SOME VERTEX-DEGREE-BASED TOPOLOGICAL INDICES UNDER EDGE CORONA PRODUCT

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Abstract. A topological index is called vertex-degree-based if it can be defined by vertex degrees. The harmonic, atom-bond connectivity and Randić indices are three important examples of such topological indices. The aim of this paper is to find lower and upper bounds for Randić, harmonic and atom-bond connectivity indices of edge corona product of graphs. Some closed formulas are obtained when the factors are regular graphs.
Keywords: Edge corona product, Randić index, harmonic index, atom-bond connectivity index.

1. Introduction and preliminaries

Suppose G is a simple graph with vertex set $V(G)$ and edge set $E(G)$. The Randić index of G, $R(G)$, is defined as the sum of $\frac{1}{\sqrt{deg(v_i)deg(v_j)}}$ over all edges $uv \in E(G)$, where $deg(x)$, as a short $d(x)$, denotes the degree of a vertex x in G [10]. This parameter, sometimes referred to as connectivity index, has been used to characterize the degree of branching of organic compounds. As an example, this number successfully explained the occurrence of critical alloy compositions in 18 different binary alloys [9]. The higher order Randić indices are also of interest in chemical graph theory. For $h \geq 1$, the h-th order Randić

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index $R_h(G)$ of G is the sum of the term $\frac{1}{\sqrt{deg(v_{i_1})deg(v_{i_2})...deg(v_{i_{h+1}})}}$ overall paths $v_{i_1}, v_{i_2}, ..., v_{i_{h+1}}$ of length h contained as a subgraph in G [5, 6]. The case that $h = 1$ is ordinary Randić index. We encourage the reader to consult [7, 8, 15] and references therein for more information on this topic. The Harmonic index of a graph G, $H(G)$, is defined as $H(G) = \sum_{uv\in E(G)} \frac{2}{d_G(u)+d_G(v)}$. As far as we know, this graph invariant first appeared in [3]. The atom-bond connectivity index (ABC index for short) was introduced by Ernesto Estrada et al. for studying the stability of alkanes and the strain energy of cycloalkanes [2]. This index can be defined as $ABC(G) = \sum_{uv\in E(G)} \sqrt{\frac{d(u)+d(v)-2}{d(u).d(v)}}$.

Shwetha Shetty [11] obtained exact formulas for the Harary index of join, corona product, Cartesian product, composition and symmetric difference of graphs. Zhong [16] obtained the minimum and maximum values of the harmonic index for simple connected graphs and trees. He also characterized the corresponding extremal graphs. Xu [12] established some relationships between harmonic, Randić and ABC indices of graphs.

The edge corona product of two graphs G and H, $G \diamond H$, is a graph obtained by taking a copy of G and $|E(G)|$ copies of H and joining each end vertex of i-th edge of G to every vertex in the i-th copy of H [1, 4]. Following Yan et al. [13], the graph $R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G, then joining each new vertex to the end vertices of the corresponding edge. Another way to describe $R(G)$ is to replace each edge of G by a triangle. It is clear that if G is a graph and H is a trivial graph, then $G \diamond H \cong R(G)$. In [14], Yero et al. studied the Randić index of corona product of graphs. In this paper, we continue this work by computing the Randić index of edge corona product of graphs.

Throughout this paper our notation is standard. A k-regular graph is a graph in which degree of each vertex is equal to k. In a graph G with at least one cycle, the length of a longest cycle is called its circumference and the length of a shortest cycle its girth. Our other notions are standard and can be taken from the standard books on graph theory.

2. Main results

The aim of this section is computing the Randić , atom bond connectivity and harmonic indices of edge corona product of graphs. The following simple lemma is crucial in our results.

Lemma 2.1. *By definition of edge corona we have:*
If $x \in V(G_1)$ *then* $d_{G_1\diamond G_2}(x) = d_{G_1}(x)(n_2 + 1)$,
If $x \in V(G_2)$ *then* $d_{G_1\diamond G_2}(x) = d_{G_2}(x) + 2$.

