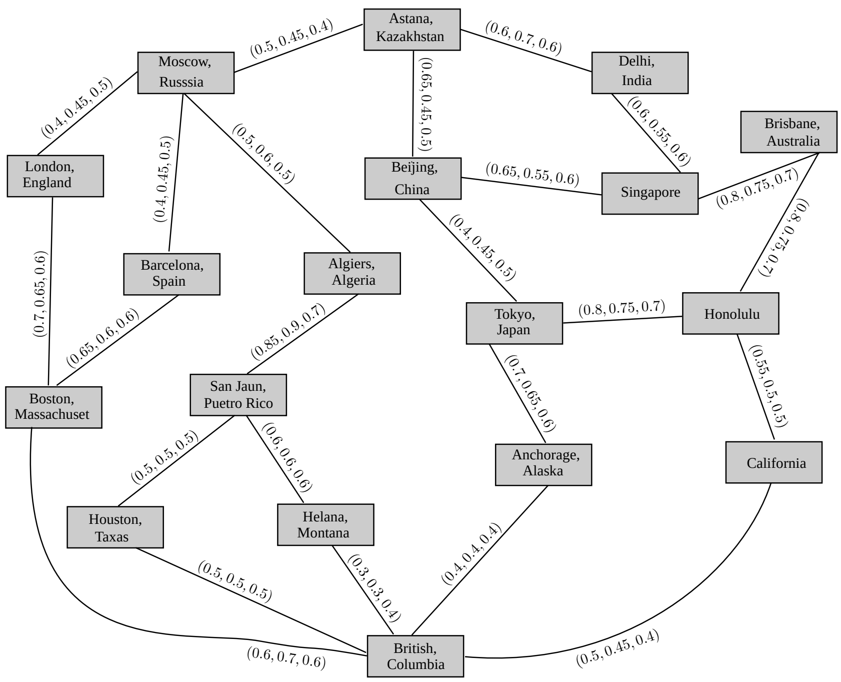



Figure 13: 3-polar fuzzy graph

through the path with least degree of "distance travelled, fuel used and danger of robbery". For this, we calculate the length of all paths from Astana to British

Columbia and travel through a path with minimum length called distance. By routine calculations, it can be easily seen that the minimum distance between Astana and British Columbia is $(2.15, 1.9, 2.0)$ which is obtained through the path Astana−Beijing−Tokyo−Alaska−British Columbia.

## 3. Conclusion and future work

An $m-$polar fuzzy model is a generalization of the bipolar fuzzy model. The $m-$polar fuzzy models give more precision, flexibility and compatibility to the system when more than one agreements are to be dealt with. In this paper, we have applied the concept of $m-$polar fuzzy model to graphs. We have presented an algorithm for computing the distance matrix, eccentricity of the vertices, radius and diameter in $m-$polar fuzzy graphs. We have also discussed applications of $m-$polar fuzzy graphs in traveling and product manufacturing. We are planing to extend our research work to (1) $m-$polar fuzzy soft hypergraphs, (2) Roughness in $m-$polar fuzzy hypergraphs, (3) $m-$polar fuzzy soft graphs.

**Conflict of interest.** The authors declare that they have no conflict of interest.

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Accepted: 27.03.2017