SOLUTION OF STEADY-STATE HAMILTON-JACOBI EQUATION BASED ON ALTERNATING EVOLUTION METHOD

A. Tongxia Li
Department of Mathematics
Hulunbeier Vocational Technical College
Hulunbeie
China
olfmyq@163.com

Abstract. Hamilton-Jacobi equation is a kind of highly nonlinear partial differential equation which is difficult to be solved. The boundary value problem of steady-state Hamilton-Jacobi equation is supposed as $H(x, \nabla x\phi(x)) = 0$, $x \in \Omega/\Gamma$; $\phi(x) = q(x)$, $x \in \Gamma(\Omega \in Rd, d$ stands for the space dimensionality, Ω stands for a bounded open set with a boundary of Γ, and H stands for a given non-linear function, called Hamiltonian). Even though Hamiltonian function is smooth, the derivative of its solution may be disconnected at some cuspidal points. There are many ways to solve a steady-state Hamilton- Jacobi equation, among which, fast marching method (FMM) and fast sweeping method (FSM) are famous. This study solved Hamilton-Jacobi equation using alternating evolution method (AE). Firstly, an initial Hamilton-Jacobi equation was described using AE; then polynomials were constructed to approach the Hamilton-Jacobi equation and the equation was finally solved by selecting proper iterative methods and correct boundary conditions.An artificial parameter was generated in the process of construction of iterative format; the selection of the parameter could directly affect the stability and convergence of the iterative format. On account of this, the stability and convergence of the first-order AE algorithm was analyzed and the effectiveness and accuracy of the algorithm was proved by a numerical experiment.

Keywords: Hamilton-Jacobi equation, alternating evolution method, viscosity solution, convergence.

1. Introduction

Hamilton-Jacobi equation is a kind of highly non-linear hyperbolic partial differential equation which was applied in mechanical studies carried out by engineers and physicists at first and then extensively applied for optimum control and differential game.With the development of computer technology, mathematicians have paid more attentions to the solution of Hamilton-Jaconi equation using numerical calculation. ϕ is generally Lipschitz continuous, but not C^1 smooth; hence there is usually no classical solution for full nonlinear partial differential equation. The concept of weak solution is proposed when solving equations lacking of smoothness. Weak solution refers to a solution that satisfies equation at points which can be derived and are continuous, but weak solution is not unique. To solve the non-uniqueness of weak solution, some experts proposed the definition of viscosity solution to illustrate the existence and uniqueness of viscosity solution.

There are many ways to solve Hamilton-Jacobi equations. This study explored the solution of steady-state Hamilton-Jacobi equation using alternating evolution

$$\phi = \phi^{SN} - \varepsilon H(\nabla_x \phi^{SN}).$$

In the formula, ε stands for an artificial parameter, whose selection should satisfy the condition of iterative stability. AE was proposed by Liu HL in 2008 and then applied in hyperbolic conservation equations. In 2011, Saran H, et al. applied AE to solve Hamilton-Jacobi equation containing time parameters and gained certain achievement. Hamilton-Jacobi equation containing time parameters was firstly converted into a new form using AE and then decomposed using discontinuous finite element; finally good numerical results could be obtained. The purpose of this study was to calculate the numerical solution of non-linear steady-state Hamilton-Jacobi equation using high-efficient and high-order AE algorithm.

2. Solution of first-order steady-state Hamilton-Jacobi equation based on AE

2.1 The construction of AE system

Before solving Hamilton-Jacobi equation containing time parameters based on AE, the following AE system was constructed:

$$H(x, \nabla_x v) = \frac{1}{\varepsilon}(v - u), \quad H(x, \nabla_x u) = \frac{1}{\varepsilon}(u - v). \tag{2.1}$$

