STOCHASTIC FINANCIAL MODEL BASED ON FRACTIONAL BROWN MOTION

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Abstract. Fractional brown motion (FBM) is featured by long-term memory and selfsimilarity compared to standard brown motion. Because of the properties, it can be used to describe the phenomenon (e.g. seasonal effect, scale effect and sharp peak and heavy tail) which cannot be described by some typical analytical methods in financial market. The fractal features of fractional brown motion make it a more suitable tool in financial studies. This study simulated the increment of FBM and the square of the increment using extended Maruyama symbols as well as the change path of underlying asset price and obtained the formula for European option pricing using insurance actuary pricing.
Keywords: frictional Brown motion, simulation, underlying asset price, share option.

1. Introduction

Fractional Brown motion (FBM) was first studied by Kolmogorov using Hilbert spatial framework and introducing the definition of Wiener helix in the 1940s. Mandelbrot deeply discussed FBM and formally proposed the concept of fractional Brown motion along with Van Ness in 1960s. They gave out the accurate definition of FBM and introduced its properties such as self-similarity, non-independence and differentiability [1] and used covariance function to express the correlation between increments of FBM. With the constant development of modern financial market, it has been found that, the initial assumption and actual situation of Black-Scholes model have difference [2] and the standard Brown motion motivation model has not been able to give reasonable explanations for more and more phenomena such as seasonal effect, scale effect and sharp peak and heavy tail [3]. Researchers also found that describing the change of asset price with FBM was more suitable than with standard Brown motion. The

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reasons for the above conclusion are as follows. Asset price will show the properties of Brown motion (martingale property, Markov property, etc.) when the standard Brown motion is used to describe the change of asset price, suggesting the price of underlying asset at a certain time point in the future is associated to the present rather than the past [4]. Obviously, the conclusion is unpractical and disobeys the thought of people; compared to standard Brown motion, FBM does not have Markov property and semimartingale [5], thus it is more suitable for describing the change of asset price. Long-term memory and self-similarity are two properties of FBM, i.e., the price of underlying asset at some time point is associated to both the present and the past in words of financial language, which has been recognized by people.

The properties of FBM make FBM be able to describe the evolution process of asset price in financial market better, but FBM has its limitations [6]. In general sense of integral, FBM does not have martingale property, and moreover market has the phenomenon of interest arbitrage. Thus the studies on the application of FBM in financial system mainly have two orientations, the first is to study the memory of stock based on FBM (the memory of yield rate and fluctuation rate) and the second is to study the integral significance under which FBM has no interest arbitrage [7, 8] and its option pricing.

This study firstly introduced the definition and properties of FBM and then deduced that the model of $dS_t^\varepsilon = \mu S_t^\varepsilon dt + \sigma S_t^\varepsilon ddB_t^\varepsilon$, $S_t^\varepsilon|_{t=0} = S_0$ was arbitrage-free for $\forall\varepsilon > 0$ based on the approach of FBM. Besides, the change path of underlying asset and European option pricing were studied.

2. The concept and properties of FBM

The concept of FBM

In 1905, Albert Einstein made a physical analysis on the motion of Brownian particles as the pioneer of the dynamic theory of Brown motion and proposed the mathematical model which was applicable to Brown motion. In 1923, Norbert Wiener proposed the concept of Wiener space based on the definition of measure and integral on Brown motion space and moreover gave a strict mathematical definition for Brown motion [9]. In 1968, Van Ness and Benoit Mandebrot proposed the concept of FBM; FBM is the most extensively used model and its specific definition is as follows.

