CERTAIN PROPERTIES ASSOCIATED WITH B-PREINVEX FUZZY MAPPINGS

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Abstract. We establish several new characterizations for B-preinvex fuzzy mappings. Under the condition of upper or lower semi-continuity and the well known Condition C introduced by Mohan and Neogy [J. Math. Anal. Appl., 189 (1995) 901-908], we obtain a sufficient condition for B-preinvex fuzzy mappings. Several necessary conditions for differentiable and twice differentiable B-preinvex fuzzy mappings are also presented and proved.

Keywords: Fuzzy numbers, B-preinvex fuzzy mappings, semi-continuity.

1. Introduction

Owing to the importance of the generalized convexity of fuzzy mappings and the generalized fuzzy convexity in the search for optimal conditions to solve the fuzzy optimization problems, many authors paid special attention to the research of fuzzy mappings, especially for generalized convex fuzzy mappings. For example, in earlier papers, the concept of fuzzy mappings was introduced by Chang and Zadeh (1972) [3]. Nanda and Kar (1992) [11] proposed the concept of convex fuzzy mappings and proved that a fuzzy mapping is convex if and only if its epigraph is a convex set. Noor(1994) [10] introduced the concept of fuzzy preinvex functions over the field of real numbers $\mathbb{R}$, and obtained some properties of fuzzy preinvex functions. Syau introduced concepts of all kinds of generalized convexity for fuzzy mappings of one variable such as convex and concave fuzzy mappings, preinvex fuzzy mapping (1999a) [16]. In the meantime, Syau also discussed many important properties of these generalized convex fuzzy mappings.

In recent ten years, Yan and Xu (2002) [21] discussed the convexity and quasiconvexity of fuzzy mappings by considering the concept of ordering. Mishra et al.(2006) [9] introduced concept of an explicitly B-preinvex fuzzy mapping

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and presented some properties of explicitly B-preinvex fuzzy mappings. Wu and Xu (2008) [20] introduced concepts of fuzzy pseudoconvex, fuzzy invex, fuzzy pseudoinvex, and fuzzy preinvex mapping from $\mathbb{R}^n$ to the set of fuzzy numbers based on the concept of differentiability of fuzzy mappings. Very recently, Li and Noor (2010) [7] studied the necessary and sufficient conditions for differentiable and twice differentiable preinvex fuzzy mapping by using the given equivalent condition of preinvex fuzzy mapping and established the semicontinuity of preinvex fuzzy mappings. Tang and Ding (2013) [19] introduced concepts of semilocally b-preinvex fuzzy mappings, semilocally quasi b-preinvex fuzzy mappings, semilocally pseudo b-preinvex fuzzy mappings and semilocally strongly pseudo b-preinvex fuzzy mappings, they also studied a fuzzy nonlinear programming which be considered involving generalized convex fuzzy mappings with η-semidifferentiability. Rufián-Lizana et al. (2014) [15] showed, by means of counterexamples, the characterizations given by Li et al. [7] were incomplete and provided valid characterizations. For more results on generalized fuzzy mappings, one can see the contributions [1, 2, 12, 13] and references therein.

Through the above researches, quite a number of contributions about preinvex fuzzy mappings was made, however, some new properties of B-preinvex fuzzy mappings should be worth studied. So we turn our attention to this new research.

Motivated by the works [6, 7, 9, 14, 15, 17, 19] going on in these areas, on the basis of the concept of parameterized triples of fuzzy numbers and by using the convexity, preinvexity and B-preinvexity, the purpose in the present paper is to study several important characterizations about B-preinvex fuzzy mappings. (i) We mainly study the sufficient conditions for B-preinvex fuzzy mappings under the semi-continuity conditions. (ii) We present and prove the necessary conditions for differentiable and twice differentiable B-preinvex fuzzy mappings. Compared with the results of Li et al. (2010) [7] and Rufián-Lizana et al. (2014) [15], the focus of our research is on the B-preinvex fuzzy mappings. That is to say, the result of B-preinvex fuzzy mappings in this paper can also be applied to preinvex fuzzy mappings, so our work has a generalized research significance.

The present paper is built up as follows. In Section 2, some preliminaries, including concepts of fuzzy numbers, preinvex fuzzy mappings, B-vex fuzzy mappings and differentiable fuzzy mappings of several variables are first reviewed. The generalized B-preinvex fuzzy mappings are then recalled, the basic properties of semi-continuity of B-preinvex fuzzy mappings are studied in Section 3. We provide and prove some results involving differentiable and twice differentiable B-preinvex fuzzy mappings in Section 4. Finally conclusions are given in Section 5.

2. Preliminaries

In this section, for convenience, several definitions and results with respect to fuzzy numbers and fuzzy mappings, which will be needed in sequel, from Diamond and Kloeden [4], Goetschel et al. [5], Noor [10] and Syau [16, 17] are summarized below.

We denote by $\mathbb{R}$ the set of all real numbers. A fuzzy number is a mapping $\mu : \mathbb{R} \to [0, 1]$ with the following properties:

(1) μ is upper semi-continuous.

(2) μ is normal, that is, there exists a $x \in \mathbb{R}$ such that $\mu(x) = 1$.

(3) μ is convex, namely, $\mu\big(\lambda x + (1 - \lambda)y\big) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$.

(4) the support of μ, $supp(\mu) = \{x \in \mathbb{R} : \mu(x) > 0\}$ and its closure $cl(supp\ \mu)$ is compact.