Theorem 2.2. *Let* G_1 *and* G_2 *be graphs. Thus, we have*

$$R_1(G_1 \diamond G_2) \geq \frac{m_1}{\Delta_1(1 + n_2)} + \frac{m_1 m_2}{\Delta_2 + 2} + \frac{2m_1 n_2}{\sqrt{\Delta_1(\Delta_2 + 2)(1 + n_2)}},$$

$$R_1(G_1 \diamond G_2) \leq \frac{m_1}{\delta_1(1+n_2)} + \frac{m_1 m_2}{\delta_2+2} + \frac{2m_1 n_2}{\sqrt{\delta_1(\delta_2+2)(1+n_2)}}.$$

Proof. Let

$$A_1 = \sum_{ab\in E(G_1)} \frac{1}{\sqrt{d(a)d(b)(n_2+1)^2}} \geq \frac{m_1}{\Delta_1(1+n_2)}.$$

$$A_2 = \sum_{uv\in E(G_2)} \frac{1}{\sqrt{(d(u)+2)(d(v)+2)}} \geq \frac{m_1 m_2}{\Delta_2+2}.$$

$$A_3 = \sum_{a\in V(G_1),u\in V(G_2)} \frac{1}{\sqrt{d(a)(n_2+1)(d(u)+2)}} \geq \frac{2m_1 n_2}{\sqrt{\Delta_1(\Delta_2+2)(1+n_2)}}.$$

By summation of A_1, A_2 and A_3, the result can be proved.

Corollary 2.3. *For $i \in \{1,2\}$, if G_i be δ_i-regular graph of order n_i, then*

$$R_1(G_1 \diamond G_2) = \frac{m_1}{\delta_1(1+n_2)} + \frac{m_1 m_2}{\delta_2+2} + \frac{2m_1 n_2}{\sqrt{\delta_1(\delta_2+2)(1+n_2)}}.$$

Theorem 2.4. *Let G_1 and G_2 be graphs. Then*

$$\begin{aligned}
R_2(G_1 \diamond G_2) \geq{}& \frac{n_1}{\Delta_2+2}\sqrt{\frac{\delta_1}{n_2+1}}\left(\frac{n_2(n_2-1)}{2} + 2m_2 + \frac{(\delta_1-1)n_2^2}{2}\right)\\
&+ \frac{m_1 n_2}{(n_2+1)\Delta_1\sqrt{\Delta_2+2}}(2\delta_1+1)\\
&+ \frac{1}{2(n_2+1)^{3/2}\Delta_1}\Big(\sum_{d_{G_1}(v_i)\geq 2} \frac{d_{G_1}(v_i)(d_{G_1}(v_i)-1)}{\sqrt{d_{G_1}(v_i)}}\Big)\\
&+ \frac{1}{2(\Delta_2+2)}\Big(\sum_{d_{G_2}(u_i)\geq 2} \frac{d_{G_2}(u_i)(d_{G_2}(u_i)-1)}{\sqrt{d_{G_2}(u_i)+2}}\Big).
\end{aligned}$$

$$\begin{aligned}
R_2(G_1 \diamond G_2) \leq{}& \frac{n_1}{\delta_2+2}\sqrt{\frac{\Delta_1}{n_2+1}}\left(\frac{n_2(n_2-1)}{2} + 2m_2 + \frac{(\Delta_1-1)n_2^2}{2}\right)\\
&+ \frac{m_1 n_2}{(n_2+1)\delta_1\sqrt{\delta_2+2}}(2\Delta_1+1)\\
&+ \frac{1}{2(n_2+1)^{3/2}\delta_1}\Big(\sum_{d_{G_1}(v_i)\geq 2} \frac{d_{G_1}(v_i)(d_{G_1}(v_i)-1)}{\sqrt{d_{G_1}(v_i)}}\Big)\\
&+ \frac{1}{2(\delta_2+2)}\Big(\sum_{d_{G_2}(u_i)\geq 2} \frac{d_{G_2}(u_i)(d_{G_2}(u_i)-1)}{\sqrt{d_{G_2}(u_i)+2}}\Big).
\end{aligned}$$

Proof. Suppose $V_1 = \{v_1, v_2, ..., v_{n_1}\}$ and $V_2 = \{u_1, u_2, ..., u_{n_2}\}$ are the vertex set of G_1 and G_2, respectively. For $v \in V_i$, $N_{G_i}(v)$ is the set of all adjacent vertices to v in G_i. Suppose $\rho_h(G)$ denotes the set of all paths of length h in G and Consider the following partition of paths of length two in $G_1 \diamond G_2$.