Numerical solutions around grid points were updated based on the above equation and using the numerical solutions of points around grid points, shown as below

$$\phi(x)\phi(x)^{SN} - \varepsilon H(x, \nabla_x \phi(x)^{SN}). \tag{2.2}$$

Considering uniform partition $\{x_k, k \in Z\}$, its grid diameter was Δx. Suppose the real solution of the equation at grid point xk as ϕ and the numerical value as Ek. r-order polynomial was constructed to approach ϕ^{SN} using $E_{k\pm l}$ and l depended on the number of orders of the polynomial. If the polynomial was supposed as $p_k[E](x)$, then Ek could be expressed as:

$$E_k = p_k^r[E](x) - \varepsilon H(x_k, \partial_x p_k^r[E](x)). \tag{2.3}$$

Next, $p_k^r[E](x)$ could be constructed using equation (2.3). If $I_k : [x_{k-1}, x_{k+1}]$, then non-oscillatory polynomial $p_k[E](x)$ was reconstructed on each grid point

xk. We have:

$$p_k^r[E](x_{k\pm 1}) = E_{k\pm 1}. \tag{2.4}$$

Then first-order polynomial constructed based on $E_{k\pm 1}$ was:

$$p_k^1[E](x) = E_{k-1} + s_k(x - x_{k-1}), s_k = \frac{E_{k+1} - E_{k-1}}{2\Delta x} \tag{2.5}$$

s_k stands for the approximate value of first-order derivative $\partial_x \phi$. The following is a second-order AE polynomial constructed based on Newton divided difference interpolation mathematics

$$p_k^2[E](x) = p_k^1[E](x) + \frac{s_k'}{2}(x - x_{k-1})(x - x_{k+1}) \tag{2.6}$$

s_k' stands for the approximate value of second-order derivative ϕ_{xx}. Based on the above AE form, second-order AE system was:

$$E_k = \frac{E_{k+1} + E_{k-1}}{2} - \frac{s_k'}{2}(\Delta x)^2 - \varepsilon H(x_k, s_k). \tag{2.7}$$

The major characteristic of essentially non-oscillatory (ENO) method is that it adopts self-adaptive template, which avoids offset and ensures the non-oscillatory property of AE format. A triangular unit was selected randomly from nonstructural grid, denoted as Δ_0. Three vertexes of Δ_0 were $i(x_i, y_i), j(x_j, y_j)$ and $k(x_k, y_k)$. ENO difference value polynomial of Δ_0 had 12 fundamental points. Those fundamental points could also be called the maximum template for numerical format construction, as shown in figure 1. When second-order derivative was calculated, the non-oscillatory property of viscosity solution of equation was

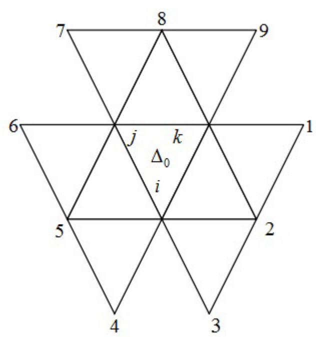

Figure 1: Template for the construction of ENO polynomial of Δ_0

calculated using ENO principle, shown as below

$$s_k' = \begin{cases} \frac{s_{k+2} - s_k}{2\Delta x}, & \text{if } \left|\frac{s_{k+2} - s_k}{2\Delta x}\right| \le \left|\frac{s_k - s_{k-2}}{2\Delta x}\right| \\ \frac{s_k - s_{k-2}}{2\Delta x}, & \text{if } \left|\frac{s_{k+2} - s_k}{2\Delta x}\right| \le \left|\frac{s_k - s_{k-2}}{2\Delta x}\right| \end{cases}. \tag{2.8}$$

Third-order AE system could be obtained similarly. Third-order AE polynomial was:

$$p_k^3[E](x) = p_k^2[E](x) + \frac{s_x''}{6}(x - x_{k+1})(x - x_{k+3}). \tag{2.9}$$

Third-order AE system was as follows:

$$E_k = \frac{E_{k+1} + E_{k-1}}{2} - \frac{s_k'}{2}(\Delta x)^2 + \frac{s_k''}{6}(\Delta x)^2(x_k - x_{k+3})$$
$$- \varepsilon H(x_k, s_k - \frac{s_k''}{6}(\Delta x)^2). \tag{2.10}$$

2.2 Hamilton-Jacobi equation

Hamilton-Jacobi equation has two formats, shown as below [15, 16].

