ON SOME SUBCLASSES OF MEROMORPHIC FUNCTIONS DEFINED BY FRACTIONAL DERIVATIVE OPERATOR

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Abstract. In this paper, we use fractional derivative operator to define new classes of meromorphic functions related to conic domains. Using convolution and differential subordination techniques, we prove some interesting properties of these newly defined classes.
Keywords: Fractional derivative operator, conic domain, bounded boundary rotation, coefficient bounds.

1. Introduction

Goodman [4] originated the idea of conic domains by introducing the classes UCV and US^*T. Further Ronning [19], Ma and Minda [10] gave a well ordered one variable characterization of these classes. Later on Kanas and Wisnniowska [6, 7] introduced $k-$uniformly convex and starlike functions. Similarly Rønning and several others [11, 12] studied the class of close to convex functions related

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with conic domains. Acu [1] extended this concept by using Salagean operator [21]. Lowner [9] introduced the class V_m of functions of bounded boundary rotations which was improved by Paatero [17]. Noor connected functions with bounded boundary rotations with conic domains, see for some details [13, 14, 15]. The class of alpha-quasi convex functions was introduced and studied by Noor et-al. [16]. Recently Haq et-al. [5] related this class with conic domains. We generalize this idea to the space of meromorphic functions and define some new classes of meromorphic functions by using fractional derivative of order α. We prove inclusion results, coefficient problems and some other interesting properties.
Let M denote the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}a_k z^{k-1}, \tag{1}$$

which are analytic and univalent in $E^*=\{z:0<|z|<1\}=E\backslash\{0\}$. Let $MS^*(\gamma), MC(\gamma)$ be the subclasses of M that are meromorphic starlike and convex of order $\gamma\,(0\leq\gamma<1)$ respectively. A function $f\in M$ is said to belong to the class $k-MS^*(\gamma)$ of meromorphic uniformly starlike of order γ, if

$$-\Re\mathrm{e}\left\{\frac{zf'(z)}{f(z)}\right\}>k\left|\frac{zf'(z)}{f(z)}-1\right|+\gamma,\quad k\geq 0,\quad 0\leq\gamma<1.$$

It is clear from the definition that for $k=0$, we get the class $MS^*(\gamma)$. Next we define the meromorphic analogue of the class $k-K(\gamma,\beta)$ defined by Noor et-al. [11] as follows:

A function $f\in M$ is in the class $k-MK(\gamma,\beta)$, if

$$\begin{aligned}-\Re\mathrm{e}\left\{\frac{zf'(z)}{g(z)}\right\} &> k\left|\frac{zf'(z)}{g(z)}-1\right|+\beta,\quad \text{for some } g\in k-MS^*(\gamma),\\ k &\geq 0,\quad 0\leq\beta<1.\end{aligned}$$

Further an analytic function p with $p(0)=1$ is said to belong to the class $P(p_{k,\gamma})$ if and only if $p(z)$ takes all values in the conic domain $\Lambda_{k,\gamma}$, $k\geq 0$, $\gamma\in[0,1)$, such that

$$\Lambda_{k,\gamma}=(1-\gamma)\Lambda_k+\gamma,$$

where

$$\Lambda_k=\left\{u+iv:u>k\sqrt{(u-1)^2+v^2}\right\},\quad k\geq 0.$$

The extremal functions $p_{k,\gamma}(\mathrm{z})$ for $P(p_{k,\gamma})$ are given by

$$p_{k,\gamma}(\mathrm{z})=\begin{cases}\frac{1+(1-2\gamma)\mathrm{z}}{1-\mathrm{z}}, & k=0,\\ 1+\frac{2(1-\gamma)}{\pi^2}\left(\log\frac{1+\sqrt{\mathrm{z}}}{1-\sqrt{\mathrm{z}}}\right)^2, & k=1,\\ \frac{1-\gamma}{1-k^2}\cosh\left\{\left(\frac{2}{\pi}\arccos k\right)\log\frac{1+\sqrt{\mathrm{z}}}{1-\sqrt{\mathrm{z}}}\right\}-\frac{k^2-\gamma}{1-k^2}, & 0<k<1,\\ 1+\frac{1-\gamma}{k^2-1}\sin\left(\frac{\pi}{2K(\kappa)}\int_0^{\frac{u(\mathrm{z})}{\sqrt{\kappa}}}\frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-\kappa^2t^2}}\right)+\frac{k^2-\gamma}{k^2-1}, & k>1.\end{cases} \tag{2}$$