Let $\mathscr{F}_0$ denote the family of fuzzy numbers on $\mathbb{R}$. Since each $r \in \mathbb{R}$ can be considered as a fuzzy number $\overline{r}$ defined as

$$\overline{r} = \begin{cases} 1, & x = r, \\ 0, & x \neq r, \end{cases}$$

so $\mathbb{R}$ can be embedded in $\mathscr{F}_0$. It is well known that α-level set of a fuzzy set $\mu : \mathbb{R} \to [0, 1]$, $\alpha \in [0, 1]$, denoted by $[\mu]_\alpha$, is defined as

$$[\mu^*(\alpha), \mu_*(\alpha)] = [\mu]_\alpha = \begin{cases} \{x \in \mathbb{R} : \mu(x) > \alpha\}, & \text{if } 0 < \alpha \leq 1, \\ cl(supp\ \mu), & \text{if } \alpha = 0. \end{cases}$$

It can be seen easily that the α-level set of a fuzzy number is closed and bounded interval $[\mu^*(\alpha), \mu_*(\alpha)]$, where $\mu^*(\alpha)$ denotes the left-hand end point of $[\mu]_\alpha$ and $\mu_*(\alpha)$ denotes the right-hand end point of $[\mu]_\alpha$. Thus, a fuzzy number μ can be identified by a parameterized triples

$$\big\{\big(\mu^*(\alpha), \mu_*(\alpha), \alpha\big) : \alpha \in [0, 1]\big\}.$$

Let $\mu, \nu \in \mathscr{F}_0$ represented parametrical by

$$\big\{\big(\mu^*(\alpha), \mu_*(\alpha), \alpha\big) : \alpha \in [0, 1]\big\} \ \ and \ \ \big\{\big(\nu^*(\alpha), \nu_*(\alpha), \alpha\big) : \alpha \in [0, 1]\big\},$$

respectively. It is said that $\mu \preceq \nu$, if for every $\alpha \in [0, 1]$, $\mu^*(\alpha) \leq \nu^*(\alpha)$ and $\mu_*(\alpha) \leq \nu_*(\alpha)$. It is said that $\mu \prec \nu$, if $\mu \preceq \nu$ and there exists a $\alpha_0 \in [0, 1]$ such that $\mu^*(\alpha_0) < \nu^*(\alpha_0)$ and $\mu_*(\alpha_0) < \nu_*(\alpha_0)$. It is said that $\mu = \nu$, if $\mu \preceq \nu$ and $\nu \preceq \mu$.

Note that $\preceq$ is a partial order on $\mathscr{F}_0$, and it is often convenient to write $\nu \succeq \mu$ instead of $\mu \preceq \nu$.

A subset S^* of $\mathscr{F}_0$ is said to be bounded above if there exists a fuzzy number $\mu \in \mathscr{F}_0$, called an upper bound of S^*, such that $\nu \preceq \mu$ for every $\nu \in S^*$. Further, a fuzzy number $\mu_0 \in \mathscr{F}_0$ is called the least upper bound (sup in short) for S^* if (i) μ_0 is an upper bound of S^*, and (ii) $\mu_0 \preceq \mu$ for every upper bound μ of S^*. A lower bound and the greatest lower bound (inf in short) are defined similarly.

For fuzzy numbers μ, ν and each nonnegative real number k, the addition $\mu\tilde{+}\nu$ and nonnegative scalar multiplication $k\mu$ are defined as follows:

$$(\mu\tilde{+}\nu)(x) = \sup_{y+z=x} \min\{\mu(y), \nu(z)\},$$

$$(k\mu)(x) = \begin{cases} \mu(k^{-1}x), & \text{if } k \neq 0, \\ 0, & \text{if } k = 0, \end{cases}$$

and

$$\begin{aligned} \mu\tilde{+}\nu =& \{\big(\mu^*(\alpha) + \nu^*(\alpha), \mu_*(\alpha) + \nu_*(\alpha), \alpha\big) : \alpha \in [0,1]\}, \\ k\mu =& \{\big(k\mu^*(\alpha), k\mu_*(\alpha), \alpha\big) : \alpha \in [0,1]\}. \end{aligned}$$

It is obvious that concepts of addition and the nonnegative scalar multiplication on $\mathscr{F}_0$ defined by the above two equations are equivalent to those derived from the universal extension criterion. And it is easy to see that $\mathscr{F}_0$ is closed under addition and nonnegative scalar multiplication. So it should be noted that $k\mu$ is not a fuzzy number for $k < 0$ and $\mu_0 \in \mathscr{F}_0$. The family of parametric representations of members of $\mathscr{F}_0$ and the parametric representations of their negative scalar multiplications form subsets of the vector space

$$\mathscr{U} = \big\{\{\big(\mu^*(\alpha), \mu_*(\alpha), \alpha\big) : \alpha \in [0,1]\} : \mu^*, \mu_* : [0,1] \to \mathbb{R}\big\},$$

where μ^*, μ_* are bounded functions. $\mathscr{U}$ is metricized by the metric as follows:

$$\begin{aligned} d\Big(\big\{\big(\mu^*(\alpha), \mu_*(\alpha), \alpha\big) : \alpha \in [0,1]\big\}, \big\{\big(\nu^*(\alpha), \nu_*(\alpha), \alpha\big) : \alpha \in [0,1]\big\}\Big) \\ = \sup\big\{\max\{|\mu^*(\alpha) - \nu^*(\alpha)|, |\mu_*(\alpha) - \nu_*(\alpha)|\} : \alpha \in [0,1]\big\}. \end{aligned}$$

Let

$$\mathscr{U}_0 = \big\{\{\big(\mu^*(\alpha), \mu_*(\alpha), \alpha\big) : \alpha \in [0,1]\} : \mu^*, \mu_* : [0,1] \to [0, +\infty)\big\},$$

where μ^*, μ_* are bounded functions. It is easy verified that $\mathscr{U}_0$ is a closed convex cone in the topological vector space $(\mathscr{U}_0, d)$.

We now turn to review definitions of preinvex fuzzy mappings and B-vex fuzzy mappings in $(\mathscr{U}_0, d)$.

