CRITICAL SEMIMODULES

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Abstract. In the paper, a particular class of semimodules (so called critical semimodules) typical for additively idempotent semirings possessing at least two right multiplicatively absorbing elements is investigated.

Keywords: semiring, semimodule, ideal, characteristic, critical.

The present note is a direct continuation of [1] and [2] and the reader is fully referred to [1], [2] as concerns notation, terminology and further references. Here, we introduce and study a certain type of (left) semimodules that are typical for additively idempotent semirings possessing at least two right multiplicatively absorbing elements.

1. Preliminaries

Let $A = A(*)$ be a groupoid. An element $a \in A$ is called *left* (*right*) *neutral* if $a * x = x$ ($x * a = x$) for all $x \in A$, and *left* (*right*) *absorbing* if $a * x = a$ ($x * a = a$) for all $x \in A$. If $A = A(+)$ then $0_A \in A$ ($o_A \in A$) means that 0_A (o_A) is (the unique) left and right neutral (absorbing) element of $A(+)$ and $0_A \notin A$ ($o_A \notin A$) denotes the fact that $A(+)$ has no (left and right) neutral (absorbing) element. Similarly, if $A = A(\cdot)$ then $1_A \in A$ means that 1_A is (the unique) left and right neutral element of $A(\cdot)$.

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A semiring is a non-empty set equipped with two associative binary operations that are usually written as addition and multiplication. The addition is commutative and the multiplication distributes over the addition. Given a semiring S, a (left S-)semimodule ($_SM =$) M is a commutative semigroup $M(+)$ together with a scalar multiplication $S \times M \to M$ such that $(a+b)x = ax + bx$, $a(x+y) = ax + ay$ and $a(bx) = (ab)x$ for all $a, b \in S$ and $x, y \in M$. If S is a semiring then $R = \underline{R}(S) = \{\, a \in S \mid Sa = \{a\} \,\}$ denotes the set of right multiplicatively absorbing elements. If $a \in \underline{R}(S)$ then $a + a = aa + aa = (a+a)a = a$ and $a(b+b) = ab + ab = ab$ for every $b \in S$. Consequently, the semiring S is additively idempotent, provided that the right semimodule $\underline{R}(S)_S$ is faithful, i.e., for all $a, b \in S$, $a \neq b$, there is at least one $x \in \underline{R}(S)$ with $xa \neq xb$.

Let S be a semiring. A non-empty subset I of S is a *left* (right) *ideal* if $SI \cup (I+I) \subseteq I$ ($IS \cup (I+I) \subseteq I$). A left (right) ideal I is called *minimal* if $|I| \geq 2$ and $J = I$ whenever J is a left (right) ideal with $|J| \geq 2$ and $J \subseteq I$. A non-empty subset I of S is an *ideal* if $SI \cup IS \cup (I+I) \subseteq I$ and it is a *bi-ideal* if $SI \cup IS \cup (I+S) \subseteq I$. In the latter case, the relation $(I \times I) \cup \mathrm{id}_S$ is a congruence of the semiring S. Finally, S is called

– *simple* (more precisely: *congruence–simple*) if S has just two congruence relations (then these are id_S and $S \times S$ and $|S| \geq 2$);

– *(bi–)ideal–simple* if $S = I$ whenever I is an (bi–)ideal of S with $|I| \geq 2$.

Throughout the paper, all semirings and semimodules are assumed to be additively idempotent. It means that the respective additive semigroups $M(+)$ are semilattices, where the basic order relation is given by $\alpha \leq \beta$ iff $\alpha + \beta = \beta$.

2. Semirings possessing at least two right multiplicatively absorbing elements

In this section, let S be a semiring such that $|R| \geq 2$. Notice that the set R is an ideal of the semiring S and it is the smallest right ideal of S. The set $R + S$ is the smallest bi–ideal of S. Consequently, the semiring S is bi–ideal–simple if and only if $S = R + S$.

Lemma 2.1. *If S is simple then the right semimodule R_S is faithful.*

Proof. Easy to see. □

Lemma 2.2. (*i*) *If $0_S \in S$ then $0_R \in R$, $R0_S = \{0_R\}$ and $S0_S \leq 0_R$.*

(*ii*) *If $o_S \in S$ then $o_R \in R$, $Ro_S = \{o_R\}$ and $o_R \leq So_S$.*

Proof. (i) We have $a0_S + b = a0_S + ab = a(0_S + b) = ab = b$ for all $a \in S$ and $b \in R$.

(ii) We have $a0_S + b = ao_S + ab = a(o_S + b) = ao_S$ for all $a \in S$ and $b \in R$. □

Proposition 2.3. *Assume that either $S = R + S$ or the right semimodule R_S is faithful or the semiring S is simple. Then:*

(*i*) *If* $0_S \in S$ *then* $0_S = 0_R \in R$.
(*ii*) *If* $0_R \in R$ *then* $0_R = 0_S \in S$.

Proof. (i) Using 2.2(i), we have $ab0_S = 0_R = a0_S$ for all $a \in R$ and $b \in S$. Then $b0_S = 0_S$, provided that R_S is faithful. Furthermore, $0_R + S = 0_R + R + S$, provided that $R + S = S$. The rest is clear.