Suppose $B_H = \{B_H(t,\omega), t > 0\}$ as a random process. When $0 < H < 1$, is a set of random function values and $B_H(0,\omega) = b_0$ is an arbitrary real number, then the random process $\{B_H(t,\omega), t > 0\}$ is a FBM with Hurst parameter H:

$$B_H(t,\omega) - B_H = \frac{1}{\Gamma(1+\alpha)}\{\int_{-\infty}^{0}[(t-s)^\alpha - (-\sigma)^\alpha]dB(s,\omega) + \int_0^t (t-s)^\alpha dB(s,\omega)\}$$

in which, $\alpha = H - \frac{1}{2}$ and $\Gamma(1+|\alpha) = \int_0^\infty x^\alpha e^{-x}$; if $b_0 = 0$, $H = \frac{1}{2}$, then $B_H(t,\omega)$ is a standard Brown motion, reflected as random walk; FBM can be discussed by being divided into three sets, i.e., $0 < H < \frac{1}{2}$, $H = \frac{1}{2}$ and $\frac{1}{2} < H < 1$.

The property of FBM

When $0 < Hurstindex(H) < 1$, FMB has the following properties:

(1) Hurst index is quite important for FBM as it determines the covariance in the past and future. Covariance function is $E[B_H(t)B_H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$;

(2) FBM is a self-similar process.

(3) The increments of FBM are not independent [10]. When $0 < H < \frac{1}{2}$, the correlation coefficient is negative and time sequence is anti-persistent; when $H = \frac{1}{2}$, time sequence is independent identically distributed random sequence, obeying standard normal distribution [11] and showing the feature of random walk; when $\frac{1}{2} < H < 1$, it has long-term memory.

3. The approach and increment of FBM

Approach

As FBM with $H1/2$ is neither a semimartingale process nor a Markoff process, to make it adapt to financial theory application, Thao proposed a semimartingale process to approach FBM.

Lemma. For $\forall\varepsilon > 0$, $B_t^\varepsilon = \int_0^t (t-s+\varepsilon)^{H-\frac{1}{2}}$, $H \neq \frac{1}{2}$, $0 < H < 1$, was defined; then the process was a semimartingale and for $\forall t \in [0,T]$, B_t^ε uniformly converged to Bt in $L^2(\Omega)$ space when $\varepsilon \to 0$.

Lemma. For $\forall\varepsilon > 0$, the approximation FBM model $dS_t^\varepsilon = \mu S_t^\varepsilon dt + \sigma S_t^\varepsilon dB_t^\varepsilon$, $S_t^\varepsilon|_{t=0} = S_0$ is arbitrage-free.

Increment

Maruyama proposed to express the increment of the standard Brown motion with a symbol of $db(t) = \omega(t)(dt)^{1/2}$. In 2004, Guy Jumarie extended the symbol to H-order FBM derivatively defined by Riemann-Liouville using fractional-order Taylor expansion [12]. If $B(t,H)$ stands for H-order FBM, the increment of FBM can be expressed as $dB(t,H) = \omega(t)(dt)^H$, $0 < H < 1$, after the extension of $db(t) = \omega(t)(dt)^{1/2}$ proposed by Maruyama.

4. Simulation method

Because of the features of FBM, the model motivated by FBM can describe data flow more real compared to traditional models [13]. In the environment of FBM, queuing systems which are concerned more have already output some results;

however, how to simulate the process, for example, simulating the change of underlying asset price to obey a random differential equation model , should be known to solve practical problems. Usually, researchers simulate FBM firstly and then its increments. Simulation process can illustrate some simulation methods of FBM through exact and approach [14]. Exact methods used for simulating FBM mainly include Cholesky method, Hosking method, Davies method and Harte method. Approach methods for simulating FBM include random midpoint displacement, wavelet transform method, spectrum analysis method and random expression method. Random expression method and spectrum analysis method are frequently used [15].

Random expression method refers to directly discretize expression of FBM proposed by Mandelbrot

$$\underset{\widetilde{B}}{H}(n) = C_H\{\sum_{k=-b}^{0}[(n-k)^{\alpha} - (-k)^{\alpha}]B_1(k) + \sum_{k=0}^{n}(n-k)^{\alpha}B_2(k)\}/.$$

The procedures of spectrum analysis are as follows. Firstly, time interval is divided to discretize the increment of FBM, i.e. $\underset{\widetilde{B}}{H}(n) = C_H\{\sum_{k=-b}^{0}[(n-k)^{\alpha} - (-k)^{\alpha}]B_1(k) + \sum_{k=0}^{n}(n-k)^{\alpha}B_2(k)\}$, then variance function is calculated, followed by spectral density function and the increment of FBM.