Definition 2.1 ([16]). *A set $X \subseteq \mathbb{R}^n$ is said to be invex with respect to mapping $\eta : X \times X \to \mathbb{R}^n$, if for every $x, y \in X$, $y + \lambda\eta(x,y) \in X$ and $0 \leq \lambda \leq 1$.*

Definition 2.2 ([16])**.** *Let $F : X \to \mathscr{F}_0$ be a fuzzy mapping defined on an invex set $X \subseteq \mathbb{R}^n$ and $X \neq \emptyset$, with respect to a mapping $\eta : X \times X \to \mathbb{R}^n$. F is said to be preinvex on X (with respect to η), if*

$$\lambda F(x) + (1-\lambda)F(y) - F\big(y + \lambda\eta(x,y)\big) \in \mathscr{U}_0, \tag{2.1}$$

for $\lambda \in [0,1]$ and $x, y \in X$.

Definition 2.3 ([16])**.** *At a point $x_0 \in S$, the fuzzy mapping $F : S \to \mathscr{F}_0$ is said to be:*

(1) *B-vex with respect to $b(x, x_0, \lambda)$ if, for all $x \in S$ and $\lambda \in [0,1]$,*

$$\lambda b(x,x_0,\lambda)F(x) + \big(1 - \lambda b(x,x_0,\lambda)\big)F(y) - F\big(\lambda x + (1-\lambda)x_0\big) \in \mathscr{U}_0, \tag{2.2}$$

(2) *strictly B-vex, with respect to $b(x, x_0, \lambda)$) if, for all $x \in S$, $x \neq x_0$ and $\lambda \in (0,1)$,*

$$\lambda b(x,x_0,\lambda)F(x) + \big(1 - \lambda b(x,x_0,\lambda)\big)F(y) - F\big(\lambda x + (1-\lambda)x_0\big) \in \mathscr{U}_0 \setminus \{\tilde{0}\}, \tag{2.3}$$

(3) *B-linear with respect to $b(x, x_0, \lambda)$ if, for all $x \in S$ and $\lambda \in [0,1]$,*

$$\lambda b(x,x_0,\lambda)F(x) + \big(1 - \lambda b(x,x_0,\lambda)\big)F(y) = F\big(\lambda x + (1-\lambda)x_0\big). \tag{2.4}$$

To end this section, let us recall some concepts of differentiability of a fuzzy mapping.

Definition 2.4 ([7])**.** *Let $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ and $\mu = (\mu_1, \mu_2, \cdots, \mu_n) \in \mathscr{F}^n$ be an n-dimensional real vector and an n-dimensional fuzzy vector, respectively. The product of a fuzzy vector is defined with a real vector as $\mu x^T = \sum_{i=1}^{n} \mu_i x_i$, which is a fuzzy number.*

Let $F : T \to \mathscr{F}_0$ be a fuzzy mapping. For any $\alpha \in [0,1]$, denote $[F(x)]^\alpha = [F^*(x,\alpha), F_*(x,\alpha)]$, where for each $\alpha \in [0,1]$, $F^*(\cdot,\alpha)$ and $F_*(\cdot,\alpha) : T \to \mathbb{R}$ are upper and lower functions of F, respectively.

Definition 2.5 ([7])**.** *Let $F : T \to \mathscr{F}_0$ be a fuzzy mapping, where $T \subset \mathbb{R}^n$ is an open set. Let $x = (x_1, x_2, \cdots, x_n) \in T$, and D_{x_i}, $i = 1, 2, \cdots, n$ stand for the partial differentiation with respect to the ith variable x_i. Assume that for all $\alpha \in [0,1]$, $F^*(x,\alpha)$ and $F_*(x,\alpha)$ have continuous partial derivatives so that $D_{x_i}F^*(x,\alpha)$ and $D_{x_i}F_*(x,\alpha)$ are continuous. Define*

$$[D_{x_i}F(x)]^\alpha = [D_{x_i}F^*(x,\alpha), D_{x_i}F_*(x,\alpha)],$$

for $i = 1, 2, \cdots, n$ and $\alpha \in [0,1]$. If for each i, $[D_{x_i}F(x)]^\alpha$ defines the α-level of a fuzzy number, then it is said that F is differentiable at x, and $\tilde{\nabla}F(x)$ is wrote by

$$\tilde{\nabla}F(x) = \big(D_{x_1}F(x,\alpha), D_{x_2}F(x,\alpha), \cdots, D_{x_n}F(x,\alpha)\big).$$

The partial derivative $\tilde{\nabla}F(x)$ is said to be the gradient of the fuzzy mapping F at x.

Definition 2.6 ([7]). *Let $F : T \to \mathscr{F}_0$ be a fuzzy mapping, where $T \subset \mathbb{R}^n$ is an open set. Let $x = (x_1, x_2, \cdots, x_n) \in T$, and $D_{x_i x_j}$, $i, j = 1, 2, \cdots, n$ stand for the second-order partial with respect to the ith variable x_i and jth variable x_j. Assume that $\tilde{\nabla} F(x)$ exists and for all $\alpha \in [0,1]$, $F^*(x,\alpha)$ and $F_*(x,\alpha)$ have continuous second-order partial derivatives so that $D_{x_i x_j} F^*(x,\alpha)$ and $D_{x_i x_j} F_*(x,\alpha)$ are continuous. Define*

$$[D_{x_i x_j} F(x)]^\alpha = [D_{x_i x_j} F^*(x,\alpha), D_{x_i x_j} F_*(x,\alpha)],$$

for $i, j = 1, 2, \cdots, n$ and $\alpha \in [0,1]$. If for each i, j, $[D_{x_i x_j} F(x)]^\alpha$ defines the α-level of a fuzzy number, then the Hessian of the fuzzy mapping (in the matrix notation) is defined as follows:

$$\tilde{\nabla}^2 F(x) = \big(D_{x_i x_j} F(x,\alpha)\big)_{i,j=1,2,\cdots,n}.$$

The fuzzy mapping F is said to be twice differentiable at x if the Hessian of the fuzzy mapping exists.