(ii) We have $b(a + 0_R) = ba + b0_R = ba + 0_R = ba$ for all $a \in S$ and $b \in R$. If R_S is faithful then $a + 0_R = a$. The rest is clear. □

Proposition 2.4. *Assume that either* $R \cap (S + a) \neq \emptyset$ *for every* $a \in S$ *or the right semimodule* R_S *is faithful or the semiring* S *is simple. Then:*
(*i*) *If* $o_S \in S$ *then* $o_S = o_R \in R$.
(*ii*) *If* $o_R \in R$ *then* $o_R = o_S \in S$.

Proof. We proceed similarly as in the proof of 2.3. □

Lemma 2.5. *Let* $e \in S$ *be a right multiplicatively neutral element. Then:*

(*i*) $e \notin R$.

(*ii*) *If* $e = a + b$, $a \in R$, $b \in S$, *then* $a = 0_S$.

(*iii*) *If* $e + a \in R$ *for some* $a \in S$ *then* $e + a = o_S$.

(*iv*) *If the right semimodule* S_S *is faithful then* $e = 1_S$ *is multiplicatively neutral.*

Proof. (i) Obvious.

(ii) For every $c \in S$, $c = ce = ca + cb = a + cb$, so that $a + S = S$ and $a = 0_S$.

(iii) We have $e + a = b(e + a) = be + ba = b + ba$ for every $b \in S$.

(iv) We have $aeb = ab$ for all $a, b \in S$. □

Corollary 2.6. *Assume that the semiring* S *is simple. If* $e \in S$ *is right multiplicatively neutral then* $e = 1_S$ *is multiplicatively neutral and* $0_S \in S$. *If, moreover,* $(1_S + S) \cap R \neq \emptyset$ *then* $o_S \in S$.

Lemma 2.7. *Let* $e \in S$ *be left multiplicatively neutral. If the left semimodule* ${}_S S$ *is faithful then* $e = 1_S$ *is multiplicatively neutral.*

Proof. Easy to see. □

Corollary 2.8. *Assume that the semiring* S *is simple. If* $e \in S$ *is left multiplicatively neutral then* $0_S \in S$ *and if, moreover,* $|S| \geq 3$ *then* $e = 1_S$ *is multiplicatively neutral.*

Proposition 2.9. *Assume that* $o_S \in S$. *The semiring* S *is simple if and only if* $o_S \in R$, $S = R + S$ *and the right semimodule* R_S *is faithful and simple.*

Proof. First, assume that S is simple. Then S is bi–ideal–simple, and hence $R + S = S$. The right semimodule R_S is faithful by 2.1. If α is a congruence of R_S then σ is a congruence of S, where $(a, b) \in \sigma$ iff $(ca, cb) \in \alpha$ for every $c \in R$, and we have $\alpha = \sigma \cap (R \times R)$. Thus $\alpha = \mathrm{id}_R$ or $\alpha = R \times R$.

Conversely, assume that $S = R+S$ and R_S is faithful and simple. Let $\varrho \neq \mathrm{id}_S$ be a congruence of the semiring S and $(a, b) \in \varrho$, $a \neq b$. Since R_S is faithful, we have $ca \neq cb$ for at least one $c \in R$, and hence $\beta = \varrho \cap (R \times R) \neq \mathrm{id}_R$. Clearly, β is a congruence of R_S, so that $\beta = R \times R$ and $R \times R \subseteq \varrho$. If $a \in S$ then $a = b+c$ for some $b \in S$, $c \in R$, $(c, o_S) \in \varrho$, and hence $(a, o_S) = (b + c, b + o_S) \in \varrho$. □

Lemma 2.10. *Assume that $0_S \in S$ and the right semimodule R_S is faithful. Then $0_S \in R$ and the following conditions are equivalent:*

(*i*) *$\alpha_1 = (R_1 \times R_1) \cup \mathrm{id}_R$, where $R_1 = R \setminus \{0_S\}$, is a congruence of R_S.*

(*ii*) *$ab \neq 0_S$ for all $a \in R_1$ and $b \in S_1 = S \setminus \{0_S\}$.*

(*iii*) *$cd \neq 0_S$ for all $c, d \in S_1$.*

(*iv*) *S_1 is a subsemiring of S.*

Proof. It is easy. □

Lemma 2.11. *Assume that $o_S \in S$ and the right semimodule R_S is faithful. Then $o_S \in R$ and the following conditions are equivalent:*

(*i*) *$\alpha_2 = (R_2 \times R_2) \cup \mathrm{id}_R$, where $R_2 = R \setminus \{o_S\}$, is a congruence of R_S.*

(*ii*) *$a + b \neq o_S \neq ac$ for all $a, b \in R_2$ and $c \in S_2 = S \setminus \{o_S\}$.*

(*iii*) *S_2 is a subsemiring if S.*

Proof. It is easy. □

Corollary 2.12. *Let the semiring S be simple and $|R| \geq 3$. Then:*

(*i*) *If $0_S \in S$ then $ab = 0_S$ for some $a \in R \setminus \{0_S\}$ and $\in S \setminus \{0_S\}$.*

