ON AN EXTENSION TO KHAN'S FIXED POINT THEOREM

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Abstract. In this present paper, the fixed point theorem that was proved by Khan [2] is extended to sequences of self maps through rational expressions. The present theorem includes non-continuous maps also.
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1. Introduction

The notion of 2-metric space was introduced by Gahler [1] in 1963 as a generalization of area function for Euclidean triangles. Many fixed point theorems were established by various authors like Khan[2], Rhoades [4], etc. A point $x \in X$ is said to be a fixed point of a self-map $f : X \to X$ if $f(x) = x$, where X is a non-empty set. Theorems concerning fixed points of self-maps are known as fixed point theorems. Most of the fixed point theorems were proved for contraction mappings. It is well known that every contraction on a metric space is continuous. The converse is not necessarily true. The identity mapping on [0,1] simply serves the counter example.

In this present work, the Khans fixed point theorem [2] is extended to a more general form in 2-metric space setting. Our generalized theorem holds for non-continuous selfmaps also.

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2. Preliminaries

In this section, we present some basic definitions which are needed for the further study of this paper.

Definition 2.1. Let X be a non-empty set and $d : X \times X \times X \to \mathbb{R}$. For all x, y, z and u in X, if d satisfies the following conditions:

(a) $d(x, y, z) = 0$ if at least two of x, y, z are equal

(b) $d(x, y, z) = d(x, z, y) = d(y, z, x) = \ldots$

(c) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$.

Then d is called a 2-metric on X and the pair (X, d) is called a 2-metric space.

Definition 2.2. Let (X, d) be a 2-metric space. A sequence $\{X_n\}$ in X is called a *Cauchy sequence*, if $d(x_m, x_n, a) \to 0$ as $m, n \to 0$, for all $a \in X$.

Definition 2.3. Let (X, d) be a 2-metric space. A sequence $\{x_n\}$ is said to converge to a point x in X if $\lim_{n\to\infty} d(x_n, x, a) = 0$, for every a in X.

Definition 2.4. A 2-metric space (X, d) is said to be a complete 2-metric space if every Cauchy sequence in X converges in X.

3. Generalized fixed point theorem

The following Theorem 3.2 is an extension of Khan's Fixed point theorem for non-continuous maps. We present Khan's result as Theorem 3.1 for completeness.

In what follows X stands for a complete 2-metric space with 2-metric d.

Theorem 3.1. *Suppose that $f : X \to X$,*

$$d(f(x), f(y), a) \leq \alpha \frac{[d(x, f(x), a)]^{r+w} [d(y, f(y), a)]^{1-r}}{[d(x, y, a)]^w} + \beta [d(x, y, a)]^{1-r-w} [d(f(x), f(y), a)]^{r+w},$$

for all x, y, a in X and for some $\alpha, \beta, w \in [0, 1)$, $r \in (0, 1)$ such that $a + \beta < 1$ and $2r + w = 1$ when $w \neq 0$. Then f has a unique fixed point in X.

Theorem 3.2. *Let $\{f_n\}$ and $\{g_n\}$ be two sequences of self-maps on X satisfying*

$$d(f_n^p(x), g_n^q(y), a) \leq \alpha \frac{[d(x, f_n^p(x), a)]^{r+w} [d(y, g_n^q(y), a)]^{1-r}}{[d(x, y, a)]^w} + \beta [d(x, y, a)]^{1-r-w} [d(f_n^p(x), g_n^q(y), a)]^{r+w}, \tag{1}$$

for all x, y, a in X and for some $\alpha, \beta, w \in [0, 1), r \in (0, 1)$ such that $\alpha + \beta < 1$ and $2r + w = 1$ when $w \neq 0$, p and q are positive integers then f_n and g_n have a unique common fixed point in X.

Proof. Let $x_0 \in X$. Define a sequence $\{x_n\}$ in X as follows $x_{2n-1} = f_n^p(x_{2n-2})$ and $x_{2n} = g_n^q(x_{2n-1})$ for $n = 1, 2, 3, \ldots$. Then we have