3. Properties about B-preinvex fuzzy mappings

Before approaching properties of B-preinvex fuzzy mappings, we first review some definitions of B-preinvex fuzzy mappings and fuzzy B-invex sets.

Definition 3.1 ([16]). *Let $F : X \to \mathscr{F}_0$ be a fuzzy mapping defined on an invex set $X \subseteq \mathbb{R}^n$ and $X \neq \emptyset$, with respect to a mapping $\eta : X \times X \to \mathbb{R}^n$. F is said to be B-preinvex on X (with respect to η) with respect to a mapping $b : X \times X \times [0,1] \to (0,1]$, if*

$$F\big(y + \lambda\eta(x,y)\big) \preceq \lambda b(x,y,\lambda) F(x) \tilde{+} \big(1 - \lambda b(x,y,\lambda)\big) F(y), \tag{3.1}$$

for $\lambda \in [0,1]$ and $x, y \in X$; and strictly B-preinvex with respect to η and b, if

$$F\big(y + \lambda\eta(x,y)\big) \prec \lambda b(x,y,\lambda) F(x) \tilde{+} \big(1 - \lambda b(x,y,\lambda)\big) F(y), \tag{3.2}$$

for $\lambda \in (0,1)$ and $x \neq y \in X$.

If (3.1) and (3.2) are reversed, then F is said to be B-preincave and strictly B-preincave on X with respect to η and b, respectively.

Definition 3.2 ([18]). *Let $F : X \to \mathscr{F}_0$ be a fuzzy mapping, where $X \subset \mathbb{R}^n$ is an open set, parameterized by*

$$F(x) = \Big\{\big(F^*(\alpha,x), F_*(\alpha,x), \alpha\big) : \alpha \in [0,1]\Big\}.$$

(1) *It is said that F is upper semi-continuous at $x_0 \in X$, if both $F^*(\alpha,x)$ and $F_*(\alpha,x)$ are upper semi-continuous at x_0 uniformly in $\alpha \in [0,1]$. F is upper semi-continuous on X, if it is upper semi-continuous at each point of X.*

(2) *It is said that* F *is lower semi-continuous at* $x_0 \in X$*, if both* $F^*(\alpha, x)$ *and* $F_*(\alpha, x)$ *are lower semi-continuous at* x_0 *uniformly in* $\alpha \in [0,1]$*.* F *is lower semi-continuous on* X*, if it is lower semi-continuous at each point of* X*.*

Lemma 3.1. *Let* X *be a non-empty invex set in* $\mathbb{R}^n$ *with respect to* $\eta : X \times X \to \mathbb{R}^n$*, and* $F : X \to \mathscr{F}_0$ *be a fuzzy mapping parameterized by*

$$F(x) = \Big\{\big(F^*(\alpha, x), F_*(\alpha, x), \alpha\big) : \alpha \in [0,1]\Big\}.$$

Then F *is B-preinvex on* X *with respect to* η *and* b*, if and only if for any* $\alpha \in [0,1]$*,* $F^*(\alpha, x)$ *and* $F_*(\alpha, x)$ *are B-preinvex on* X *with respect to* η *and* b*.*

The demonstration of Lemma 3.1 is analogous to the proof of Lemma 3.1 which provided by Li and Noor (2010) [7].

Definition 3.3 ([16])**.** *Given* $S \subseteq \mathbb{R}^n \times \mathscr{F}_0$*,* S *is said to be a fuzzy B-invex set with respect to mappings* $\eta : X \times X \to \mathbb{R}^n$ *and* $b : X \times X \times [0,1] \to (0,1]$*, if* $(x, \mu), (y, \nu) \in S$ *and* $\lambda \in [0,1]$ *implies that*

$$\big(y + \lambda\eta(x, y), \lambda b(x, y, \lambda)\mu \tilde{+} \big(1 - \lambda b(x, y, \lambda)\big)\nu\big) \in S. \tag{3.3}$$

In the following, some basic results of B-preinvex fuzzy mappings are presented without proof.

Theorem 3.1. *Let* X *be a non-empty invex subset of* $\mathbb{R}^n$ *with respect to* $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$*, and* $F_i : X \to \mathscr{F}_0$*,* $(i = 1, 2, \cdots, n)$ *be B-preinvex fuzzy mappings with respect to the same mappings* η *and* b*. Then the function* $F : X \to \mathscr{F}_0$ *defined by*

$$F(x) = \sum_{i=1}^{n} a_i F_i(x), \quad a_i \geq 0$$

is a B-preinvex fuzzy mapping on X *with respect to the same mappings* η *and* b*.*

Theorem 3.2. *Let* X *be a non-empty invex subset of* $\mathbb{R}^n$ *with respect to* $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$*, and* $F_i : X \to \mathscr{F}_0$*,* $(i \in I = \{1, 2, \cdots, n\})$ *be B-preinvex fuzzy mappings with respect to the same mappings* η*,* b*. Then the function* $F : X \to \mathscr{F}_0$ *defined by* $F(x) = \sup_{i \in I} F_i(x)$ *is a B-preinvex fuzzy mapping on* X *with respect to the same mappings* η *and* b*.*

Theorem 3.3. *Let* X *be a non-empty invex subset of* $\mathbb{R}^n$ *with respect to* $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$*. A fuzzy mapping* $F : X \to \mathscr{F}_0$ *is a B-preinvex fuzzy mapping on* X *with respect to mappings* η*,* b*, if and only if for all* $x, y \in X$*,* $\mu, \nu \in \mathscr{F}_0$ *and* $\lambda \in (0,1)$ *such that* $F(x) \prec \mu$*,* $F(y) \prec \nu$ *and* $F\big(y + \lambda\eta(x, y)\big) \preceq \lambda b(x, y, \lambda)\mu \tilde{+} \big(1 - \lambda b(x, y, \lambda)\big)\nu$*.*

After the above results are presented, we now give the following conclusion, which a characterization of B-preinvex fuzzy mappings F in terms of its epigraph $epi(F)$ is given by $epi(F) = \{(x, \mu) : x \in X, \mu \in \mathscr{F}_0, F(x) \preceq \mu\}$.