(*ii*) *If $o_S \in S$ then either $a + b = o_S$ or $ac = o_S$ for $a, b \in R \setminus \{o_S\}$ and $c \in S \setminus \{o_S\}$.*

Lemma 2.13. *Assume that $0_S, o_S \in R$. Then $\alpha_3 = (R_3 \times R_3) \cup \mathrm{id}_R$, where $R_3 = R \setminus \{0_S, o_S\}$, is a congruence of R_S iff the following three conditions are satisfied:*

1. *$R_3 + R_3 = R_3$ (equivalently, $R_2 + R_2 = R_2$).*
2. *If $ab = 0_S$ for some $a \in R_1$ and $b \in S_1$ then $R_2 b = \{0_S\}$.*
3. *If $cd = o_S$ for some $c \in R_2$ and $d \in S_2$ then $R_1 d = \{o_S\}$.*

Proof. It is easy. □

3. Critical semimodules (a)

In this section, let S be a non-trivial semiring and M be a *precharacteristic* (left S-)semimodule, i.e. $|M| \geq 2$, $0_M, o_M \in M$, $S0_M = \{0_M\}$ and $So_M = \{o_M\}$. Put $N = M \setminus \{o_M\}$ and $K = M \setminus \{0_M, o_M\}$.

Lemma 3.1. *Let $|M| \geq 4$. The following conditions are equivalent:*

(*i*) *$Sx = M$ for every $x \in K$.*

(*ii*) *M has (at most) four subsemimodules (and these are $\{0_M\}$, $\{o_M\}$, $\{0_M, o_M\}$ and M).*

Proof. (i) implies (ii). This implication is easy.

(ii) implies (i). Put $F = \{\, x \in M \mid Sx \subseteq \{0_M, o_M\}\,\}$. Then F is a subsemimodule of M, $\{0_M, o_M\} \subseteq F$ and (i) is clear, provided that $F = \{0_M, o_M\}$. On the other hand, if $F = M$ then the set $G_x = \{0_M, x, o_M\}$ is a proper subsemimodule of M for every $x \in K$, a contradiction. □

A semimodule M will be called *almost minimal* if $|M| \geq 3$, M is precharacteristic and $Sx = M$ for every $x \in K$. Further, M will be called *almost critical* if it is almost minimal, faithful and simple.

Lemma 3.2. *Let M be almost minimal. Define a relation α on M by $(x, y) \in \alpha$ iff $\{\, a \in S \mid ax = 0_M \,\} = \{\, a \in S \mid ay = 0_M \,\}$. Then:*

(*i*) *α is a congruence of M and M/α is almost minimal.*

(*ii*) *α is the (unique) greatest proper congruence of M and M/α is simple.*

(*iii*) *$(x, y) \in \alpha$ iff $\{\, a \in S \mid ax = o_M \,\} = \{\, a \in S \mid ay = o_M \,\}$.*

Proof. It is easy to see that α is a congruence of the semimodule M and $(0_M, o_M) \notin \alpha$, $(0_M, x) \notin \alpha$ and $(x, o_M) \notin \alpha$ for every $x \in K$. Thus $|M/\alpha| \geq 3$ and M/α is almost minimal. Now, let β be a congruence of M such that $\beta \nsubseteq \alpha$. If $(u, v) \in \beta \setminus \alpha$ and $a \in S$ is such that $au = 0_M$ and $w = av \neq 0_M$ then $(0_M, w) = (au, av) \in \beta$, $w \in M \setminus \{0_M\}$ and there is $b \in S$ with $bw = o_M$. Consequently, $(0_M, o_M) = (b0_M, bw) \in \beta$ and we conclude that $\beta = M \times M$. Thus we have proved assertions (i) and (ii).

Finally, define a relation γ on M by $(x, y) \in \gamma$ iff $\{\, a \in S \mid ax = o_M \,\} = \{\, a \in S \mid ay = o_M \,\}$. From (ii) it follows that $\gamma \subseteq \alpha$. On the other hand, if $(u, v) \in \alpha \setminus \gamma$ and $a \in S$ is such that $au = o_M \neq av$ then $(o_M, av) \in \alpha$, a contradiction. □

Lemma 3.3. *Assume that M is almost minimal and S is simple. Then M is faithful.*

Proof. If M were not faithful then we get $ax = bx$ for all $a, b \in S$ and $x \in M$. But this is a contradiction with $Sx = M$ for $x \in K$. □

Proposition 3.4. *Assume that the semiring S is simple. If there is at least one almost minimal semimodule then there is an almost critical semimodule.*

Proof. Combine 3.2 and 3.3. □

In the reamining part of this section, assume that M is simple and $\{0_M, o_M\} \subseteq Sx$ for every $x \in K$. If I is an ideal of the semiring S and $y \in M$, we put $J_{I,y} = \{\, a \in I \mid ay = 0_M \,\}$.