5. Random simulation and European option pricing

The simulation of underlying asset price

The price of underlying asset under the condition of $1/2 < H < 1/2$ was simulated. Its price change satisfies fractional stochastic differential equation:

$$dS = \mu S dt + \sigma S dB_H + \lambda S(dB_H)^2$$

μ stands for the expected return rate of stock, σ stands for the fluctuation rate of stock yield, and λ stands for the disturbance term of stock.

Compared to the standard Brown motion motivated model, FBM model can describe the long-term memory of stock yield and its increment is not independent.

The simulation method used in this study was different from the methods mentioned above. The extended symbol $dB(t, H) = \omega(t)(dt)^H$ was used to simulate the increment of FBM. Hurst index H was specified as 0.42 and $S(t_j) = 60$. Firstly, the increment of FBM was simulated and then the change path of asset price was simulated using Monte Carlo simulation method. Figure 1 shows the flow of the simulation. (1) The simulation of FBM increments It has been mentioned above that, Hurst index H was specified as 0.42 and time interval $(0, T)$ was divided into N parts, for each part. For each equidistant interval, the increment of FBM was discretized, i.e., $\Delta B(t_j, H) = B(t_j + \Delta t, H)B(t_j, H) = \omega(t)(\Delta t)^H$.

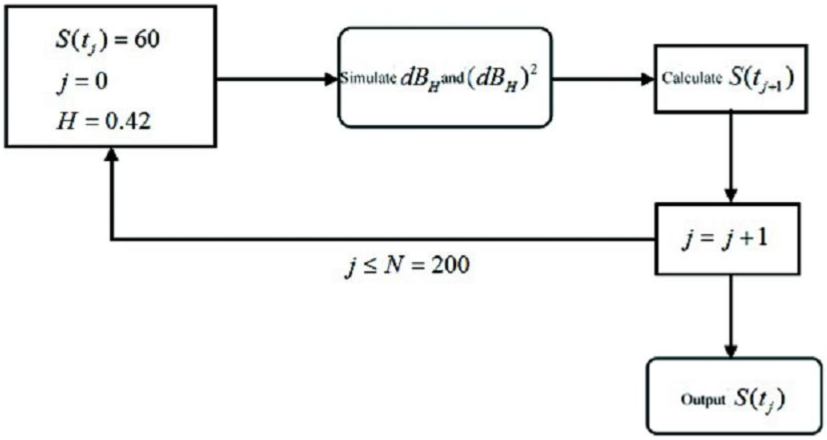

Figure 1: The construction process of decision tree

If the validity period T = 1;N = 200 and t = 0:005, the increment of FBM was simulated according to the above formula. (2) The simulation of underlying asset price

Suppose that the price change of underlying asset satisfies the following fractional stochastic differential equation:

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dB_H + \lambda_1 S_1 (dB_H)^2, 1/3 < H < 1/2. \tag{5.1}$$

As $dB(t,H) = \omega_1(t)(dt)^H$ and $(dB(t,H))^2 = \omega_2(t)(dt)^{2H}$, the formula (1) could be written as:

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 \omega_1(t)(dt)^H + \lambda_1 S_1 \omega_2(t)(dt)^{2H}, 0 < H < 1 \tag{5.2}$$

in which, $\omega_1(t)$ and $\omega_2(t)$ are independent, obeying standard normal distribution with an average value of 0 and a variance of 1.

The validity period of contract $[0,T]$ was divided into N small equidistant intervals, $t_0, t_1, t_2, t_3, \ldots, t_N$ and the length of the interval was expressed as Δt. Then formula (2) was discretized, i.e.