1. Paths $u_iv_ju_k, i \neq k$ where $u_i, u_k \in V_2$ and $v_j \in V_1$,
2. Paths $u_iv_jv_k, j \neq k$ where $v_j, v_k \in V_1$ and $u_i \in V_2$,
3. Paths $v_iu_ju_k, j \neq k$ where $u_j, u_k \in V_2$ and $v_i \in V_1$,
4. Paths $u_iv_ju_k$, where $u_i, u_k \in V_2$(two different copies of G_2) and $v_j \in V_1$,
5. Paths $v_iu_jv_k, i \neq k$ where $u_j \in V_2$, $v_i, v_k \in V_1$ and $v_iv_k \in E(G_1)$,
6. Paths of length two belonging to G_1,
7. Paths of length two belonging to m_1 copies of G_2.

Define T_i to be the set of all paths of type i, $1 \leq i \leq 7$. Therefore, $R_2(G_1 \diamond G_2) = \sum_{i=1}^{7} A_i$, where

$$\begin{aligned}
A_1 &= \sum_{u_{i_1}u_{i_2}u_{i_3} \in \{P \in \rho_2(G_1 \diamond G_2) \mid P \in T_1\}} \frac{1}{\sqrt{d(u_{i_1})d(u_{i_2})d(u_{i_3})}} \\
&= \sum_{u_i, u_k \in V_2, v_j \in V_1} \frac{1}{\sqrt{(d_{G_2}(u_i)+2)(d_{G_2}(u_k)+2)(n_2+1)d_{G_1}(v_j)}} \\
&= \sum_{j=1}^{n_1} \frac{d_{G_1}(v_j)}{\sqrt{d_{G_1}(v_j)(n_2+1)}} . \sum_{i=1}^{n_2-1} \sum_{k=i+1}^{n_2} \frac{1}{\sqrt{(d_{G_2}(u_i)+2)(d_{G_2}(u_k)+2)}} \\
&\geq \frac{n_1n_2(n_2-1)}{2(\Delta_2+2)} . \sqrt{\frac{\delta_1}{n_2+1}}, \\
A_2 &= \sum_{u_{i_1}u_{i_2}u_{i_3} \in \{P \in \rho_2(G_1 \diamond G_2) \mid P \in T_2\}} \frac{1}{\sqrt{d(u_{i_1})d(u_{i_2})d(u_{i_3})}} \\
&= \sum_{v_j, v_k \in V_1, u_i \in V_2} \frac{1}{\sqrt{(d_{G_2}(u_i)+2)d_{G_1}(v_j)d_{G_1}(v_k)(n_2+1)^2}} \\
&= \frac{1}{n_2+1} \left(\sum_{i=1}^{n_2} \frac{1}{\sqrt{d_{G_2}(u_i)+2}} . \sum_{j=1}^{n_1} d_{G_1}(v_j) \sum_{v_k \in N_{G_1}(v_j)} \frac{1}{\sqrt{d_{G_1}(v_j)d_{G_1}(v_k)}} \right) \\
&\geq \frac{2\delta_1 m_1 n_2}{(n_2+1)\Delta_1\sqrt{\Delta_2+2}}, \\
A_3 &= \sum_{u_{i_1}u_{i_2}u_{i_3} \in \{P \in \rho_2(G_1 \diamond G_2) \mid P \in T_3\}} \frac{1}{\sqrt{d(u_{i_1})d(u_{i_2})d(u_{i_3})}} \\
&= \sum_{u_j, u_k \in V_2, v_i \in V_1} \frac{1}{\sqrt{d_{G_1}(v_i)(n_2+1)(d_{G_2}(u_j)+2)(d_{G_2}(u_k)+2)}}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n_1} \frac{d_{G_1}(v_i)}{\sqrt{d_{G_1}(v_i)(n_2+1)}}.\sum_{j=1}^{n_2}\sum_{u_k\in N_{G_2}(u_j)} \frac{1}{\sqrt{(d_{G_2}(u_j)+2)(d_{G_2}(u_k)+2)}}\\