$$\begin{aligned} d(x_{2n-1}, x_n, a) &= d(f_n^p(x_{2n-2}), g_n^q(x_{2n-1}), a) \\ &\leq \alpha \frac{[d(x_{2n-2}, f_n^p(x_{2n-2}), a)]^{r+w}[d(x_{2n-1}, g_n^q(x_{2n-1}), a)]^{1-r}}{[d(x_{2n-2}, x_{2n-1}, a)]^w} \\ &+ \beta d(x_{2n-2}, x_{2n-1}, a)]^{1-r-w}[d(f_n^p(x_{2n-2}), g_n^q(x_{2n-1})), a)]^{r+w} \quad (2) \\ &\alpha \frac{[d(x_{2n-2}, x_{2n01}, a)^{r+w}][d(x_{2n-1}, x_{2n}, a)]^{1-r}}{[d(x_{2n-2}, x_{2n-1}, a)]^w} \\ &= \beta[d(x_{2n-2}, x_{2n-1}, a)]^{1-r-w}[d(x_{2n-1}, x_{2n}, a)]^{r+w} \\ &= \alpha[d(x_{2n-2}, x_{2n-1}, a)]^r[d(x_{2n-1}, x_{2n}, a)]^{1-r} \\ &+ \beta[d(x_{2n-2}, x_{2n-1}, a)]^{1-r-w}[d(x_{2n-1}, x_{2n}, a)]^{r+w} \end{aligned}$$

or

$$\begin{aligned} &d(x_{2n-1}, x_{2n}, a) \leq \alpha[d(x_{2n-2}, x_{2n-1}, a)]^r[d(x_{2n-1}, x_{2n}, a)]^{1-r} \\ &+ \beta[d(x_{2n-2}, x_{2n-1}, a)]^{1-r-w}[d(x_{2n-1}, x_{2n}, a)]^{r+w}. \quad (3) \end{aligned}$$

Now

$$\begin{aligned} &d(x_{2n}, x_{2n+1}, a) = d(g_n^q(x_{2n-1}), f_n^p(x_{2n}), a) \\ &= d(f_n^p(x_{2n}), g_n^p(x_{2n-1}), a) \\ &\leq \alpha \frac{[d(x_{2n}, f_n^p(x_{2n}), a)]^{r+w}[d(x_{2n-1}, g_n^q(x_{2n-1}), a)]^{1-r}}{[d(x_{2n}, x_{2n-1}, a)]^w} \\ &+ \beta[d(x_{2n}, x - 2n - 1, a)]^{1-r-w}[d(f_n^p(x_{2n}), g_n^q(x_{2n-1}), a)]^{r+w} \\ &= \alpha \frac{[d(x_{2n}, x_{2n+1}, a)]^{r+w}[d(x_{2n-1}, x_{2n}, a)]^{1-r}}{[d(x_{2n}, x_{2n-1}, a)]^w} \\ &+ \beta[d(x_{2n-1}, x_{2n}, a)]^{1-r-w}[d(x_{2n+1}, x_{2n}, a)]^{r+w} \quad (4) \\ &= \alpha[d(x_{2n}, x_{2n+1}, a)]^{r+w}[d(x_{2-1}, x_{2n}, a)]^{1-r-w} \\ &+ \beta[d(x_{2n-1}, x_{2n}, a)]^{1-r-w}[d(x_{2n}, x_{2n+1}, a)]^{r+w} \\ &= (\alpha + \beta)[d(x_{2n-1}, x_{2n}, a)]^{1-r-w}[d(x_{2n, x_{2n+1}, a})]^{r+w} \\ &\Rightarrow d(x_{2n}, x_{2n+1}, a) \leq (\alpha + \beta)[d(x_{2n-1}, x_{2n}, a)]^{1-r-w}[d(x_{2n}, x_{2n-1}, a)]^{r+w} \\ &\Rightarrow [d(x_{2n}, x_{2n+1}, a)]^{1-r-w} \leq (a + \beta)[d(x_{2n-1}, x_{2n}, a)]^{1-r-w} \\ &\Rightarrow d(x_{2n}, x_{2n+1}, a) \leq (a + \beta)^{1/1-r-w} d(x_{2n-1}, x_{2n}, a). \end{aligned}$$

Case (i). When $w \neq 0$, we obtain

$$d(x_{2n}, x_{2n+1}, a) \leq (\alpha + \beta)^{1/r} d(x_{2n-1}, x_{2n}, a)$$

and from (2), we have

$$
\begin{aligned}
d(x_{2n-1}, x_{2n}, a) &\le \alpha[d(x_{2n-2}, x_{2n-1}, a)]^r[d(x_{2n-1}, x_{2n}, a)]^{1-r} \\
&+ \beta[d(x_{2n-2}, x_{2n-1}, a)]^r[d(x_{2n-1}, x_{2n}, a)]^{1-r} \\
&= (\alpha+\beta)[d(x_{2n-2}, x_{2n-1}, a)^r][d(x_{2n-1}, x_{2n}, a)]^{1-r} \\
&\Rightarrow [d(x_{2n-1}, x_{2n}, a)]^r \le (\alpha+\beta)[d(x_{2n-2}, x_{2n-1}, a)]^r \\
&\Rightarrow d(x_{2n-1}, x_{2n}, a) \le k_1 d(x_{2n-2}, x_{2n-1}, a)
\end{aligned} \tag{5}
$$