Theorem 3.4. *Let X be a non-empty invex subset of $\mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. A fuzzy mapping $F : X \to \mathscr{F}_0$ is a B-preinvex fuzzy mapping on X with respect to mappings η and b, if and only if its epigraph $epi(F)$ is a fuzzy B-invex set in $\mathbb{R}^n \times \mathscr{F}_0$ with respect to η and b.*

Proof. Let $F : X \to \mathscr{F}_0$ be a B-preinvex fuzzy mapping and (x, μ), $(y, \nu) \in epi(F)$ with $x, y \in X$ and $\mu, \nu \in \mathscr{F}_0$, then $F(x) \preceq \mu$ and $F(y) \preceq \nu$. Since F is B-preinvex on X and addition and nonnegative scalar multiplication preserve the order on $\mathscr{F}_0$, it follows that

$$\begin{aligned} F\big(y + \lambda\eta(x, y)\big) &\preceq \lambda b(x, y, \lambda)F(x)\tilde{+}\big(1 - \lambda b(x, y, \lambda)\big)F(y) \\ &\preceq \lambda b(x, y, \lambda)\mu\tilde{+}\big(1 - \lambda b(x, y, \lambda)\big)\nu, \end{aligned}$$

for $\lambda \in [0, 1]$, which implies that

$$\big(y + \lambda\eta(x, y), \lambda b(x, y, \lambda)\mu\tilde{+}\big(1 - \lambda b(x, y, \lambda)\big)\nu\big) \in epi(F).$$

Thus, $epi(F)$ is a fuzzy B-invex set with respect to η, b.

Conversely, assume that $epi(F)$ is a fuzzy B-invex set and $x, y \in X$, so $\big(x, F(x)\big)$, $\big(y, F(y)\big) \in epi(F)$. Meanwhile, $epi(F)$ is a fuzzy B-invex set with respect to η, b, it yields that

$$\big(y + \lambda\eta(x, y), \lambda b(x, y, \lambda)F(x)\tilde{+}\big(1 - \lambda b(x, y, \lambda)\big)F(y)\big) \in epi(F).$$

which implies that, for $\lambda \in [0, 1]$,

$$F\big(y + \lambda\eta(x, y)\big) \preceq \lambda b(x, y, \lambda)F(x)\tilde{+}\big(1 - \lambda b(x, y, \lambda)\big)F(y).$$

Hence, F is a B-preinvex fuzzy mapping with respect to η and b, which completes the proof.

Now, we begin to study the sufficient conditions of B-preinvex fuzzy mappings. To discuss this problem, we need the following well known Condition C which introduced by Mohan and Neogy (1995) [8].

Condition C: It is said that the function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ satisfied Condition C, if for any $x, y \in \mathbb{R}^n$,

$$\eta(y, y + \lambda\eta(x, y)) = -\lambda\eta(x, y), \qquad \eta(x, y + \lambda\eta(x, y)) = (1 - \lambda)\eta(x, y)$$

are satisfied for $\lambda \in [0, 1]$.

Theorem 3.5. *Let X be a non-empty invex subset of $\mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, where η satisfies Condition C. Assume that $F : X \to \mathscr{F}_0$ is an upper semi-continuous fuzzy mapping and satisfy $F\big(y+\eta(x, y)\big) \preceq b(x, y, \lambda)F(x)$ for $\forall x, y \in X$, and if there exists a $t \in (0, 1)$ such that*

$$F\big(y + t\eta(x, y)\big) \preceq tb(x, y, \lambda)F(x)\tilde{+}\big(1 - tb(x, y, \lambda)\big)F(y),$$

for all $x, y \in X$, then F is a B-preinvex fuzzy mapping on X with respect to mappings η and b.

Proof. The proof is by contradiction. Suppose that F is not a B-preinvex fuzzy mapping, so there exist $x, y \in X$ and $\bar{\lambda} \in (0,1)$ such that

$$F\big(y + \bar{\lambda}\eta(x,y)\big) \succ \bar{\lambda}b(x,y,\lambda)F(x)\tilde{+}\big(1-\bar{\lambda}b(x,y,\lambda)\big)F(y), \tag{3.4}$$

i.e., there exist $x, y \in X$ and $\bar{\lambda} \in (0,1)$ for some $\alpha_0 \in [0,1]$ such that

$$F^*\big(\alpha_0, y + \bar{\lambda}\eta(x,y)\big) > \bar{\lambda}b(x,y,\lambda)F^*(\alpha_0,x) + \big(1-\bar{\lambda}b(x,y,\lambda)\big)F^*(\alpha_0,y), \tag{3.5}$$

or

$$F_*\big(\alpha_0, y + \bar{\lambda}\eta(x,y)\big) > \bar{\lambda}b(x,y,\lambda)F_*(\alpha_0,x) + \big(1-\bar{\lambda}b(x,y,\lambda)\big)F_*(\alpha_0,y). \tag{3.6}$$

Now let

$$g(\lambda) = F^*\big(\alpha_0, y + \lambda\eta(x,y)\big) - \lambda b(x,y,\lambda)F^*(\alpha_0,x) - \big(1-\lambda b(x,y,\lambda)\big)F^*(\alpha_0,y).$$