&\geq \frac{2n_1m_2\sqrt{\delta_1}}{(\Delta_2+2)\sqrt{1+n_2}},\\
A_4 &= \sum_{u_{i_1}u_{i_2}u_{i_3}\in\{P\in\rho_2(G_1\diamond G_2)\mid P\in T_4\}} \frac{1}{\sqrt{d(u_{i_1})d(u_{i_2})d(u_{i_3})}}\\
&= \sum_{u_i\in V_{2_t},u_k\in V_{2_l}(l\neq t),v_j\in V_1} \frac{1}{\sqrt{(d_{G_2}(u_i)+2)(n_2+1)d_{G_1}(v_j)(d_{G_2}(u_k)+2)}}\\
&= \sum_{j=1}^{n_1} \frac{d_{G_1}(v_j)(d_{G_1}(v_j)-1)}{2\sqrt{d_{G_1}(v_j)(n_2+1)}}.\sum_{i=1}^{n_2}\sum_{k=1}^{n_2} \frac{1}{\sqrt{(d_{G_2}(u_i)+2)(d_{G_2}(u_k)+2)}}\\
&\geq \frac{n_1n_2^2(\delta_1-1)}{2(\Delta_2+2)}.\sqrt{\frac{\delta_1}{(n_2+1)}},\\
A_5 &= \sum_{u_{i_1}u_{i_2}u_{i_3}\in\{P\in\rho_2(G_1\diamond G_2)\mid P\in T_5\}} \frac{1}{\sqrt{d(u_{i_1})d(u_{i_2})d(u_{i_3})}}\\
&= \sum_{u_j\in V_2,v_iv_k\in E(G_1)} \frac{1}{\sqrt{d_{G_1}(v_i)d_{G_1}(v_k)(n_2+1)^2(d_{G_2}(u_j)+2)}}\\
&= \frac{1}{n_2+1}\sum_{v_iv_k\in E(G_1)} (\frac{1}{\sqrt{d_{G_1}(v_i)d_{G_1}(v_k)}}.\sum_{j=1}^{n_2}\frac{1}{\sqrt{d_{G_2}(u_j)+2}})\\
&\geq \frac{m_1n_2}{(n_2+1)\Delta_1\sqrt{\Delta_2+2}},\\
A_6 &= \sum_{u_{i_1}u_{i_2}u_{i_3}\in\{P\in\rho_2(G_1\diamond G_2)\mid P\in T_6\}} \frac{1}{\sqrt{d(u_{i_1})d(u_{i_2})d(u_{i_3})}}\\
&= \sum_{v_iv_jv_k\in\rho_2(G_1)} \frac{1}{\sqrt{d_{G_1}(v_i)d_{G_1}(v_j)d_{G_1}(v_k)(n_2+1)^3}}\\
&= \frac{1}{(n_2+1)^{3/2}}\sum_{v_iv_jv_k\in\rho_2(G_1)} \frac{1}{\sqrt{d_{G_1}(v_i)d_{G_1}(v_j)d_{G_1}(v_k)}}\\
&\geq \frac{1}{2(n_2+1)^{3/2}\Delta_1}\sum_{d_{G_1}(v_i)\geq 2} \frac{d_{G_1}(v_i)(d_{G_1}(v_i)-1)}{\sqrt{d_{G_1}(v_i)}},\\
A_7 &= \sum_{u_{i_1}u_{i_2}u_{i_3}\in\{P\in\rho_2(G_1\diamond G_2)\mid P\in T_7\}} \frac{1}{\sqrt{d(u_{i_1})d(u_{i_2})d(u_{i_3})}}\\
&= \sum_{u_iu_ju_k\in\rho_2(G_2)} \frac{1}{\sqrt{(d_{G_2}(u_i)+2)(d_{G_2}(u_j)+2)(d_{G_2}(u_k)+2)}}
\end{aligned}
$$

$$\geq \frac{1}{2(\Delta_2+2)} \sum_{d_{G_2}(u_i)\geq 2} \frac{d_{G_2}(u_i)(d_{G_2}(u_i)-1)}{\sqrt{d_{G_2}(u_i)+2}}.$$

Now a simple calculations will complete the proof.