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 \omega_1(t)(dt)^H + \lambda_1 S_1 \omega_2(t)(dt)^{2H}, \tag{5.3}$$

$\Delta S_1(t_j) = S_1(t_{j+1}) S_1(t_j)$. For each small interval, when $j = 1,2,\ldots,N$, there was

$$\begin{aligned} S_1(t_{j+1}) &= S_1(t_j) + \mu_1 S_1(t_j)\Delta t + \sigma_1 S_1(t_j)\omega_1(t)(\Delta t)^H \\ &\quad + \lambda_1 S_1(t_j)\omega_2)(t)(\Delta t)^{2H}. \end{aligned} \tag{5.4}$$

Then Monte Carlo simulation method was used to simulate the price change path of underlying asset [16]. The values of relevant parameters were as follows:

$H = 0.42, T = 1$, S_0 (the initial price of underlying asset) $= 60, \mu_1$ (the expected yield rate of stock) $= 0.11$, σ_1 (the fluctuation rate of stock) $= 0.36$ and λ_1 (the small disturbance term of stock yield) $= 0.082$.

Matlab software was used to perform Monte Carlo simulation on the change path of underlying asset price. Then a sample path ($H = 0.42$) was obtained. The sample path of the price of such underlying asset was simulated for several times. The approximate price of underlying asset at each time point, i.e., the change path of the price of underlying asset, could be obtained based on the average value of the price of underlying asset at each time point [17].

The actuarial method for European style option

Compared to the traditional option pricing method, the actuarial method is in no need of any assumptions about financial market while being used to study option pricing. Therefore, it is effective for any market (arbitrage, arbitrage-free, complete or incomplete markets) and it can also make option pricing be understood easily [18].

Definition 5.1. The expected yield rate of stock price $S(t)$ was $e^{\beta t} = \frac{ES(t)}{S(0)}$ in the time period of $[0, t]$.

Definition 5.2. European options would be exercised only when the difference between the present worth of stock which was discounted at due date according to the expected rate and the present worth of exercise price which was discounted according to risk-free interest rate was larger than 0, i.e.,

$$\begin{aligned} C(K,T) &= E[(e^{-\beta T}S(T) \\ &\quad - \exp[-\int_t^T r(s)ds]K)I_{\{e^{-\beta T}S(T)>\exp[-\int_t^T r(s)ds]K\}}]. \end{aligned} \tag{5.5}$$

Suppose that the pricing process of stock $\{S(t) : t \geq 0\}$ satisfied FBM and moreover $\sigma(t) = \sigma$ and $d\mu(t)$ were constants, then we had:

$$dS(t) = \mu S(t)dt + \sigma S(t)dB_H(s). \tag{5.6}$$

Theorem 5.3. For European call option whose due date was T and exercise price was K, when r(t) was a non-random function and $\sigma(t) = \sigma$ was a constant, the price was:

$$C(K,T) = S(0)N(-d_1) - \exp\{-\int_t^T r(s)ds\}KN(-d_2) \tag{5.7}$$

and

$$\begin{aligned} d_1 &= \frac{\ln\frac{K}{S(0)} - \int_t^T r(s)ds - \frac{1}{2}\sigma^2 T^{2H}}{\sigma T^H}, \\ d_2 &= \frac{\ln\frac{K}{S(0)} - \int_t^T r(s)ds + \frac{1}{2}\sigma^2 T^{2H}}{\sigma T^H}. \end{aligned} \tag{5.8}$$

Proof. It could be known from $S(T)=S(0)\exp\{\mu t\frac{1}{2}\sigma^2T^{2H}+\sigma B_H(T)\}$ that,