Lemma 3.5. *If $w \in N$ then $J = J_{I,w}$ is a left ideal of S. If $a_0 = o_J \in J$ and $x, y \in M$ are such that $x \leq w$ and $y \not\leq w$ then $a_0x = 0_M$ and $a_0y = o_M$ (and hence $a_0 \in R$, provided that M is faithful).*

Proof. We have $0_M \in Sw$, so that $J \neq \emptyset$ and J is a left ideal of S. If $x \leq w$ then $a_0x \leq a_0w = 0_M$ and $a_0x = 0_M$. Now, let $y \not\leq w$. If $ay = o_M$ for some $a \in J$ then $o_M = ay \leq a_0y$ and $a_0y = o_M$. Assume, therefore, that $o_M \notin Jy$. Then $(w, w + y) \in \alpha$, where α is the relation defined on M by $(u, v) \in \alpha$ iff $\{0_M, o_M\} \not\subseteq \{au, av\}$ for every $a \in I$.Clearly, the relation α is reflexive, symmetric and if $(u, v) \in \alpha$ then $(bu, bv) \in \alpha$ for every $b \in S$. If, moreover, $c \in I$ and $z \in M$ are such that $c(u + z) = 0_M$ and $c(v + z) = o_M$ then $cu = 0_M = cz$ and $c(v + z) = o_M$, a contradiction with $(u, v) \in \alpha$. Thus $(u + z, v + z) \in \alpha$ and we see that β is a congruence of the semimodule M, where β denotes the transitive closure of α. Since $(w, w+y) \in \alpha \subseteq \beta$, we have $\beta \neq \mathrm{id}_M$ and it follows that $\beta = M \times M$, the semimodule M being simple. In particular, $(0_M, t) \in \alpha$ for at least one $t \in M \setminus \{0_M\}$. It means that $\{0_M, o_M\} \not\subseteq \{a0_M, at\} = \{0_M, at\}$ for every $a \in I$ and $o_M \notin It$. But $o_M = dt$ for some $d \in S$, and if $e \in I$ then $edt = o_M$ and $ed \in I$, a contradiction. □

Lemma 3.6. *If M is faithful and $a_0 = o_J \in J = J_{I,0_M}$ then $a_0 = o_S = o_I$.*

Proof. We have $J_{I,0_M} = I$, and hence $a_0 = o_I$. Now, $(a + a_0)y = ay + a_0y = ay + o_M = o_M = a_0y$ for all $a \in S$ and $y \in M \setminus \{0_M\}$. Of course, $(a + a_0)0_M = 0_M = a_0M$. Since M is faithful, we get $a + a_0 = a_0$. Thus $a_0 = o_S$. □

Lemma 3.7. *Assume that $R \neq \emptyset$ and that the following condition is true:*

(ε) *If $a_1 < a_2 < a_3 \ldots$ is an infinite strictly increasing sequence of elements from R then for every $a \in R \setminus \{o_R\}$ there is $i \geq 1$ with $a \leq a_i$ (see* [2, 4.18]*).*

Let $w \in K$ be such that $o_J \notin J = J_{R,w}$. Then:
(i) w is the smallest element of the set $M \setminus \{0_M\}$.
(ii) $o_R \in R$, $o_R(M \setminus \{0_M\}) = \{o_M\}$ and $(R \setminus \{o_R\})w = \{0_M\}$.

Proof. The left ideal J has no greatest element, and hence it has no maximal element, either. Using (ε), we deduce that $R \setminus \{o_R\} \subseteq J$. Since $o_M \in Sw$, we have $o_M \in Rw$, and therefore $o_R \in R$ and $o_Rw = o_M$. Now, if $w' \in K$ is such that $J_{R,w'}$ has no greatest element then $J_{R,w} = R \setminus \{o_R\} = J_{R,w'}$ and

$aw = aw'$ for every $a \in R$. Thus $(w, w') \in \alpha$, where α is the relation defined on M by $(x, y) \in \alpha$ iff $ax = ay$ for every $a \in R$. Clearly, α is a congruence of the semimodule M and $(0_M, o_M) \notin \alpha$. Since M is simple, we get $\alpha = \mathrm{id}_M$ and $w = w'$. It means that the element w is unique and it is a minimal element of K. Moreover, if $v \in K$ then $o_M \in Sv$, and hence $o_R v = o_M$. If $v \neq w$ then the left ideal $J_{R,v}$ has the greatest element, say a_0, and we have $a_0(w + v) = a_0 w + 0_M = a_0 w$. If $a_0 w = 0_M$ then $w \leq v$ follows from 3.5. If $a_0 w \neq 0_M$ then $a_0 = o_R$ and $0_M = a_0 v = o_R v = o_M$, a contradiction. □

4. Critical semimodules (b)

Let S be a non-trivial semiring. A semimodule M will be called *characteristic* if M is faithful (i.e., for all $a, b \in S$, $a \neq b$, there is $x \in M$ with $ax \neq bx$), precharacteristic and there is a mapping $\underline{\varepsilon} : N \to S$ such that $\underline{\varepsilon}(x)y = 0_M$ and $\underline{\varepsilon}(x)z = o_M$ for all $x, y, z \in M$, $y \leq x$, $z \not\leq x$. Further, M will be called *critical* if it is both almost critical and characteristic (see [2]).

In this section, we assume that $|R| \geq 2$ and $o_R \in R$. Assume also that the condition (ε) is satisfied in 4.1,...,4.4.