Corollary 2.5. *For $i \in \{1,2\}$, if G_i be δ_i-regular graph of order n_i, then*

$$\begin{aligned} R_2(G_1 \diamond G_2) = & \frac{n_1}{\delta_2+2}\sqrt{\frac{\delta_1}{n_2+1}}(2m_2 + \frac{n_2(n_2-1)}{2} + \frac{(\delta_1-1)n_2^2}{2}) \\ & + \frac{m_1 n_2}{(n_2+1)\delta_1\sqrt{\delta_2+2}}(2\delta_1+1) \\ & + \frac{1}{2(n_2+1)^{3/2}\delta_1} \sum_{d_{G_1}(v_i)\geq 2} \frac{d_{G_1}(v_i)(d_{G_1}(v_i)-1)}{\sqrt{d_{G_1}(v_i)}} \\ & + \frac{1}{2(\delta_2+2)} \sum_{d_{G_2}(u_i)\geq 2} \frac{d_{G_2}(u_i)(d_{G_2}(u_i)-1)}{\sqrt{d_{G_2}(u_i)+2}}. \end{aligned}$$

To prove our main result, we state an important result of [14].

Lemma 2.6 ([14]). *Let $G = (V,E)$ be a graph with girth $g(G)$. If $\delta \geq 2$ and $g(G) > h$, then the number of paths of length h in G is bounded by*

$$\frac{(\delta-1)^{h-2}}{2} \sum_{u\in V} d(u)(d(u)-1) \leq |\rho_h(G)| \leq \frac{(\Delta-1)^{h-2}}{2} \sum_{u\in V} d(u)(d(u)-1).$$

In what follows, let N_k denote the empty graph of order k.

Theorem 2.7. *Let $G = (V,E)$ be a graph with girth $g(G)$, minimum degree δ, and maximum degree Δ. If $\delta \geq 2$ and $g(G) > h \geq 3$, then*

$$\begin{aligned} R_h(G \diamond N_k) \leq & \Big(\frac{(\Delta-1)^{h-2}}{2\sqrt{\delta^{h-1}(k+1)^{h-1}}}(\frac{k^2}{2} + \frac{1}{\delta(k+1)} + \frac{2k(h+1)}{\sqrt{2(k+1)\delta}}) \\ & + \frac{k(k-1)(\Delta-1)^{h-3}}{8\sqrt{(k+1)^{h-1}\delta^{h-1}}} \Big). \sum_{u\in V} d(u)(d(u)-1), \\ R_h(G \diamond N_k) \geq & \Big(\frac{(\delta-1)^{h-2}}{2\sqrt{\Delta^{h-1}(k+1)^{h-1}}}(\frac{k^2}{2} + \frac{1}{\Delta(k+1)} + \frac{2k(h+1)}{\sqrt{2(k+1)\Delta}}) \\ & + \frac{k(k-1)(\delta-1)^{h-3}}{8\sqrt{(k+1)^{h-1}\Delta^{h-1}}} \Big). \sum_{u\in V} d(u)(d(u)-1). \end{aligned}$$

Proof. The paths of length h in G contribute to $R_h(G \diamond N_k)$ in

$$\sum_{v_{i_1}v_{i_2}\dots v_{i_{h+1}} \in \rho_h(G)} \frac{1}{\sqrt{(k+1)^{h+1}\prod_{l=1}^{h+1} d_G(v_{i_l})}}.$$

Moreover, each path of length h in G leads to $2k+k^2$ paths of length h in $G\diamond N_k$; thus, the paths of length h in G contribute to $R_h(G\diamond N_k)$ in

$$\sum_{v_{i_1}v_{i_2}\dots v_{i_{h+1}}\in\rho_h(G)}\Big(\frac{k}{\sqrt{2(k+1)^h\prod_{l=1}^{h}d(v_{i_l})}}+\frac{k}{\sqrt{2(k+1)^h\prod_{l=2}^{h+1}d(v_{i_l})}}+\frac{k^2}{\sqrt{4(k+1)^{h-1}\prod_{l=2}^{h}d(v_{i_l})}}\Big).$$