$$
\begin{aligned}
ES(T) &= \int_{-\infty}^{+\infty} S(0)\exp\{\mu T-\frac{1}{2}\sigma^2T^{2H}+\sigma x\}\frac{1}{\sqrt{2\pi T^{2H}}}\exp\{\frac{-x^2}{2T^{2H}}\}dx \\
&= S(0)\int_{-\infty}^{+\infty}\exp\{\mu T-\frac{1}{2}\sigma^2T^{2H}+\sigma x-\frac{x^2}{2T^{2H}}\}\frac{1}{\sqrt{2\pi T^{2H}}}dx \\
&= \frac{S(0)e^{\mu T}}{\sqrt{2\pi T^{2H}}}\int_{-\infty}^{\infty}\exp\{-\frac{1}{2}\sigma^2T^{2H}+\sigma x-\frac{x^2}{2T^{2H}}\}dx \\
&= \frac{S(0)e^{\mu T}}{\sqrt{2\pi T^{2H}}}\int_{-\infty}^{+\infty}\exp\{\frac{-x^2+2\sigma xT^{2H}-\sigma^2T^{4H}}{2T^{2H}}\}dx \\
&= \frac{S(0)e^{\mu T}}{\sqrt{2\pi T^{2H}}}\int_{-\infty}^{\infty}\exp\{\frac{-(x-\sigma T^{2H})^2}{2T^{2H}}\}dx.
\end{aligned} \tag{5.9}
$$

Suppose $y=\frac{x-\sigma T^{2H}}{T^H}$, then

$$
ES(T)=\frac{S(0)e^{\mu T}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\{\frac{-y^2}{2}\}dyS(0), i.e. \tag{5.10}
$$

$e^{\beta T}=\frac{ES(T)}{S(0)}=e^{\mu T}$. As

$$
e^{-\beta T}S(T)>\exp\{-\int_t^T r(s)ds\}K,
$$

then

$$
B_H(t)>\frac{\ln\frac{K}{S(0)-\int_t^T r(s)ds+\frac{1}{2}\sigma^2T^{2H}}}{\sigma}. \tag{5.11}
$$

Suppose $y=\frac{x-\sigma T^{2H}}{T^H}$ and

$$
d_1=\frac{\ln\frac{K}{S(0)}-\int_t^T r(s)ds-\frac{1}{2}\sigma^2T^{2H}}{\sigma T^H},
$$

then

$$
\begin{aligned}
E[e^{-\beta T}S(T)I_{\{e^{-\beta T}S(T)>\exp[-\int_t^T r(s)ds]K\}}] &= S(0)\int_{d_1}^{+\infty}\exp\{-\frac{x^2}{2}\}\frac{1}{\sqrt{2\pi}} \\
&= S(0)N(-d_1).
\end{aligned} \tag{5.12}
$$

Suppose $\frac{x}{T^H}=z$ and

$$
d_2=\frac{\ln\frac{K}{S(0)}-\int_t^T r(s)ds+\frac{1}{2}\sigma^2T^{2H}}{\sigma T^H},
$$

then

$$E[\exp\{-\int_t^T r(s)ds\}KI_{\{e^{-\beta T}S(T)>\exp[-\int_t^T r(s)ds]K\}}]$$
$$= \exp[-\int_0^T r(s)ds]KN(-d_2). \tag{5.13}$$

Thus

$$C(K,T) = E[(e^{-\beta T}S(T) - \exp\{-\int_t^T r(s)ds\}K)I_{\{e^{-\beta T}S(T)>\exp[-\int_t^T r(s)ds]K\}}]$$
$$= S(0)N(-d_1) - \exp\{-\int_t^T r(s)ds\}KN(-d_2). \tag{5.14}$$

The above is the proof process of the theorem. □

6. Conclusions

The changing process of underlying asset price is featured by self-similarity and long-term memory. Thus researchers realize that describing the process using FBM is more in line with the practical situation of financial market. Through the approach of FBM, this study deduced that model $dS_t^\varepsilon = \mu S_t^\varepsilon dt + \sigma S_t^\varepsilon dB_t^\varepsilon$, $S_t^\varepsilon|_{t=0} = S_0$ was arbitrage free for $\forall\varepsilon > 0$. FBM considered both the past price and the present price while describing the change of underlying asset price; in some sense, it was more exact in describing objects. Through establishing financial model, simulating the increment of FBM and making a fitting analysis on sample paths, we obtained the change path of underlying asset price. Moreover, the pricing formula for European option was also obtained using actuarial approach.

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Accepted: 3.01.2017