Proposition 4.1. *A precharacteristic semimodule M is characteristic if and only if the following three conditions are satisfied:*

1. *M is simple and faithful.*
2. *$\{0_M, o_M\} \subseteq Sx$ for every $x \in K$.*
3. *If the set K has the smallest element then the set $R \setminus \{o_R\}$ has at least one maximal element (the greatest element, resp.).*

Proof. If M is characteristic then our conditions follow from [2, 2.2, 2.3(i), 2.4(ii),(iii), 2.8, 2.9]. Now, let the three conditions be satisfied. For any $w \in N$, we have to find $\underline{\varepsilon}(w) \in S$ such that $\underline{\varepsilon}(w)x = 0_M$ and $\underline{\varepsilon}(w)y = o_M$, whenever $x \leq w$, $y \not\leq w$. If $w = 0_M$ then $\underline{\varepsilon}(w) = o_R = o_S$ (use 3.5, 3.6, (1) and (2)). If $w \in K$ and $o_J \in J = J_{r,w}$ then $\underline{\varepsilon}(w) = o_J$ (use 3.5). If $o_J \notin J$ then w is the smallest element of K by 3.7(i), $J = R \setminus \{o_S\}$ by 3.7(ii), J is a left ideal and has no maximal element, a contradiction with (3). □

Corollary 4.2. *An almost minimal semimodule M is critical if and only if M is faithful, simple and the condition* 4.1(3) *is true.*

Corollary 4.3. *An almost critical semimodule M is critical if and only if the condition* 4.1(3) *is true.*

Proposition 4.4. *Assume that the semiring S is simple, the set $R \setminus \{o_R\}$ has at least one maximal element and there is at least one almost minimal semimodule. Then there is a critical semimodule.*

Proof. By 3.4, there is an almost critical semimodule and it remains to use 4.3. □

Proposition 4.5. *Assume that the semiring S is simple, $|S| \geq 3$ and the following condition is satisfied:*

(γ) *If $a_1 > a_2 > a_3 > \dots$ is an infinite strictly decreasing sequence of elements form R then for every $a \in R \setminus \{0_R\}$ there is $i \geq 1$ with $a \geq a_i$.*

Let M be a characteristic semimodule that is not critical. Then:
(i) *Both S and M are infinite.*
(ii) *The set N has the greatest element w and $\underline{\varepsilon}(w) = 0_R = 0_S \in S$.*
(iii) *$\underline{\varepsilon}(N) = R$ and the set $N \setminus \{w\}$ has no maximal element.*
(iv) *The set $R \setminus \{0_S\}$ has no minimal element.*
(v) *$G = M \setminus \{w\}$ is a subsemimodule of M.*
(vi) *G is a critical semimodule, $o_M = o_G$ and $\underline{\varepsilon}(G \setminus \{o_M\}) = R \setminus \{0_S\}$.*

Proof. M is characteristic and almost critical, and it means that M is not almost minimal and $Sx_0 \neq M$ for at least one $x_0 \in K$. If $|M| = 3$ then $M = \{0_M, x_0, o_M\}$, $|S| = 3$, $1_S \in S$, a contradiction with $Sx_0 \neq M$. Thus $|M| \geq 4$ and, using 3.1, we see that M has a proper subsemimodule G such that $G \nsubseteq \{0_M, 0_M\}$. By [2, 2.14, 2.15], G is a characteristic semimodule. By [2, 4.6], $w = o_N \in N$, $\underline{\varepsilon}(w) = 0_S \in S$, $\underline{\varepsilon}(N) = R$, $G = M \setminus \{w\}$ and $\underline{\varepsilon}(G \setminus \{o_M\}) = R \setminus \{0_S\}$. If a_0 is minimal in $R \setminus \{0_S\}$ then $0_S y = a_0 y$ for every $y \in G$, a contradiction with the fact that G is faihtful. Thus $R \setminus \{0_S\}$ has no minimal element and $N \setminus \{w\}$ has no maximal element. By [2, 2.14, 4.6] and 3.1, the semimodule G is critical. □

5. Summary

In this section, let S be a simple semiring such that $|S| \geq 3$, $|R| \geq 2$ and $o_S \in S$. Assume, furthermore, that every infinite strictly increasing (decreasing, resp.) sequence of elements from the set $R \setminus \{0_R, o_R\}$ is upwards (downwards, resp.) cofinal in that set (i.e., the conditions (ε) and (γ) are satisfied).

By 2.1, the right semimodule R_S is faithful. By 2.4(i), $o_S = o_R \in R$. By 2.3, if $0_S \in S$ ($0_R \in R$, resp.) then $0_S = 0_R \in R$ ($0_R = 0_S \in S$, resp.).

Theorem 5.1. *Assume that $0_S \notin S$. The following conditions are equivalent:*

(i) *There is at least one critical semimoodule.*

(ii) *The is at least one critical semimodule M such that $\underline{\varepsilon}(M \setminus \{o_M\}) = R$.*

(iii) *If the set $R_2 = R \setminus \{o_S\}$ has no maximal element and if $a \in R_2$, $b \in S$ are such that $R_2 b \leq a$ then $o_S b \leq a$ (and hence $Rb \leq a$).*

Proof. (i) implies (ii). Use [2, 4.2].