Furthermore, each cycle of length h in G leads to $2kh$ paths of length h in $G\diamond N_k$ and also each cycle of length $h-1$ in G leads to $(h-1)\binom{k}{2}$ paths of length h in $G\diamond N_k$; thus, the cycle of length h and $h-1$ in G contribute to $R_h(G\diamond N_k)$ in

$$\sum_{v_{i_1}v_{i_2}\dots v_{i_h}v_{i_1}\in\zeta_h(G)}\frac{2kh}{\sqrt{2(k+1)^h\prod_{l=1}^{h}d(v_{i_l})}},$$
$$\sum_{v_{i_1}v_{i_2}\dots v_{i_{h-1}}v_{i_1}\in\zeta_{h-1}(G)}\frac{(h-1)\binom{k}{2}}{\sqrt{4(k+1)^{h-1}\prod_{l=1}^{h-1}d(v_{i_l})}},$$

respectively, where $\zeta_h(G)$ denotes the set of cycles of length h contained (as subgraphs) in G. So,

$$\begin{aligned}
&R_h(G\diamond N_k)\\
&=\sum_{v_{i_1}v_{i_2}\dots v_{i_{h+1}}\in\rho_h(G)}\left(\frac{1}{\sqrt{(k+1)^{h+1}\prod_{l=1}^{h+1}d(v_{i_l})}}+\frac{k}{\sqrt{2(k+1)^h\prod_{l=1}^{h}d(v_{i_l})}}\right.\\
&\left.+\frac{k}{\sqrt{2(k+1)^h\prod_{l=2}^{h+1}d(v_{i_l})}}+\frac{k^2}{\sqrt{4(k+1)^{h-1}\prod_{l=2}^{h}d(v_{i_l})}}\right)\\
&+\sum_{v_{i_1}v_{i_2}\dots v_{i_h}v_{i_1}\in\zeta_h(G)}\frac{2kh}{\sqrt{2(k+1)^h\prod_{l=1}^{h}d(v_{i_l})}}\\
&+\sum_{v_{i_1}v_{i_2}\dots v_{i_{h-1}}v_{i_1}\in\zeta_{h-1}(G)}\frac{(h-1)\binom{k}{2}}{\sqrt{4(k+1)^{h-1}\prod_{l=1}^{h-1}d(v_{i_l})}}\\
&\le|\rho_h(G)|\left(\frac{1}{\sqrt{(k+1)^{h+1}\delta^{h+1}}}+\frac{2k}{\sqrt{2(k+1)^h\delta^h}}+\frac{k^2}{\sqrt{4(k+1)^{h-1}\delta^{h-1}}}\right)\\
&+|\zeta_h(G)|\frac{2kh}{\sqrt{2(k+1)^h\delta^h}}+|\zeta_{h-1}(G)|\frac{(h-1)\binom{k}{2}}{\sqrt{4(k+1)^{h-1}\delta^{h-1}}}.
\end{aligned}$$

By taking into account that $|\zeta_h(G)|\le|\rho_h(G)|$, $|\zeta_{h-1}(G)|\le|\rho_{h-1}(G)|$ and Lemma 2.6 we obtain the upper bound and the lower bound.

Theorem 2.8. *Let G_1 and G_2 be graphs. Thus, we have*

$$H(G_1 \diamond G_2) \geq \frac{1}{1+n_2} H(G_1) + \frac{m_1 m_2}{\Delta_2 + 2} + \frac{2m_1 n_2}{\Delta_1(n_2+1) + \Delta_2 + 2},$$
$$H(G_1 \diamond G_2) \leq \frac{1}{1+n_2} H(G_1) + \frac{m_1 m_2}{\delta_2 + 2} + \frac{2m_1 n_2}{\delta_1(n_2+1) + \delta_2 + 2}.$$

Proof. The edges of $G_1 \diamond G_2$ are partitioned into three subsets E_1, E_2 and E_3 as follows:

$$E_1 = \{e \in E(G_1 \diamond G_2) | e \in E(G_1)\},$$
$$E_2 = \{e \in E(G_1 \diamond G_2) | e \in E(G_{2_i}), i = 1, 2, ..., |E(G_1)|\},$$
$$E_3 = \{e \in E(G_1 \diamond G_2) | e = uv,\ u \in V(G_{2_i}), i = 1, 2, ..., |E(G_1)|, v \in V(G_1)\}.$$

Therefore,

$$H(G_1 \diamond G_2) = \sum_{uv \in E(G_1 \diamond G_2)} \frac{2}{d_{G_1 \diamond G_2}(u) + d_{G_1 \diamond G_2}(v)} = A_1 + A_2 + A_3.$$