$$\begin{cases} H(x, u, Du) = 0 \\ u = z \end{cases} . \tag{2.11}$$

The first format means within Ω and the second format means on $\partial\Omega$. The current Hamilton-Jacobi equation was a steady-state equation. The following equation is called Cauchy equation or developmental Hamilton-Jacobi equation

$$\begin{cases} u_t + H(x, t, u, Du) = 0 \\ u = z \\ u(x, 0) = u_0(x) \end{cases} . \tag{2.12}$$

The first format means within $\Omega \times [0, T]$, the second format means on $\partial\Omega \times [0, T]$, and the third format means within Ω. The current Ω stands for the open set of R^N, z and o stand for given boundary and initial value condition, and Du stands for the gradient of with regard to x. $H(x, u, Du)$ was a given function defined on $\Omega \times R \times R^N$ and $H(x, t, tu, Du)$ was a function defined on $H(x, t, tu, Du)$, both of them were Hamilton function.

3. Algorithm for solving steady-state Hamilton-Jacobi equation based on AE

The construction of algorithm includes initialization, iteration process and ending condition:

1. Initialization of algorithm: The initial value was supposed as E^0, boundary points were assigned, non-boundary points were assigned with arbitrary initial values. The values were updated in the following iteration process.

2. E^{n+1} was calculated based on E^n iteration and it satisfied the following relation expression

$$E_k^{n+1} = p_k^r[E^n](x_k) - \varepsilon H(x_k, \partial_x p_k^r[E^n](x_k)) \tag{3.1}$$

3. After the fixation of grid size, the ending condition for algorithm was:

$$\|E^{n+1} - E^{n}\| \leq \xi. \tag{3.2}$$

ξ was small enough and it showed different degrees of changes according to different precision. Simultaneous iteration could be divided into two parts, odd part and even part. The grid values of odds after $(n+1)$ times of iteration were calculated based on the even grid values obtained after n times of iteration; the rest could be deduced by analogy till the end.

When numerical solutions are calculated using AE format, mistakes may appear in the process of solution if proper values of calculation region boundary and values beyond the involved calculation region are not selected. The calculation region was supposed as a compact set, boundary as Γ, and some grids as even grids. If $\Gamma = \partial\Omega$, the exact solutions of boundary points could be used to calculate points inside region and extract boundary conditions were unnecessary to be given additionally, because the boundary values have been given.

4. Analysis of stability and convergence of AE

Suppose $E = (E_1, E_2, \ldots, E_n)$, then first-order AE format was:

$$E = F(E) \tag{4.1}$$

$$F_k(E) = \frac{E_{k+1} + E_{k-1}}{2} - \varepsilon H\left(\frac{E_{k+1} - E_{k-1}}{2\Delta x}\right). \tag{4.2}$$

The following was a proof for the uniqueness of AE format.

Theorem 3.1. *If there was*

$$\frac{\varepsilon}{\Delta x} \max |H'(\cdot)| < 1 \tag{4.3}$$

then $E = F(E)$ had at most only one solution.

Proof. Proof The following was a proof for theorem 1 using proof by contradiction. Suppose that equation (15) had two different solutions, i.e., ϕ^* and ϕ^x. The proof was considered tenable if $\zeta \equiv 0$ and $\zeta \equiv \phi^* - \varphi^*$

$$\zeta_k = \phi_k^* - \varphi_k^* = F_k(\phi^*) - F_k(\varphi^*) = \frac{\zeta_{k+1} + \zeta_{k-1}}{2} - \varepsilon H'(\theta_k)\frac{\zeta_{k+1} - \zeta_{k-1}}{2\Delta x}. \tag{4.4}$$

According to the boundary condition, we have:

$$\begin{aligned} \zeta_1 = a_1\zeta_2, \zeta_2 &= \frac{1}{2}\left(1 - \frac{\varepsilon H'(\theta_k)}{\Delta x}\right)\zeta_{k+1} + \frac{1}{2}\left(1 + \frac{\varepsilon H'(\theta_x)}{\Delta x}\right)\zeta_{k-1} \\ \zeta_N &= b_N\zeta_{N-1}; 2 \leq k \leq N-1. \end{aligned} \tag{4.5}$$

Suppose $\|\zeta\|_\infty = C > 0$, then:

(1) $C = |\zeta_1|$, then $|\zeta_1| \geq |\zeta_2|$. But it was inconsistent with). Similarly, $C \neq |\zeta_N|$.