where $k_1 = (\alpha+\beta)^{1/r} < 1$. Therefore we have

$$
\begin{aligned}
d(x_{2n}, x_{2n+1}, a) &\le k_1 d(x_{2n-1}, x_{2n}, a) \le k_1^2 d(x_{2n-2}, x_{2n-1}, a) \\
&\le \ldots \le k_1^{2n} d(x_0, x_1, a) \\
&\Rightarrow d(x_{2n}, x_{2n+1}, a) \le k_1^{2n} d(x_0, x_1, a).
\end{aligned}
$$

Since $k_1 < 1$, $k_1^{2n} \to 0$ as $n \to \infty$. Hence $d(x_{2n}, x_{2n+1}, a) = 0$ as $n \to \infty$

Case (ii)

Suppose that $w = 0$

From (3) and (2), we have $d(x_{2n}, x_{2n+1}, a) \le (\alpha+\beta)^{1/1-r} d(x_{2n-1}, x_{2n}, a)$ and

$$
\begin{aligned}
d(x_{2n-1}, x_{2n}, a) &\le \alpha[d(x_{2n-2}, x_{2n-2}, a)]^r[d(x_{2n-1}, x_{2n}, a)]^{1-r} \\
&+ \beta[d(x_{2n-2}, x_{2n-1}, a)]^{1-r} d[(x_{2n-1}, x_{2n}, a)]^r.
\end{aligned} \tag{6}
$$

We claim that $d(x_{2n-1}, x_{2n}, a) \le d(x_{2n-2}, x_{2n-1}, a)$.

If it is not so, suppose that $d(x_{2n-1}, x_{2n}, a) > d(x_{2n-2}, x_{2n-1}, a)$. Then from (4), we have

$$
\begin{aligned}
d(x_{2n-1}, x_{2n}, a) &\le \alpha[d(x_{2n-1}, x_{2n}, a)]^r[d(x_{2n-1}, x_{2n}, a)]^{1-r} \\
&+ \beta[d(x_{2n-1}, x_{2n}, a)]^{1-r}[d(x_{2n-1}, x_{2n}, a)]^r \\
&= (\alpha+\beta) d(x_{2n-1}, x_{2n}, a)
\end{aligned}
$$

or

$$d(x_{2n-1}, x_{2n}, a) \le (\alpha+\beta) d(x_{2n-1}, x_{2n}, a)$$

Which is a contradiction, since $\alpha+\beta < 1$.

Therefore

$$
\begin{aligned}
d(x_{2n-1}, x_{2n}, a) &\le d(x_{2n-2}, x_{2n-1}, a) \le d(x_{2n-3}, x_{2n-2}, a) \\
&\le k_2^2 d(x_{2n-4}, x_{2n-3}, a) \le \ldots \le k_2^{2n-2} d(x_0, x_1, a) \\
&\Rightarrow d(x_{2n-1}, x_{2n}, a) \le k_2^{2n-2} d(x_0, x_1, a).
\end{aligned}
$$

Where $k_2 = (\alpha+\beta)^{1/r} < 1$. Since $k_2 < 1, k_2^{2n-2} \to 0$ as $n \to \infty$.

Hence $d(x_{2n-1}, x_{2n}, 0) = 0$ as $n \to \infty$.

Hence $d(x_{2n}, x_{2n+1}, a) = 0$ as $n \to \infty$.

Thus in two cases $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, $\{X_n\}$ converges to some point, say, u in X.

Now we prove that u is a unique common fixed point of f_n and g_n, $(n = 1, 2, 3, \ldots)$.