where

$$u(\mathrm{z}) = \frac{\mathrm{z} - \sqrt{\kappa}}{1 - \sqrt{\kappa}\mathrm{z}}, \quad \mathrm{z} \in \mathrm{E},$$

and $\kappa \in (0,1)$ is selected in such a way that $k = \cosh(\pi K'(\kappa)/(4K(\kappa)))$. Here $K(\kappa)$ is Legendre's complete elliptic integral of first kind and $K'(\kappa) = K(\sqrt{1-\kappa^2})$ and $K'(t)$ is the complementary integral of $K(t)$.

Remark 1. *If* $p_{k,\gamma}(\mathrm{z})$, *be given by*

$$p_{k,\gamma}(\mathrm{z}) = 1 + \delta_1 \mathrm{z} + \delta_2 \mathrm{z}^2 + ...,$$

then

$$\delta_1 = \delta(k,\gamma) = \begin{cases} 8(1-\gamma)(arc\cos k)^2, & 0 \le k < 1 \\ \frac{8(1-\gamma)}{\pi^2}, & k = 1, \\ \frac{\pi^2(1-\gamma)}{4\sqrt{x(k^2-1)K^2(x)(x+1)}}, & k > 1. \end{cases} \tag{3}$$

The meromorphic analogue of the fractional derivative of order α, $0 \le \alpha < 1$, is defined in [2] for a function $f(z)$ by

$$D_z^\alpha f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \left\{ z^{\alpha-1} \int_0^z (z-\xi)^{-\alpha} {}_2F_1 \left(1-\alpha, 1, 1-\alpha; 1 - \frac{\xi}{2}\right) \xi^2 f(\xi) \right\},$$

where $f(z)$ is analytic function in a simply connected domain of the $z-$plane containing the origin and the multiplicity of $(z-\xi)^{-\alpha}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi) > 0$. Using $D_z^\alpha f(z)$, we define an operator $\Omega_z^\alpha f(z) : M \to M$, as follows:

$$\begin{aligned} \Omega_z^\alpha f(z) &= \frac{\Gamma(2-\alpha)}{\Gamma(2)} z D_z^\alpha f(z) \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(2)_k}{(2-\alpha)_k} a_k z^{k-1} \\ &= \phi(2, 2-\alpha; z) * f(z), \quad \alpha \neq 2, 3, 4, .., \end{aligned} \tag{4}$$

where

$$\phi(2, 2-\alpha; z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(2)_k}{(2-\alpha)_k} z^{k-1}.$$

We now define the following classes of functions.

Definition 2. *Let* $f \in M$. *Then* $f \in k - MK(m, \gamma, \beta)$, $k \ge 0, \gamma, \gamma \in (0,1)$, $m \ge 2$ *if and only if there exists* $g \in k - MS^*(\gamma)$ *such that*

$$-\frac{zf'(z)}{g(z)} \in \mathcal{P}_m(\mathrm{p}_{k,\beta}), \quad z \in E^*.$$

Also $f \in k - MK_{\mu}^{\alpha}(m,\gamma,\beta)$ *if and only if*

$$-\left\{(1-\mu)\,\Omega_z^{\alpha} f(z) + \mu\left(z\left(\Omega_z^{\alpha} f(z)\right)'\right)\right\} \in k - MK(m,\gamma,\beta).$$

It can easily be seen that $f \in k - MUK_{\mu}^{\alpha}(m,\gamma,\beta)$ *implies that*

$$-\left\{(1-\mu)\frac{z\left(\Omega_z^{\alpha} f(z)\right)'}{\Omega_z^{\alpha} g(z)} + \mu\frac{z\left(z\left(\Omega_z^{\alpha} f(z)\right)'\right)'}{\Omega_z^{\alpha} g(z)}\right\} \in \mathcal{P}_m(\mathrm{p}_{k,\beta}), \tag{5}$$

where $g \in k - MS^{*}(\gamma), k \geq 0, \gamma, \beta \in (0,1), m \geq 2, \mu \in \mathbb{R}$ *and* $z \in E$.