(ii) implies (iii). We have $R_2 \subseteq Q = \{ c \in R \,|\, cb \le a \}$. Thus $Q \neq \emptyset$ and, by [2, 3.3(ii)], we have $o_Q \in Q$.Now, if R_2 has no maximal element then $Q \neq R_2$, and hence $Q = R$.

(iii) implies (ii). Using (ε), we see that $R \cup \{\omega\}$, where $\omega < R$, is a lattice. Now, let $a \in R$ and $b \in S$ be such that $Q = \{ c \in R \,|\, cb \le a \} \neq \emptyset$. If $o_Q \notin Q$ then $R_2 = Q$ follows from (ε). Thus $R_2 b \le a$ and $o_S b \not\le a$, a contradiction. It means that $o_Q \in Q$ and it remains to use [2, 3.3]and 4.5. □

Remark 5.2. If the conditions of 5.1 are satisfied then $M \cong \frac{1}{S}R$ (see [1, 7.2]), and hence the critical semimodule $_SM$ is determined uniquely up to isomorphism.

Now, assume that 5.1(ii) is true and the set R_2 has no maximal element. Let $a \in R_2$ and $b \in S$ be such that $R_2 b \le a$. We have $a = \underline{\varepsilon}(w)$ for some $w \in K = M \setminus \{0_M, o_M\}$. If $x \in K$ then $\underline{\varepsilon}(x) \in R_2$, $\underline{\varepsilon}(x)b \le a$ and $\underline{\varepsilon}(x)bw \le aw = 0_M$. Thus $bw \le x$ and either $bw = 0_M$, or bw is the smallest element of K and $\underline{\varepsilon}(bw)$ is the greatest element of R_2, a contradiction. Thus $bw = 0_M$, $(a+b)y = ay + by \le aw + bw = 0_M = ay$ for $y \le w$ and $(a+b)z = az + bz = o_M = az$ for $z \not\le w$. Since $_SM$ is faithful, we get $a + b = a$ and $b \le a$ (conversely, if $b \le a$ then $o_S b \le o_S a = a$ anyway).

Theorem 5.3. *Assume that $|S| \ge 4$ and $0_S \in S$. The following conditions are equivalent:*

(*i*) *There is at least one critical semimodule M such that $0_S \in \underline{\varepsilon}(M \setminus \{o_M\})$ or, equivalently, such that the set $M \setminus \{o_M\}$ has the greatest element.*

(*ii*) *There is at least one critical semimodule M such that $\underline{\varepsilon}(M \setminus \{o_M\}) = R$.*

(*iii*) *The condition* 5.1(*iii*) *holds and, besides, at least one of the following three conditions is satisfied:*

(1) *There are $a, b \in R_1 = R \setminus \{0_S\}$ such that for every $c \in R_1$ either $c \not\le a$ or $c \not\le b$.*

(2) *There are $a \in R_1$ and $c \in S$ such that $0_S c = 0_S$ and $bc \not\le a$ for every $b \in R_1$.*

(3) *There is $c \in S$ such that $0_S c \neq 0_S$ and $0_S c \neq bc$ for every $b \in R_1$.*

Proof. (i) implies (ii). Use [2, 4.2] and [2, 2.11].

(ii) implies (iii). The condition 5.1(iii) follows from [2, 3.3(ii)] (see the proof of 5.1). By [2, 2.11(iv)], $\underline{\varepsilon}(w) = 0_S$, where w is the greatest element of N. Since $|S| \ge 4$, we have $|M| \ge 4$ and $|N_1| \ge 2$, where $N_1 = N \setminus \{w\}$. Since M is almost minimal, the set $M \setminus \{w\}$ is not a subsemimodule of M (use 3.1).

Assume, first, that $N_1 + N_1 \not\subseteq N_1$. There are $u, v \in N_1$ such that $u + v = w$. Then $a = \underline{\varepsilon}(u) \in R_1$, $b = \underline{\varepsilon}(v) \in R_1$, and if $c \in R_1$ then $c = \underline{\varepsilon}(z)$ for some $z \in N_1$ and either $u \not\le z$ and $c \not\le a$, or $v \not\le z$ and $c \not\le b$. Thus (1) is true.

Assume, next, that $N_1 + N_1 \subseteq N_1$. Since $M \setminus \{w\}$ is not a subsemimodule, there are $e \in S$ and $x \in N_1$ such that $cx = w$. Then $x \neq 0_M$, $a = \underline{\varepsilon}(x) \in R_1$, $a \neq 0_S$ and $0_S cx = 0_S w = \underline{\varepsilon}(w)w = 0_M$. Consequently, $0_S c \leq a$ (see [2, 2.7]). If $b \in R_1$ then $bcx = bw = o_M$ (see [2, 2.4, 2.7]), and so $bc \nleq a$. Now, it is clear that either (2) or (3) is true.