According to the Definition 3.2 and from the assumption F is an upper semi-continuous fuzzy mapping, $F^*(\alpha,x)$ is an upper semi-continuous real-valued function. Then, $g(\lambda)$ also is upper semi-continuous real-valued function in interval $[0,1]$. Therefore, $g(\lambda)$ exists maximum $M_0 > 0$ (due to (3.5)) in interval $[0,1]$. Let $\lambda_0 = \max\{\lambda \in [0,1] : g(\lambda) = M_0\}$, it follows that

$$\begin{aligned} g(0) =&0, \\ g(1) =&F^*(\alpha_0, y+\eta(x,y)) - b(x,y,1)F^*(\alpha_0,x). \end{aligned}$$

By the conditions $F(y+\eta(x,y)) \preceq b(x,y,\lambda)F(x)$ for $\forall x, y \in X$, $\lambda \in [0,1]$ and maximum $M_0 > 0$. It can easily be shown that $g(1) \le 0$ and $\lambda_0 \in (0,1)$. For simplicity, let $b(x,y,\lambda) \triangleq b$. Choose a $\delta > 0$ such that

$$\big(b\lambda_0 - (1-tb)\delta, b\lambda_0 + tb\delta\big) \subset (0,1).$$

Let $b\lambda_2 = b\lambda_0 - (1-tb)\delta$ and $b\lambda_1 = b\lambda_0 + tb\delta$, $\tilde{x} = y + \lambda_2\eta(x,y)$ and $\tilde{y} = y + \lambda_1\eta(x,y)$. Obviously, $\lambda_0 = tb\lambda_2 + (1-tb)\lambda_1$, $\lambda_1 = \lambda_2 + \delta$ and $\lambda_1, \lambda_2 \ne \lambda_0$. By the Condition C, we have

$$\tilde{y} + t\eta(\tilde{x},\tilde{y}) = y + \lambda_0\eta(x,y).$$

Combining Lemma 3.1, it is easy to see that $F^*(\alpha,x)$ is B-preinvex on X with respect to b. Thus,

$$\begin{aligned} M_0 = g(\lambda_0) &= F^*\big(\alpha_0, y+\lambda_0\eta(x,y)\big) - \lambda_0 bF^*(\alpha_0,x) - (1-\lambda_0 b)F^*(\alpha_0,y) \\ &= F^*\big(\alpha_0, \tilde{y}+t\eta(\tilde{x},\tilde{y})\big) - \lambda_0 bF^*(\alpha_0,x) - (1-\lambda_0 b)F^*(\alpha_0,y) \\ &\le tbF^*(\alpha_0,\tilde{x}) - (1-\lambda_0 b)F^*(\alpha_0,\tilde{y}) - \lambda_0 bF^*(\alpha_0,x) - (1-\lambda_0 b)F^*(\alpha_0,y) \\ &= tbF^*(\alpha_0,\tilde{x}) - (1-tb)F^*(\alpha_0,\tilde{y}) \\ &- \big(tb\lambda_2 + (1-tb)\lambda_1\big)bF^*(\alpha_0,x) - \big(1-\big(tb\lambda_2+(1-tb)\lambda_1\big)b\big)F^*(\alpha_0,y) \\ &= tb[F^*(\alpha_0,\tilde{x}) - \lambda_2 bF^*(\alpha_0,x) - (1-\lambda_2 b)F^*(\alpha_0,y)] \\ &+ (1-tb)[F^*(\alpha_0,\tilde{y}) - \lambda_1 bF^*(\alpha_0,x) - (1-\lambda_1 b)F^*(\alpha_0,y)] \\ &= tb[F^*\big(\alpha_0, y+\lambda_2\eta(x,y)\big) - \lambda_2 bF^*(\alpha_0,x) - (1-\lambda_2 b)F^*(\alpha_0,y)] \\ &+ (1-tb)[F^*\big(\alpha_0, y+\lambda_1\eta(x,y)\big) - \lambda_1 bF^*(\alpha_0,x) - (1-\lambda_1 b)F^*(\alpha_0,y)] \\ &= tbg(\lambda_2) + (1-tb)g(\lambda_1) < tbM_0 + (1-tb)M_0 = M_0. \end{aligned}$$

This contradiction proves the result that F is a B-preinvex fuzzy mapping on X with respect to mappings η and b.

By a similar way, using Lemma 3.1 and Definition 3.2, it is easy to deduce the following Theorem 3.6.

Theorem 3.6. *Let X be a non-empty invex subset of $\mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, where η satisfies Condition C. Assume that $F : X \to \mathscr{F}_0$ is an lower semi-continuous fuzzy mapping and satisfy $F(y+\eta(x,y)) \preceq b(x,y,\lambda)F(x)$ for $\forall x, y \in X$. If there exists a $t \in (0,1)$ such that*

$$F\big(y + t\eta(x,y)\big) \preceq tb(x,y,\lambda)F(x)\tilde{+}\big(1 - tb(x,y,\lambda)\big)F(y),$$

for all $x, y \in X$, then F is a B-preinvex fuzzy mapping on X with respect to mappings η and b.

4. Main results

In this section, we will present and prove several necessary conditions for differentiable and twice differentiable of B-preinvex fuzzy mappings satisfied the famous condition C which introduced by Mohan and Neogy (1995) [8].