Special cases

i) For $\alpha = k = \gamma = \beta = 0, m = 2, \mu = 1$, the class $0 - MUK_1^0(2,0,0)$ reduces to the class MC^{*} of meromorphic quasi convex functions.

ii) For $\alpha = k = \gamma = \beta = 0, m = 2, \mu = 0$, the class $0 - MUK_0^0(2,0,0)$ reduces to the class MK, see [8].

Remark 3. Note that if f is given by (1) then from (4), we can write

$$(1-\mu)\,\Omega_z^{\alpha}\mathrm{f}(\mathrm{z}) + \mu \mathrm{z}\left(\Omega_z^{\alpha}\mathrm{f}(\mathrm{z})\right)' = \mathrm{z} + \sum_{j=2}^{\infty}\mathcal{A}_j \mathrm{z}^j,$$

where

$$\mathcal{A}_j = \frac{\Gamma(j+1)\,\Gamma(2-\alpha)}{\Gamma(j+1-\alpha)}\left(1+\mu(j-1)\right)a_j. \tag{6}$$

2. Preliminaries

Lemma 4 ([20])**.** *Let f and* g *be convex and starlike univalent functions respectively. Then, for any analytic function* F *in E*

$$\frac{\mathrm{f} * \mathrm{Fg}}{\mathrm{f} * \mathrm{g}}(\mathrm{E}) \subset \overline{co}\left(\mathrm{F}(\mathrm{E})\right),$$

where $\overline{co}(\mathrm{F}(\mathrm{E}))$ *denotes the close convex hull of* $\mathrm{F}(\mathrm{E})$.

Lemma 5 ([18])**.** *Let* $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ *be subordinate to* $H(z) = 1 + \sum_{n=1}^{\infty} C_n z^n$ *in E. If H is univalent and* $H(\mathrm{E})$ *is convex, then* $|c_n| \leq |C_1|, n \geq 1$.

Now we give the following lemma concerning with the class k-$MS^{*}(\gamma)$. The proof is straight forward therefore, we omit it.

Lemma 6. *Let* $f \in k - MS^{*}(\gamma)$. *Then*

$$|b_j| \leq \frac{\left(|\delta(k,\gamma)|\right)_{j+1}}{\left(|\delta(k,\gamma)|+1\right)\left(|\delta(k,\gamma)|+2\right)(j+1)!}, \qquad j \geq 2.$$

Proceeding in a similar way as [11], yields the following result.

Lemma 7. *Let* $\Omega_z^{\alpha} f(\mathrm{z}) \in k - MS^{*}(\gamma)$. *Then* $f(\mathrm{z}) \in k - MS^{*}(\gamma)$.

3. Main results

Here we shall investigate certain properties of above defined classes. We use techniques of convolution and differential subordination to study these properties. Throughout in this section, we assume $k \geq 0$, $m \geq 2$, α, γ, $\beta \in [0,1)$ and $z \in E$, unless otherwise stated.

Theorem 8. *Let*

$$\Re e\left\{\frac{z\left(\Omega_z^\alpha g\left(z\right)\right)'}{\Omega_z^\alpha g\left(z\right)}\right\} < 2-\rho, \quad 0 \leq \rho < 1.$$

Then

$$k - MK_\mu^\alpha\left(m,\gamma,\beta\right) \subset k - MUK_\mu^0\left(m,\gamma,\beta\right).$$

Proof. Let $f \in k - MUK_\mu^\alpha\left(m,\gamma,\beta\right)$. Then

$$-\left\{\left(1-\mu\right)\frac{\mathrm{z}\left(\mathfrak{L}_\alpha \mathrm{f}\left(\mathrm{z}\right)\right)'}{\mathfrak{L}_\alpha g\left(\mathrm{z}\right)} + \mu\frac{\mathrm{z}\left(\mathrm{z}\left(\mathfrak{L}_\alpha \mathrm{f}\left(\mathrm{z}\right)\right)'\right)'}{\mathfrak{L}_\alpha g\left(\mathrm{z}\right)}\right\} \in \mathcal{P}_m\left(\mathrm{p}_{k,\beta}\right),$$

where

$$-\frac{\mathrm{z}\left(\mathfrak{L}_\alpha g\left(\mathrm{z}\right)\right)'}{\mathfrak{L}_\alpha g\left(\mathrm{z}\right)} \in \mathcal{P}\left(\mathrm{p}_{k,\gamma}\right) \quad \text{in } E^*.$$