(2) $C = |\zeta_1|$ and $2 \leq l \leq N1$, then $|\zeta_1| \geq |\zeta_{l\pm 1}|$. But the following two situations might occur:

Firstly, if $|\zeta_{l-1}| \neq |\zeta_{l+1}|$, then it was inconsistent with $|\zeta_l| < \max(|\zeta_{l-1}|, |\zeta_{l+1}|) \leq \|\zeta\|_\infty = C = |\zeta_l|$.

Secondly, if $|\zeta_{l-1}| = |\zeta_{l+1}|$, then $|\zeta_l| = |\zeta_{l\pm 1}|$. In this case, the above two steps were repeated on $C = |\zeta_{l\pm 1}|$ till the first situation appeared. Then $\|\zeta\|_\infty = 0$ was obtained, which suggested the uniqueness of the solution. □

Theorem 3.2. *Suppose E^n as the numerical solution of Hamilton-Jacobi equation $H(\partial_x \phi) = 0$ under the condition of first-order $AEE^{n+1} = F(E^n)$. If*

$$\frac{\varepsilon}{\Delta x} \max |H'(\cdot)| < 1, \tag{4.6}$$

then $\{E^n\}$ was considered as convergent.

Proof. If

$$\zeta_k^{n+1} = E_k^{n+1} - E_k^n \tag{4.7}$$

the $E^{n+1} = F(E^n)$ could be transformed into:

$$\begin{aligned}\zeta_1^{n+1} &= a_1^n \zeta_2^n, \zeta_k^{n+1} = \frac{1}{2}\left(1 - \frac{\varepsilon H'(\theta_k^n)}{\Delta x}\right)\zeta_{k+1}^n + \frac{1}{2}(1 + \frac{\varepsilon H'(\theta_k^n)}{\Delta x})\zeta_{k-1}^n,\\ \zeta_N^{n+1} &= b_N^n \zeta_{N-1}^n; 2 \leq k \leq N-1.\end{aligned} \tag{4.8}$$

If there was $\|\zeta^{n+1}\|_\infty = |\zeta_l^{n+1}|$ for $2 \leq l \leq N$, then there were two situations.

(1) For $l = 1$ or $l = N$, there was $\|\zeta^{n+1}\|_\infty = \alpha|\zeta^n|_\infty \alpha < 1$.

(2) For $2 \leq l \leq N1$, there was $|\zeta_l^{n+1}| = |b_l^n \zeta_{l-1}^n + a_l^n \zeta_{l+1}^n| < \max(|\zeta_{l-1}^n|, |\zeta_{l+1}^n|) \leq \|\zeta^n\|_\infty$.

If $\zeta_{l-1}^n \neq \zeta_{l+1}^n$, then $\alpha < 1$; if $\zeta_{l-1}^n = \zeta_{l+1}^n$, then $\alpha \leq 1$. It could also be expressed as:

$$\|\zeta^n\|_\infty = \alpha_1 \|\zeta^{n-1}\|_\infty = \alpha_1 \alpha_2 \|\zeta^{n-2}\|_\infty = \ldots = \alpha_1 \alpha_2 \ldots \alpha_{n-1} \alpha_n \|\zeta^0\|_\infty. \tag{4.9}$$

If $\alpha_m = 1$, then $\alpha_s < 1(s < m + N/2)$, we have:

$$\|\zeta^n\|_\infty \leq \alpha^{\frac{2n}{N}} \|\zeta^0\|_\infty. \tag{4.10}$$

The above testified the convergence of $\sum_{n=1}^\infty \zeta^n$ i.e., the convergence of $\{En\}$.□

5. Analysis of example

The following was an example analysis based on two-dimensional AE system, which made the application of the solution of steady-state Hamilton-Jacobi equation using AE more extensive.

The numerical results obtained using second-order AE are shown in table 1. It could be known that, L_1 error gradually approached 2 with the decrease of grid size.