(iii) implies (ii). Using [2,3.3] and proceeding similarly as in the proof of 5.1, we find a characteristic semimodule M with $\underline{\varepsilon}(N) = R$. Again, $\underline{\varepsilon}(w) = 0_S$, where w is the greatest element of N. If M is not critical then $G = M \setminus \{w\}$ is a subsemimodule of M and G is critical by 4.5. If (1) is true and $a = \underline{\varepsilon}(u)$, $b = \underline{\varepsilon}(v)$, $u, v \in N_1 = N \setminus \{w\}$, then $\underline{\varepsilon}(u + v) \leq a$, $\underline{\varepsilon}(u + v) \leq b$, and hence $\underline{\varepsilon}(u+v) = 0_S$ and $u+v = w$, a contradiction. If (2) is true then $a = \underline{\varepsilon}(z)$, $z \in N_1$, $0_S cz = 0_S z = 0_M$, $cz \in G$, $bcz = o_M$, $\underline{\varepsilon}(cz) = 0_S$ and $cz = w$, a contradiction. Finally, if (3) is true then $0_S \neq 0_S c < bc$ for every $b \in R_1$, $0_S c = \underline{\varepsilon}(t)$, $t \in N_1$, $0_S ct = 0_M$, $ct \in G$ and $bct = o_M$ for every $b \in R_1$ (see [2, 2.7]). Thus $\underline{\varepsilon}(ct) = 0_S$ and $ct = w$, a contradiction.

(ii) implies (i). It is clear. □

Remark 5.4. If the conditions of 5.3 are satisfied then ${}_S M \cong {}^{1}_{S}R$ (see [1, 7.2]).

Remark 5.5. If $|S| = 3$ then the (left S-)semimodule ${}_S S$ is critical. In fact, $S = \{0_S, 1_S, o_S\}$, $R = \{0_S, o_S\}$, $\underline{\varepsilon}(0_S) = o_S$ and $\underline{\varepsilon}(1_S) = 0_S$. On the other hand, none of the conditions 5.3(1),(2),(3) is true.

Remark 5.6. Assume that $0_S \in$ and the set $R_1 = R \setminus \{0_S\}$ has at least one minimal element.

(i) If the set R_1 has at least two minimal elements then the condition 5.3(1) is true. Consequently, assume that R_1 has just one minimal element a_0. Then a_0 is the smallest element of R_1 (use (γ)). Since S is simple and $|S| \geq 3$, the set $C = \{\, c \in S \mid 0_S c \neq a_0 c \,\}$ is non-empty. If $c \in C$ then $0_S c < a_0 c \leq bc$ for every $b \in R_1$. If $0_S c_0 \neq 0_S$ for at least one $c_0 \in C$ then the condition 5.3(3) is true. Assume, therefore, that $0_S C = \{0_S\}$. If $c \in C$ then $a_0 c \in R_1$, and if $a_0 c_1 \neq a_0$ for some $c_1 \in C$ then $bc_1 \nleq a_0$ for every $b \in R_1$ and 5.3(2) is true. Assume, finally, that $a_0 C = \{a_0\}$. Now, it is easy to see that none of the conditions 5.3(1),(2),(3) is true.

Put $D_1 = \{\, d \in S \mid 0_S d = 0_S = a_0 d \,\}$, $D_2 = \{\, d \in S \mid 0_S d = a_0 = a_0 d \,\}$, $D_3 = \{\, d \in S \mid 0_S d = a_0 d > a_0 \,\}$ and $D = D_1 \cup D_2 \cup D_3$. We get $S = C \cup D_1 \cup D_2 \cup D_3$ and this union is disjoint. Moreover, it is easy to check that $(C+C) \cup (C+D_1) = C$, $D_1 + D_1 = D_1$, $(C + D_2) \cup (D_1 + D_2) \cup (D_2 + D_2) = D_2$, $S + D_3 = D_3$, $CC \subseteq C$, $CD_1 \cup D_1 C \cup D_1 D_1 \cup D_2 D_1 \subseteq D_1$, $CD_2 \cup D_2 C \cup D_1 D_2 \cup D_2 D_2 \subseteq D_2$, $SD_3 \subseteq D_3$, $D_3 C \cup D_3 D_2 \subseteq D_2 \cup D_3$, $D_3 D_1 \subseteq D$ and $SD \cup DS \subseteq D$.

(ii) Assume that the condition 5.1(iii) is satisfied. By [2, 3.3], there is a characteristic semimodule M with $\underline{\varepsilon}(N) = R$ (see also the proof of 5.1). Let a_0 be a minimal element of R_1. The set N has the greatest element w, $\underline{\varepsilon}(w) = 0_S$ and if $v \in N$ is such that $\underline{\varepsilon}(v) = a_0$ then v is a maximal element of $N \setminus \{w\}$. Now, by 4.5, the semimodule M is critical, and hence at least one of the conditions 5.3(1),(2),(3) is satisfied due to 5.3 (cf. (i)).