From Lemma 4, we have, $g \in k - MS^*\left(\gamma\right)$ whenever $\mathfrak{L}_\alpha g\left(\mathrm{z}\right) \in k - MS^*\left(\gamma\right)$, in E^*. Now

$$\begin{aligned}
& -\left\{\left(1-\mu\right)\frac{z\mathrm{f}'\left(z\right)}{g\left(z\right)} + \mu\frac{z\left(z\mathrm{f}'\left(z\right)\right)'}{g\left(z\right)}\right\} \\
= & -\left\{\begin{array}{l}\left(1-\mu\right)\frac{\varphi(2-\alpha,2;z)*\varphi(2,2-\alpha;z)*z\mathrm{f}'(z)}{\varphi(2-\alpha,2;z)*\varphi(2,2-\alpha;z)*g(z)} \\ +\mu\frac{\varphi(2-\alpha,2;z)*\varphi(2,2-\alpha;z)*z(z\mathrm{f}'(z))'}{\varphi(2-\alpha,2;z)*\varphi(2,2-\alpha;z)*g(z)}\end{array}\right\} \\
= & -\left\{\begin{array}{l}\left(1-\mu\right)\frac{\varphi(2-\alpha,2;z)*z(\Omega_z^\alpha \mathrm{f}(z))'}{\varphi(2-\alpha,2;z)*\Omega_z^\alpha g(z)} \\ +\mu\frac{\varphi(2-\alpha,2;z)*z\left(z(\Omega_z^\alpha \mathrm{f}(z))'\right)'}{\varphi(2-\alpha,2;z)*\Omega_z^\alpha g(z)}\end{array}\right\} \\
= & \frac{\varphi\left(2-\alpha,2;z\right)*\left[-\left\{\left(1-\mu\right)\frac{z(\Omega_z^\alpha \mathrm{f}(z))'}{\Omega_z^\alpha g(z)} + \mu\frac{z\left(z(\Omega_z^\alpha \mathrm{f}(z))'\right)'}{\Omega_z^\alpha g(z)}\right\}\right]\left(\Omega_z^\alpha g\right)}{\varphi\left(2-\alpha,2;z\right)*\Omega_z^\alpha g\left(z\right)} \\
= & \frac{\varphi\left(2-\alpha,2;z\right)*F\left(\Omega_z^\alpha g\right)}{\varphi\left(2-\alpha,2;z\right)*\Omega_z^\alpha g\left(z\right)} \\
= & \left(\frac{m}{4}+\frac{1}{2}\right)\frac{\varphi\left(2-\alpha,2;z\right)*F_1\left(\Omega_z^\alpha g\right)}{\varphi\left(2-\alpha,2;z\right)*\Omega_z^\alpha g\left(z\right)} \\
& -\left(\frac{m}{4}-\frac{1}{2}\right)\frac{\varphi\left(2-\alpha,2;z\right)*F_2\left(\Omega_z^\alpha g\right)}{\varphi\left(2-\alpha,2;z\right)*\Omega_z^\alpha g\left(z\right)},
\end{aligned}$$

where $F_i \in P\left(\mathrm{p}_{k,\gamma}\right)$, $i = 1,2$ and by hypothesis of theorem, we get

$$\Re\mathrm{e}\left\{\frac{z\left(\Omega_z^\alpha g\left(z\right)\right)'}{\Omega_z^\alpha g\left(z\right)}\right\} < 2-\rho, \qquad 0 \leq \rho < 1.$$

From which we have

$$\Re\text{e}\left\{\frac{z\left(z^2\Omega_z^\alpha g(z)\right)'}{z^2\Omega_z^\alpha g(z)}\right\} > \rho.$$

This implies $z^2\Omega_z^\alpha g(z) \in S^*$, the class of usual starlike functions.