N	L_1		L_∞	
	Error	Order	Error	Order
$2^0\times10$	3.386×10^{-4}	-	3.348×10^{-3}	-
$2^1\times10$	6.606×10^{-5}	2.441	1.624×10^{-3}	1.968
$2^2\times10$	1.647×10^{-5}	2.012	4.014×10^{-4}	2.017
$2^3\times10$	4.135×10^{-6}	1.999	9.998×10^{-5}	2.009
$2^4\times10$	1.032×10^{-6}	1.999	2.486×10^{-5}	2.008
$2^5\times10$	2.574×10^{-7}	2.001	6.261×10^{-6}	1.988
$2^6\times10$	6.432×10^{-8}	2.001	1.639×10^{-6}	1.935
$2^7\times10$	1.610×10^{-8}	1.999	4.107×10^{-7}	1.997

Figure 2: The solution of example 1 $\frac{\varepsilon}{h} = \frac{1}{4}$ using second-order AE

If two-dimensional Eikonal equation was:

$$f(x,y) = \sqrt{\phi_x^2 + \phi_y^2} = \frac{\pi}{2}\sqrt{\sin^2\left(\frac{\pi x}{2}\right) + \sin^2\left(\frac{\pi y}{2}\right)}. \tag{5.1}$$

To obtain an accurate initial value, the boundary condition needed to be processed at first, because the boundary condition only involved one point. If boundary $\Gamma = \{(0,0)\}$, boundary condition was $q(x,y)|_\Gamma = 2$, and region $\Omega = [1,1] \times [1,1]$ was taken as numerical value computation region, then the exact solution of the problem was:

$$\phi(x,y) = \cos\left(\frac{\pi x}{2}\right) + \cos\left(\frac{\pi y}{2}\right). \tag{5.2}$$

Next $\varepsilon/h = 1/2$, was selected as the reference value of first-order system and $\varepsilon/h = 1/4$ as the reference value of first-order system. The numerical value results are shown in table 2. It could be seen from the data in the table that, the selected parameters satisfied the convergence condition.

6. Conclusions

This study mainly used AE to approach the viscosity solution of steady-state Hamilton-Jacobi. Firstly, proper parameters were selected by constructing polynomials; numerical solutions of surrounding grids were used to express the numerical solutions of iteration points, which avoided the problems encountered when implicit iterative expression was used. Then the stability and convergence

N	First-order AE system				Second-order AE system			
	L_1		L_∞		L_1		L_∞	
	Error	Order	Error	Order	Error	Order	Error	Order
$2^0\times10$	-	-	-	-	1.356×10^{-1}	-	3.265×10^{-1}	-
$2^1\times10$	1.907×10^{-1}	-	0.842	-	2.876×10^{-2}	2.243	7.112×10^{-2}	2.203
$2^2\times10$	6.962×10^{-2}	1.454	0.423	1.000	6.301×10^{-3}	2.196	1.603×10^{-2}	2.154
$2^3\times10$	2.474×10^{-2}	1.495	0.192	1.115	1.522×10^{-3}	2.075	3.826×10^{-3}	2.018
$2^4\times10$	9.108×10^{-3}	1.437	0.093	1.046	3.667×10^{-4}	2.034	9.682×10^{-4}	2.006
$2^5\times10$	3.326×10^{-3}	1.458	0.058	0.687	9.168×10^{-5}	2.002	2.418×10^{-5}	2.001
$2^6\times10$	1.222×10^{-3}	1.497	0.032	0.850	2.292×10^{-5}	2.000	6.046×10^{-5}	2.000
$2^7\times10$	4.337×10^{-4}	1.441	0.015	1.056	5.731×10^{-6}	2.001	1.512×10^{-5}	2.000

Figure 3: The numerical value results of first-order and second-order AE systems

of AE format were analyzed and an example analysis was made on AE format. It could be seen from the analysis results that, AE is effective in solving the viscosity solution of Hamilton-Jacobi equation and can achieve required precision.

References

[1] Q.H. Liu, X.X. Li, J. Yan, *Large time behavior of solutions for a class of time-dependent Hamilton-Jacobi equations*, Science China Mathematics, 59 (2016), 875-890.