For this first we prove that $u == f_n^p(u) = g_n^q(u)$, $n = 1, 2, 3, \ldots$

Now by the properties of 2-metric, we have

$$\begin{aligned} d(u, f_n^p(u), a) &\leq d(u, f_n^p(u), x_{2n}) + d(ui, x_{2n}, a) + d(x_{2n}, f_n^p(u), a) \\ &= d(u, f_n^p(u), x_{2n}) + d(u, x_{2n}, a) + d(f_n^p(u), g_n^q(x_{2n-1}), a). \end{aligned}$$

Taking $x = u$ and $y = x_{2n-1}$ in (1), we get

$$\begin{aligned} d(f_n^p(u), g_n^q(x_{2n-1}), a) \leq \alpha & \frac{[d(u, f_n^p(u), a)]^{r+w}[d(x_{2n-1}, g_n^q(x_{2n-1}), a)]^{1-r}}{[d(u, x_{2n-1}, a)]^w} \\ & + \beta[d(u, x_{2n-1}, a)^{1-r-w}][d(f_n^p(u), g_n^q(x_{2n-1}), a)]^{r+w}. \end{aligned}$$

Hence

$$\begin{aligned} & d(u, f_n^p, a) \leq d(u, f_n^p(u), x_{2n}) + d(u, x_{2n}, a) \\ & + \alpha \frac{[d(u, f_n^p(u), a)]^{r+w}[d(x_{2n-1}, g_n^q(x_{2n-1}), a)]^{1-r}}{[d(u, x_{2n-1}, a)]^w} \\ & + \beta[d(u, x_{2n-1}, a)]^{1-r-w}[d(f_n^p(u), g_n^q(x_{2n-1}), a)]^{r+w}. \end{aligned}$$

When as $n \to \infty$, $d(u, f_n^p(u), a) \leq 0 + \alpha(0) + \beta(0)$, which implies that $d(u, f_n^p(u), a) = 0$. Hence $f_n^p(u) = u$. Similarly $g_n^q(u) = u$. Therefore u is a common fixed point of f_n^p and g_n^q. To show that u is unique, let v be the another common fixed point of f_n^p and $g_n^q \Rightarrow v = f_n^p(v) = g_n^q(v)$, $n = 1, 2, 3, \ldots$ Then by given condition

$$\begin{aligned} & d(f_n^p(u), g_n^q(v), a) \leq \alpha \frac{[d(u, f_n^p(u), a)]^{r+w}[d(v, g_n^q(v), a)]^{1-r}}{[d(u, v, a)]^w} \\ & + \beta[d(u, v, a)]^{1-r-w}[d(f_n^p(u), g_n^q(v), a)]^{r+w} \\ & = \alpha \frac{[d(u, u, a)]^{r+w}[d(v, v, a)]^{1-r}}{[d(u, v, a)]^w} + \beta[d(u, v, a)]^{1-r-w}[d(u, v, a)]^{r+w} \end{aligned}$$

or $(1 - \beta)d(u, v, a) \leq 0$, which implies that $d(u, v, a) = 0$, for every a in X.

$\Rightarrow u = v$

Hence u is a unique common fixed point of f_n^p and g_n^q.

Finally we show that u is the only common fixed point of f_n and g_n. For $f_n^p(f_n(u)) = f_n(f_n^p(u))$ gives $f_n^p(f_n(u)) = f_n(u)$. Hence $f_n(u)$ is the fixed point of f_n^p. Since u is the unique fixed point of $f_n^p, f_n(u) = u$.

Similarly $g_n(u) = u$. To show that u is unique common fixed point of f_n and g_n, let z be the another common fixed point of f_n and g_n. That is, $z = f_n(z) =$

$g_n(z)$. From the given condition, we have

$$\begin{aligned}
d(u,z,a) &= d(f_n(u), g_n(z), a) \\
&= d(f_n^p(u), g_n^q(z), a) \\
&\le \alpha \frac{[d(u, f_n^p(u), a)]^{r+q}[d(z, g_n^q(z), a)]^{1-r}}{[d(u,z,a)]^w} \\
&+ \beta[d(u,z,a)]^{1-r-w}[d(f_n^p(u), g_n^q(z), a)]^{r+w} \\
&= \alpha \frac{[d(u,u,a)]^{r+w}[d(z,z,a)^{1-r}]}{[d(u,z,a)]^w} + \beta[d(u,z,a)]^{1-r-w}[d(u,z,a)]^{r+w}
\end{aligned}$$

or

$$(1-b)d(u,z,a) \le 0$$

which implies that $d(u,z,a) = 0$ for every a in X.

$\Rightarrow u = z$

Hence u is a unique common fixed point of f_n and g_n, $(n = 1, 2, 3, \ldots)$ □

Remark 3.3. Theorem 3.1 can be easily deduced as a corollary from Theorem 3.2 by taking $f_n^p = g_n^q = f$. Hence Theorem 3.2 is a generalized version of Khan's theorem [2] in 2-metric space.

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