Now $z^2\psi(2-\alpha,2;z) = \varphi(2-\alpha,2;z)$ is a convex function. Using Lemma 1, we have for $i = 1,2$

$$\frac{z^2\psi(2-\alpha,2;z) * G_i\left(z^2\Omega_z^\alpha g\right)}{z^2\psi(2-\alpha,2;z) * z^2\Omega_z^\alpha g(z)} \subseteq \overline{co}\left(\mathrm{p}_{k,\beta}(\mathrm{E})\right).$$

This shows that

$$-\left\{(1-\mu)\frac{z\mathbf{f}'(z)}{g(z)} + \mu\frac{z\left(z\mathbf{f}'(z)\right)'}{g(z)}\right\} \in \mathcal{P}_m\left(\mathrm{p}_{k,\beta}\right) \text{ in } E^*.$$

Thus $f \in k - MUK_\mu^0(m,\gamma,\beta)$. □

The following theorem can be proved by using the similar arguments as above.

Theorem 9. *Let* $0 \le \alpha_1 < \alpha - 2 < 1$. *Then for*

$$\Re e\left\{\frac{z\left(\Omega_z^\alpha g(z)\right)'}{\Omega_z^\alpha g(z)}\right\} < 2-\rho, \quad 0 \le \rho < 1,$$
$$k - MUK_\mu^{\alpha_2}(m,\gamma,\beta) \subset k - MUK_\mu^{\alpha_1}(m,\gamma,\beta).$$

Theorem 10. *Let* $f \in k - MUK_\mu^\alpha(m,\gamma,\beta)$ *and be given by* (1). *Then*

$$\begin{aligned} |a_j| \quad &\le \quad \frac{\lceil(j+1-\alpha)}{(1+\mu(j-1))\lceil(j+1)\lceil(2-\alpha)}\left\{\frac{2\left(\delta(k,\beta)\right)_{j+1}}{(\delta(k,\beta)+1)(\delta(k,\beta)+2)\,j\,(j+1)!}\right. \\ &\left. + \frac{m}{2j}\left|\delta(k,\beta)\right|\left(1+\sum_{l=1}^{j-1}\frac{2\left|\delta(k,\beta)\right|_{l+1}}{(\delta(k,\beta)+1)(\delta(k,\gamma)+2)(l+1)!}\right)\right\} \end{aligned}$$

Proof. Let $G(z) = \Omega_z^\alpha g(z) \in k - MS^*(\gamma)$ and write

$$G(z) = \frac{1}{z} + \sum_{j=2}^{\infty}\mathcal{B}_j z^{j-1},$$
$$g(z) = \frac{1}{z} + \sum_{j=2}^{\infty} b_j z^{j-1}.$$

Then

$$\mathcal{B}_j = \frac{\lceil(j+1)\lceil(2-\alpha)}{\lceil(j+1-\alpha)}b_j, \quad j \ge 2. \tag{7}$$

For $p \in P_m(\mathrm{p}_{k,\beta})$ and $p(z) = 1 + c_1 z + c_2 z^2 + ...$, let

$$\mathrm{p}(z) = \left(\frac{m}{4} + \frac{1}{2}\right) \mathrm{p}_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right) \mathrm{p}_2(z), \quad \mathrm{p}_i(z) \prec \mathrm{p}_{k,\beta}, \quad i = 1, 2,$$

writing

$$\mathrm{p}_i(z) = 1 + d_1 z + d_2 z^2 + ..., \qquad j \geq 1,$$

we have

$$|d_j| \leq |\delta(k, \beta)|,$$

where $\delta(k, \beta)$ is given by (3) and we have used Lemma 2. Combining these facts, we have

$$|c_j| \leq \frac{m}{2} |\delta(k, \beta)|. \tag{8}$$

Now, using (4) and (5), we have

$$j\mathcal{A}_j = \mathcal{B}_j + \sum_{l=1}^{j-1} c_{j-l} \mathcal{B}_j, \quad j \geq 2. \tag{9}$$

From (7), (8), (9) and Lemma 3, it follows that

$$|\mathcal{A}_j| \leq \frac{(|\delta(k, \beta)|)_{j-1}}{j!} + \frac{m|\delta(k, \beta)|}{2j} \sum_{l=1}^{j-1} \frac{|\delta(k, \beta)|_{l-1}}{(l-1)!}. \tag{10}$$

We obtain the desire result from (4) and (10). □

Theorem 11. *Let* $\mathbf{f} \in k - MUK_\mu^\alpha(m, \gamma, \beta)$ *and* $h \in M$ *such that* $\mathrm{z}^2 h$ *be a convex univalent function. Then for* $\Re e\{\frac{\mathrm{z}(\Omega_z^\alpha g(\mathrm{z}))'}{\Omega_z^\alpha g(\mathrm{z})}\} < 2 - \rho$, $0 \leq \rho < 1$,

$$(\mathbf{f} * h)(\mathrm{z}) \in k - MUK_\mu^\alpha(m, \gamma, \beta), \quad \mathrm{z} \in \mathrm{E}^*.$$