Theorem 4.1. *Let X be a non-empty open invex set in $\mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, where $\eta(x,y) \geq 0$ for any $x, y \in X$, satisfies Condition C. Assume that $F : X \to \mathscr{F}_0$ is a differentiable B-preinvex fuzzy mapping, the mapping $b(x,y,\lambda)$ is continuous at $\lambda = 0$ for fixed $x, y \in X$, and $b(x,y) = \lim_{\lambda\to 0^+} b(x,y,\lambda)$. Then, for any $x, y \in X$,*

$$b(x,y)F(x) \succeq \tilde{\nabla}F(y)\eta(x,y)^T\tilde{+}b(x,y)F(y). \tag{4.1}$$

Proof. Since F is a B-preinvex fuzzy mapping with respect to η and $b(x,y,\lambda)$, then for all $x, y \in X$ and $\lambda \in [0,1]$,

$$F\big(y + \lambda\eta(x,y)\big) \preceq \lambda b(x,y,\lambda)F(x)\tilde{+}\big(1 - \lambda b(x,y,\lambda)\big)F(y),$$

i.e., for all $x, y \in X$, $\alpha \in [0,1]$ and $\lambda \in [0,1]$,

$$F^*\big(\alpha, y + \lambda\eta(x,y)\big) \leq \lambda b(x,y,\lambda)F^*(\alpha,x) + \big(1 - \lambda b(x,y,\lambda)\big)F^*(\alpha,y) \tag{4.2}$$

and

$$F_*\big(\alpha, y + \lambda\eta(x,y)\big) \leq \lambda b(x,y,\lambda)F_*(\alpha,x) + \big(1 - \lambda b(x,y,\lambda)\big)F_*(\alpha,y). \tag{4.3}$$

By the differentiability of F and according to the mean-valued theorem, we have

$$F^*\big(\alpha, y + \lambda\eta(x,y)\big) = F^*(\alpha,y) + \lambda\nabla F^*(\alpha,\varepsilon)\big(\eta(x,y)\big)^T \tag{4.4}$$

and

$$F_*\big(\alpha, y + \lambda\eta(x,y)\big) = F_*(\alpha,y) + \lambda\nabla F_*(\alpha,\zeta)\big(\eta(x,y)\big)^T, \tag{4.5}$$

where $\varepsilon = y + \theta_1\lambda\eta(x,y)$, $\zeta = y + \theta_2\lambda\eta(x,y)$ and $0 < \theta_1, \theta_2 < 1$. Combining the above inequalities (4.2) and (4.3) and the equalities (4.4) and (4.5), it follows that

$$\lambda b(x,y,\lambda)F^*(\alpha,x) \geq \lambda\nabla F^*(\alpha,y)\big(\eta(x,\varepsilon)\big)^T + \lambda b(x,y,\lambda)F^*(\alpha,y) \tag{4.6}$$

and

$$\lambda b(x,y,\lambda)F_*(\alpha,x) \geq \lambda\nabla F_*(\alpha,y)\big(\eta(x,\zeta)\big)^T + \lambda b(x,y,\lambda)F_*(\alpha,y). \tag{4.7}$$

Dividing the inequalities (4.6) and (4.7) by λ and taking $\lambda \to 0^+$, then $\theta_1 \to 0^+$, $\theta_2 \to 0^+$. It is easy to verify that

$$b(x,y)F^*(\alpha,x) \geq \nabla F^*(\alpha,y)\big(\eta(x,y)\big)^T + b(x,y)F^*(\alpha,y)$$

and $b(x,y)F_*(\alpha,x) \geq \nabla F_*(\alpha,y)\big(\eta(x,y)\big)^T + b(x,y)F_*(\alpha,y)$. That is, $b(x,y)F(x) \succeq \tilde{\nabla}F(y)\eta(x,y)^T\tilde{+}b(x,y)F(y)$ for any $x,y \in X$ and $\lambda \in [0,1]$. Hence, the statement in Theorem 4.1 is proved.

Corollary 4.1. *Let X be a non-empty open invex set in $\mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, where $\eta(x,y) \geq 0$ for any $x,y \in X$, and η satisfy Condition C. Suppose that $F : X \to \mathscr{F}_0$ is a differentiable B-preinvex fuzzy mapping. Then, for all $x,y \in X$,*

$$b(y,x)\tilde{\nabla}F(y)\big(\eta(x,y)\big)^T\tilde{+}b(x,y)\tilde{\nabla}F(x)\big(\eta(y,x)\big)^T \preceq \tilde{0}. \tag{4.8}$$

Proof. Since F is a differentiable B-preinvex fuzzy mapping, and by using Theorem 4.1, it is easy to show that, for all $x,y \in X$,

$$b(x,y)F^*(\alpha,x) \geq \nabla F^*(\alpha,y)\big(\eta(x,y)\big)^T + b(x,y)F^*(\alpha,y), \tag{4.9}$$

$$b(x,y)F_*(\alpha,x) \geq \nabla F_*(\alpha,y)\big(\eta(x,y)\big)^T + b(x,y)F_*(\alpha,y), \tag{4.10}$$

$$b(y,x)F^*(\alpha,y) \geq \nabla F^*(\alpha,x)\big(\eta(y,x)\big)^T + b(y,x)F^*(\alpha,x), \tag{4.11}$$

and

$$b(y,x)F_*(\alpha,y) \geq \nabla F_*(\alpha,x)\big(\eta(y,x)\big)^T + b(y,x)F_*(\alpha,x). \tag{4.12}$$

Multiplying inequality (4.9) by $b(y,x)$ and plus multiplying inequality (4.11) by $b(x,y)$, it follows that for $x,y \in X$,

$$b(y,x)\nabla F^*(\alpha,y)\big(\eta(x,y)\big)^T + b(y,x)\nabla F^*(\alpha,x)\big(\eta(y,x)\big)^T \leq 0. \tag{4.13}$$

By the same way, Multiplying inequality (4.10) by $b(y,x)$ and plus multiplying inequality (4.12) by $b(x,y)$, it yields that for $x,y \in X$,

$$b(y,x)\nabla F_*(\alpha,y)\big(\eta(x,y)\big)^T + b(y,x)\nabla F_*(\alpha,x)\big(\eta(y,x)\big)^T \leq 0. \tag{4.14}$$

Hence, from the above two inequalities (4.13) and (4.14), the desired conclusion (4.8) is obtained.