Theorem 5.7. *Assume that $0_S \in S$. The following conditions are equivalent:*

(*i*) *There is a critical semimodule M such that $0_S \notin \underline{\varepsilon}(M \setminus \{o_M\})$ (or, equivalently, such that the set $M \setminus \{o_M\}$ has no greatest element).*

(*ii*) *There is a critical semimodule M such that $\underline{\varepsilon}(M \setminus \{o_M\}) = R \setminus \{0_S\}$ (and the set $M \setminus \{o_M\}$ has no maximal element).*

(*iii*) *$|S| \geq 4$ (S is infinite, resp.) and the following four conditions are satisfied:*

(1) *If the set $R_2 = R \setminus \{o_S\}$ has no maximal element and if $a \in R_2 \setminus \{0_S\}$ and $b \in S$ are such that $R_2 b \leq a$ then $o_S b \geq a$.*

(2) *For all $a, b \in R_1 = R \setminus \{0_S\}$ there is $c \in R_1$ with $c \leq a$ $c \leq b$.*

(3) *For all $c \in S$ such that $0_S c = 0_S$ and $a \in R_1$ there is $b \in R_1$ with $bc \leq a$.*

(4) *For every $c \in S$ such that $0_S c \neq 0_S$ there is $b \in R_1$ with $0_S c = bc$.*

Proof. (i) implies (ii). Use [2, 4.2].

(ii) implies (iii). As concerns (1), we have $R_2 \subseteq Q = \{\, c \in R \mid cb \leq a \,\}$. Thus $Q \neq \emptyset$ and, by [2, 3.4(ii)], we get $o_Q \in Q$. If R_2 has no maximal element then $Q \neq R_2$, and hence $Q = R$. The conditions (2), (3) and (4) follow from [2, 3.4] (see [2, 3.2.3, 3.2.4, 3.2.5]). By [2, 2.11(viii)], the set $M \setminus \{o_M\}$ has no maximal element, and hence both M and S are infinite.

(iii) implies (ii). Since $0_S \in S$ and (ε) is satisfied, we see that the ordered set R is a lattice in fact. If $a \in R \setminus \{0_S\}$, $b \in S$ and $o_Q \notin Q = \{\, c \in R \mid cb \leq a\} \neq \emptyset$ then $R_2 = Q$ follows from (ε). Thus $R_2 b \leq a$ and $o_S b \not\leq a$, a contradiction with (1). Now, according to [2, 3.4], we have a characteristic semimodule M such that $\underline{\varepsilon}(N) = R_1$. By 4.5, M contains a subsemimodule G such that G is critical, $o_M \in G$ and $\underline{\varepsilon}(G \setminus \{o_M\}) = R_1$. Since $\underline{\varepsilon}$ is injective, we get $G = M$. □

Remark 5.8. If the conditions of 5.7 are satisfied then $M \cong {}^2_S R$ (see [1, 7.3].

Remark 5.9. If $|S| = 3$ then the conditions 5.7(1),(2),(3),(4) are satisfied trivially.

Remark 5.10. Notice that the condition 5.7(2) (5.7(3), 5.7(4), resp.) is just the negation of the condition 5.3(1) (5.3(2), 5.3(3), resp.)

Remark 5.11. According to [2, 5.2.4], under our assumptions the conditions 5.1(iii) and 5.7(1) are equivalent.

Remark 5.12. Assume that $0_S \in S$ and the set $R_1 = R \setminus \{0_S\}$ has at least one minimal element (cf. 5.6).

(i) If R_1 has at least two minimal elements then 5.7(2) is not true. On the other hand, if R_1 has just one minimal element, say a_0, then a_0 is the smallest element of R_1 and 5.7(2) is satisfied trivially. Moreover, the condition 5.7(3) is satisfied iff, for any $c \in S$, $0_S c = 0_S$ implies $a_0 c \in \{0_S, a_0\}$. The condition 5.7(4) is satisfied iff, for any $c \in S$, $0_S c \neq 0_S$ implies $0_S c = a_0 c$.

(ii) Assume that the condition 5.1(iii) is satisfied. Then there is a characteristic semimodule M such that $\underline{\varepsilon}(N) = R$ and at least one of the conditions 5.7(2),(3),(4) is not true.

Theorem 5.13. *There is at most one critical semimodule (up to isomorphism).*

Proof. Let M and M' be critical semimodules. If $0_S \notin S$ then $M \cong {}^1_S R \cong M'$ by 5.1 and 5.2. Assume, therefore, that $0_S \in S$. If $0_S \in \underline{\varepsilon}(M \setminus \{0_M\})$ then at least one of the conditions 5.3(1),(2),(3) is satisfied and it follows that at least of the conditions 5.7(2),(3),(4) is not satisfied (see 5.10). Consequently, the (equivalent) conditions of 5.7 are not true, and hence $0_S \in \underline{\varepsilon}(M' \setminus \{o_M\})$. Now, again, $M \cong {}^1_S R \cong M'$ (see 5.3, 5.4). Finally, if $0_S \notin \underline{\varepsilon}(M \setminus \{o_M\})$ and $0_S \notin \underline{\varepsilon}(M' \setminus \{o'_M\})$ then $M \cong {}^2_S R \cong M'$ (see 5.7, 5.8). □

Theorem 5.14. *There is at least one critical semimodule if and only if either the set $R_2 = R \setminus \{o_S\}$ has a maximal element or $b \leq a$ whenever $a \in R_2$ and $b \in S$ are such that $R_2 b \leq a$.*

Proof. See 5.1, 5.2, 5.3, 5.7 and 5.11. □

References

[1] B. Batíková, T. Kepka and P. Němec, *On how to construct left semimodules from the right ones,* Ital. J. Pure Appl. Math., 32 (2014), 561–578.

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