Proof. Let f$\in k - MUK_\mu^\alpha(m, \gamma, \beta)$. Then

$$-\left\{(1-\mu) \frac{\mathrm{z}(\Omega_z^\alpha \mathrm{f}(\mathrm{z}))'}{\Omega_z^\alpha g(\mathrm{z})} + \mu \frac{\mathrm{z}\left(\mathrm{z}(\Omega_z^\alpha \mathrm{f}(\mathrm{z}))'\right)'}{\Omega_z^\alpha g(\mathrm{z})}\right\} \in \mathcal{P}_m(\mathrm{p}_{k,\beta}),$$

where

$$\Omega_z^\alpha g(\mathrm{z}) \in k - MS^*(\gamma) \subseteq MS^* \quad \text{in } \mathrm{E}^*.$$

Now

$$-\left\{(1-\mu) \frac{\mathrm{z}(\Omega_z^\alpha(\mathrm{f} * h)(\mathrm{z}))'}{\Omega_z^\alpha(g * h)(\mathrm{z})} + \mu \frac{\mathrm{z}\left(\mathrm{z}(\Omega_z^\alpha(\mathrm{f} * h)(\mathrm{z}))'\right)'}{\Omega_z^\alpha(g * h)(\mathrm{z})}\right\}$$

$$= \frac{z^2 h * \left\{-\left\{(1-\mu) \frac{\mathrm{z}(\Omega_z^\alpha \mathrm{f}(\mathrm{z}))'}{\Omega_z^\alpha g(\mathrm{z})} + \mu \frac{\mathrm{z}\left(\mathrm{z}(\Omega_z^\alpha \mathrm{f}(\mathrm{z}))'\right)'}{\Omega_z^\alpha g(\mathrm{z})}\right\}\right\} z^2 \Omega_z^\alpha g}{z^2 h * z^2 \Omega_z^\alpha g}$$

$$= \frac{z^2 h * F\left(z^2 \Omega_z^\alpha g\right)}{z^2 h * z^2 \Omega_z^\alpha g}.$$

Now from the hypothesis of theorem and applying Lemma 1, we obtain our desire result. □

The following operator was defined by Bajpai in [3]. For $\varepsilon \in \mathbb{C}$ and $\Re e\, \varepsilon > 0$, we have $I_\varepsilon : M \to M$ as

$$I_\varepsilon G(\mathrm{z}) = \frac{\varepsilon}{\mathrm{z}^{\varepsilon+1}} \int_0^{\mathrm{z}} G(t)\, t^\varepsilon dt. \tag{11}$$

Theorem 12. *Let* $G \in k - MUK_\mu^\alpha(m,\gamma,\beta)$ *and let* $f(\mathrm{z}) = I_\varepsilon G(\mathrm{z})$, *where* I_ε *is the integral operator defined by* (11). *Then* $f \in k - MUK_\mu^\alpha(m,\gamma,\beta)$ *for*

$$\Re e \left\{ \frac{\mathrm{z}\left(\Omega_z^\alpha g(\mathrm{z})\right)'}{\Omega_z^\alpha g(\mathrm{z})} \right\} < 2 - \rho, \quad 0 \le \rho < 1$$

and $\mathrm{z} \in E^*$.

Proof. As

$$I_\varepsilon G(\mathrm{z}) = \Psi(\mathrm{z}) * G(\mathrm{z}),$$

where

$$\Psi_\varepsilon(\mathrm{z}) = \frac{1}{\mathrm{z}} + \sum_{j=0}^{\infty} \frac{\varepsilon}{1+j+\varepsilon}\mathrm{z}^j, \qquad \Re e\, \{\varepsilon\} > 0.$$

Now $\mathrm{z}^2\Psi_\varepsilon(\mathrm{z}) = \phi(\mathrm{z})$, with $\varepsilon = 1 + a$ is convex in E^*, see [20]. Proof follows immediately by applying Theorem 4, and hence $f \in k - MUK_\mu^\alpha(m,\gamma,\beta)$ for z in E^*. □

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