Theorem 4.2. *Let X be a non-empty open invex subset of $\mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, where $\eta(x,y) \geq 0$ for any $x, y \in X$ and satisfies Condition C. Suppose that $F : X \to \mathscr{F}_0$ is a twice differentiable B-preinvex fuzzy mapping. Then, for any $x, y \in X$,*

$$\eta(x,y)\tilde{\nabla}^2 F(y)\eta(x,y)^T \succeq 0. \tag{4.15}$$

Proof. Suppose that F is a B-preinvex fuzzy mapping on X with respect to mappings η and b. Then, from Lemma 3.1, for any $\alpha \in [0,1]$, $F^*(\alpha, x)$ and $F_*(\alpha, x)$ are B-preinvex on X with respect to η and b. Thus, the extreme functions $F^*(\alpha, x)$ and $F_*(\alpha, x)$ verify the relation (4.15). Replacing $\tilde{\nabla}$ by ∇, for any $\alpha \in [0,1]$ and $\eta(x,y) \geq 0$, F verify the relation (4.15). The proof is completed.

A necessary and sufficient conditions for B-preinvex fuzzy mappings is stated below.

Theorem 4.3. *Let X be a non-empty invex subset of $\mathbb{R}^n$ with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, where η satisfies Condition C. Assume that $F : X \to \mathscr{F}_0$ is a fuzzy mapping and satisfies $F\big(y + \eta(x,y)\big) \preceq F(x)$ for all $x, y \in X$. Then F is a B-preinvex fuzzy mapping, if and only if, for $\forall x, y \in X$,*

$$\varphi(\lambda) = F\big(y + \lambda\eta(x,y)\big) \tag{4.16}$$

is a B-invex mapping on $[0,1]$ with respect to b.

Proof. Suppose that F is a B-preinvex fuzzy mapping on X, for any fixed $x, y \in X$ and $\lambda, \alpha_1, \alpha_2 \in [0,1]$, if $\alpha_1 = \alpha_2$, then the result is obvious.

If $\alpha_1 < \alpha_2$, then $\alpha_2 - \alpha_1 > 0$ and $\alpha_1 \neq 1$, thus we have $0 < (\alpha_2 - \alpha_1)/(1 - \alpha_1) < 1$. From Condition C, we have that, for any $x, y \in X$ and $\alpha \in [0,1]$,

$$\eta\big(y + \alpha\eta(x,y), y\big) = -\eta\big(y, y + \alpha\eta(x,y)\big) = \alpha\eta(x,y). \tag{4.17}$$

Combining the equality (4.17) and Condition C, it follows that

$$\begin{aligned} \eta\big(y + \alpha_2\eta(x,y), y + \alpha_1\eta(x,y)\big) &= (\alpha_2 - \alpha_1)/(1 - \alpha_1)\eta\big(x, y + \alpha_1\eta(x,y)\big) \\ &= (\alpha_2 - \alpha_1)\eta(x,y). \end{aligned} \tag{4.18}$$

By the equality (4.18) and according to the B-preinvexity of F, thus for any fixed $x, y \in X$,

$$\begin{aligned} &\varphi\big(\alpha_1 + \lambda(\alpha_2 - \alpha_1)\big) \\ &= F\big(y + \alpha_1\eta(x,y) + \lambda(\alpha_2 - \alpha_1)\eta(x,y)\big) \\ &= F\big(y + \alpha_1\eta(x,y) + \lambda\eta\big(y + \alpha_2\eta(x,y), y + \alpha_1\eta(x,y)\big)\big) \\ &\preceq \lambda b(x,y,\lambda)F\big(y + \alpha_2\eta(x,y)\big)\tilde{+}\big(1 - \lambda b(x,y,\lambda)\big)F\big(y + \alpha_1\eta(x,y)\big) \\ &= \lambda b(x,y,\lambda)\varphi(\alpha_2) + \big(1 - \lambda b(x,y,\lambda)\big)\varphi(\alpha_1). \end{aligned} \tag{4.19}$$

By the same way, if $\alpha_1 > \alpha_2$, we have that, for any fixed $x, y \in X$,

$$\varphi\big(\alpha_2 + \lambda(\alpha_1 - \alpha_2)\big) \le \lambda b(x,y,\lambda)\varphi(\alpha_1) + \big(1 - \lambda b(x,y,\lambda)\big)\varphi(\alpha_2). \tag{4.20}$$

Utilizing the inequalities (4.19) and (4.20), it yields that $\varphi(\lambda)$ is a B-invex function on [0,1].

Conversely, since $\varphi(\lambda) = F\big(y + \lambda\eta(x,y)\big)$ is a B-invex function and $F\big(y + \eta(x,y)\big) \preceq F(x)$, then we have, for any $x, y \in X$ and $\lambda \in [0,1]$,

$$\begin{aligned} F\big(y + \lambda\eta(x,y)\big) &= \varphi(\lambda) = \varphi\big(\lambda \cdot 1 + (1-\lambda) \cdot 0\big) \\ &\le \lambda b(x,y,\lambda)\varphi(1) + \big(1 - \lambda b(x,y,\lambda)\big)\varphi(0) \\ &= \lambda b(x,y,\lambda)F(y + \eta(x,y)) + \big(1 - \lambda b(x,y,\lambda)\big)F(y) \\ &\preceq b(x,y,\lambda)F(x) + \big(1 - \lambda b(x,y,\lambda)\big)F(y). \end{aligned}$$

Thus, F is a B-preinvex fuzzy mapping with respect to b and η. This ends the proof.

5. Conclusions

The first conclusion to be draw is that some results about sufficient conditions of B-preinvex fuzzy mappings, which be presented and proved under certain conditions. The second one is the necessary conditions for differentiable and twice differentiable B-preinvex fuzzy mappings. A point that should be stressed is that these results discussed here are on the B-preinvex fuzzy mappings. As a consequence, these results are the extension of results from Li and Noor (2010) in [7] and Rufián-Lizana et al. (2014) in [15].

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