Where

$$A_1 = \sum_{uv \in E_1} \frac{2}{d(u)+d(v)} = \sum_{uv \in E_1} \frac{2}{(n_2+1)d_{G_1}(u) + (n_2+1)d_{G_1}(v)} = \frac{1}{(1+n_2)} H(G_1),$$
$$A_2 = \sum_{uv \in E_2} \frac{2}{d(u)+d(v)} = m_1 \sum_{uv \in E_2} \frac{2}{d_{G_2}(u) + 2 + d_{G_2}(v) + 2} \geq \frac{m_1 m_2}{\Delta_2 + 2},$$
$$A_3 = \sum_{uv \in E_3} \frac{2}{d(u)+d(v)} = \sum_{uv \in E_3} \frac{2}{d_{G_2}(u) + 2 + (n_2+1)d(v)} \geq \frac{2m_1 n_2}{\Delta_1(1+n_2) + \Delta_2 + 2}.$$

By summation of A_1, A_2 and A_3, the result can be proved. Also for the reverse bound we can do analogously.

Corollary 2.9. *For $i \in \{1, 2\}$, if G_i be δ_i-regular graph of order n_i, then*

$$H(G_1 \diamond G_2) = \frac{1}{1+n_2} H(G_1) + \frac{m_1 m_2}{\delta_2 + 2} + \frac{2m_1 n_2}{\delta_1(n_2+1) + \Delta_2 + 2}.$$

Corollary 2.10. *For a graph G of size m and an empty graph N_k of order k, we have*

$$H(G \diamond N_k) \geq \frac{1}{1+k} H(G) + \frac{2mk}{\Delta_1(k+1) + 2},$$
$$H(G \diamond N_k) \leq \frac{1}{1+k} H(G) + \frac{2mk}{\delta_1(k+1) + 2}.$$

Theorem 2.11. *Let G_1 and G_2 be graphs. Thus, we have*

$$\begin{aligned} ABC(G_1 \diamond G_2) &\geq \frac{m_1}{\Delta_1(1+n_2)}\sqrt{2(n_2+1)\delta_1 - 2} + \frac{m_1m_2}{\Delta_2+2}\sqrt{2\delta_2+2} \\ &+ 2m_1n_2\sqrt{\frac{\delta_1(n_2+1)+\delta_2}{(n_2+1)(\Delta_2+2)\Delta_1}}, \\ ABC(G_1 \diamond G_2) &\leq \frac{m_1}{\delta_1(1+n_2)}\sqrt{2(n_2+1)\Delta_1 - 2} + \frac{m_1m_2}{\delta_2+2}\sqrt{2\Delta_2+2} \\ &+ 2m_1n_2\sqrt{\frac{\Delta_1(n_2+1)+\Delta_2}{(n_2+1)(\delta_2+2)\delta_1}}. \end{aligned}$$

Proof. The edges of $G_1 \diamond G_2$ are partitioned into three subsets E_1, E_2 and E_3 as follows:

$$\begin{aligned} &E_1 = \{e \in E(G_1 \diamond G_2) | e \in E(G_1)\}, \\ &E_2 = \{e \in E(G_1 \diamond G_2) | e \in E(G_{2_i}), i = 1, 2, ..., |E(G_1)|\}, \\ &E_3 = \{e \in E(G_1 \diamond G_2) | e = uv,\ u \in V(G_{2_i}), i = 1, 2, ..., |E(G_1)|, v \in V(G_1)\}. \end{aligned}$$

Therefore,

$$ABC(G_1 \diamond G_2) = \sum_{uv \in E(G_1 \diamond G_2)} \sqrt{\frac{d_{G_1 \diamond G_2}(u) + d_{G_1 \diamond G_2}(v) - 2}{d_{G_1 \diamond G_2}(u) d_{G_1 \diamond G_2}(v)}} = A_1 + A_2 + A_3.$$