[2] H. Elvang, M. Hadjiantonis, *A practical approach to the Hamilton-Jacobi formulation of holographic renormalization*, Journal of High Energy Physics, 2016 (6) (2016), 1-29.

[3] Z. Feng, G. Li, P. Jiang et al., *Deformed Hamilton-Jacobi equations and the tunneling radiation of the higher-dimensional RN-(A)dS black hole,* International Journal of Theoretical Physics, 2016, 1-9.

[4] A. Davini, A. Fathi, R. Iturriaga et al., *Convergence of the solutions of the discounted HamiltonCJacobi equation*, Inventiones Mathematicae, 105 (2016), 1-27.

[5] D. Castorina, A. Cesaroni, L. Rossi, *Large time behavior of solutions to a degenerate parabolic Hamilton-Jacobi-Bellman equation*, Communications on Pure & Applied Analysis, 40 (2015), 1042-1054.

[6] C.Y. Kao, S. Osher, Y.H. Tsai, *Fast sweeping methods for static Hamilton-Jacobi equations*, Siam Numerical Analysis, 42 (2012), 2612 2632.

[7] M. Baggio, J. Boer, K. Holsheimer, *Hamilton-Jacobi renormalization for Lifshitz spacetime*, Journal of High Energy Physics, 1 (2012), 1-25.

[8] J.M. Bioucas-Dias, M.A.T. Figueiredo, *Alternating direction algorithms for constrained sparse regression: application to hyperspectral unmixing,* Mathematics, 2012, 1-4.

[9] Christos Arvanitis, Charalambos Makridakis, Nikolaos I. Sfakianakis, *Entropy conservative schemes and adaptive mesh selection for hyperbolic conservation laws*, Journal of Hyperbolic Differential Equations, 07 (2011), 383-404.

[10] Liu Hailiang, *An alternating evolution approximation to systems of hyperbolic conservation laws*, Journal of Hyperbolic Differential Equations, 5 (2008), 421-447.

[11] H. Saran, H. Liu, *Alternating evolution schemes for hyperbolic conservation laws*, Siam Journal on Scientific Computing, 33 (2011), 3210-3240.

[12] Y.H. Zahran, *WENO-TVD schemes for hyperbolic conservation laws*, Analysis, 27 (2007), 73-94.

[13] A. Balaguer-Beser, *A new reconstruction procedure in central schemes for hyperbolic conservation laws*, International Journal for Numerical Methods in Engineering, 86 (2011), 1481-1506.

[14] S. Evje, T. Flåtten, H.A. Friis, *On a relation between pressure-based schemes and central schemes for hyperbolic conservation laws*, Numerical Methods for Partial Differential Equations, 24 (20080, 605-645.

[15] S.N. Armstrong, P.E. Souganidis, *Stochastic homogenization of level- set convex Hamilton-Jacobi equations*, International Mathematics Research Notices, 39 (2012), 3420-3449.

[16] S. Bianchini, D. Tonon, *SBV regularity for Hamilton-Jacobi equations with Hamiltonian depending on* (t, x)*,* Siam Journal on Mathematical Analysis, 44 (2012), 2179-2203.

[17] K. Debrabant, E.R. Jakobsen, *Semi-Lagrangian schemes for linear and fully non-linear Hamilton-Jacobi-Bellman equations*, Hyperbolic Problems: Theory, Numerics, Applications, 2014, 483-490.

[18] K. Alton, I.M. Mitchell, *An ordered upwind method with precomputed stencil and monotone node acceptance for solving static convex Hamilton- Jacobi equations*, Journal of Scientific Computing, 51 (2012), 313-348.

[19] Y.H. Zahran, *Central ADER schemes for hyperbolic conservation laws*, Journal of Mathematical Analysis & Applications, 346 (2008), 120-140.

[20] R. Kumar, M.K. Kadalbajoo, *A class of high resolution shock capturing schemes for hyperbolic conservation laws*, Applied Mathematics & Computation, 195 (2008), 110-126.

[21] H. Liu, J. Qiu, *Finite difference hermite WENO schemes for hyperbolic conservation laws*, Journal of Scientific Computing, 63 (2015), 548-572.

Accepted: 28.12.2016