Where

$$\begin{aligned} A_1 &= \sum_{uv \in E_1} \sqrt{\frac{(n_2+1)(d_{G_1}(u) + d_{G_1}(v)) - 2}{(n_2+1)^2 d_{G_1}(u) d_{G_1}(v)}} \geq \frac{m_1}{\Delta_1(1+n_2)}\sqrt{2(n_2+1)\delta_1 - 2}, \\ A_2 &= m_1 \sum_{uv \in E_2} \sqrt{\frac{d_{G_2}(u) + 2 + d_{G_2}(v) + 2 - 2}{(d_{G_1}(u) + 2)(d_{G_1}(v) + 2)}} \geq \frac{m_1m_2}{\Delta_2+2}\sqrt{2\delta_2+2}, \\ A_3 &= \sum_{uv \in E_3} \sqrt{\frac{(n_2+1)d_{G_1}(u) + d_{G_2}(v) + 2 - 2}{(n_2+1)d_{G_1}(u)(d_{G_2}(v)+2)}} \geq 2m_1n_2\sqrt{\frac{(n_2+1)\delta_1+\delta_2}{(n_2+1)(\Delta_2+2)\Delta_1}}. \end{aligned}$$

By summation of A_1, A_2 and A_3, the result can be proved. Also for the reverse bound we can do analogously.

Corollary 2.12. *For $i \in \{1, 2\}$, if G_i be δ_i-regular graph of order n_i, then*

$$\begin{aligned} ABC(G_1 \diamond G_2) &= \frac{m_1}{\delta_1(1+n_2)}\sqrt{2(n_2+1)\delta_1 - 2} + \frac{m_1m_2}{\delta_2+2}\sqrt{2\delta_2+2} \\ &+ 2m_1n_2\sqrt{\frac{\delta_1(n_2+1)+\delta_2}{(n_2+1)(\delta_2+2)\delta_1}}. \end{aligned}$$

Corollary 2.13. *For a graph G of size m and an empty graph N_k of order k, we have*

$$ABC(G \diamond N_k) \geq \frac{m}{\Delta_1(1+k)}\sqrt{2(k+1)\delta_1 - 2} + 2mk\sqrt{\frac{\delta_1(k+1)+2}{4(k+1)\Delta_1}},$$

$$ABC(G \diamond N_k) \leq \frac{m}{\delta_1(1+k)}\sqrt{2(k+1)\Delta_1 - 2} + 2mk\sqrt{\frac{\Delta_1(k+1)+2}{4(k+1)\delta_1}}.$$

References

[1] K. P. Chithra, K. A. Germina and N. K. Sudev, *On the Sparing Number of the Edge Corona of Graphs*, Int. J. Comput. Appl., 118 (2015), 1-5.

[2] E. Estrada, L. Torres, L. Rodriguez and I. Gutman, *An Atom-bond connectivity index: Modelling the enthalpy of formation of alkanes*, Indian J. Chem., 37 (1998), 849-855.

[3] S. Fajtlowicz, *On conjectures of Graffiti.* II, Congr., 60 (1987), 189-197.

[4] Y. Hou, W-C.Shiu, *The spectrum of the edge corona of two graphs*, Electron. J. Linear Algebra, 20 (2010), 586-594.

[5] L. B. Kier and L. H. Hall, *Molecular Connectivity in Chemistry and Drug Research*, Academic Press, New York, 1976.

[6] L. B. Kier and L. H. Hall, *Molecular Connectivity in Structure Activity Analysis*, Wiley, New York, 1986.

[7] X. Li and I. Gutman, *Mathematical Aspects of Randić -Type Molecular Structure Descriptors*, Faculty of Science, University of Kragujevac, 2006.

[8] X. Li and Y. Shi, *A survey on the Randić index*, MATCH Commun. Math. Comput. Chem., 59 (2008), 127-156.

[9] E. McCafferty, *Introduction to corrosion science*, Springer, New York , 2009.

[10] M. Randić , *On characterization of molecular branching*, J. Amer. Chem. Soc., 97 (1975), 6609-6615.

[11] B. Shwetha Shetty, V. Lokesha and P. S. Ranjini, *On the harmonic index of graphs operations*, Transactions on Combinatorics, 4 (2015), 5-14.

[12] X. Xu, *Relationships between harmonic index and other topological indices*, Appl. Math. Sci., 6 (2012), 2013-2018.

[13] W. Yan, B.-Y Yang and Y.-N Yeh, *The behavior of Wiener indices and polynomials of graphs under ve graph decorations*, Appl. Math. Lett., 20 (2007), 290-295.

[14] I. G. Yero and J. A. Rodriguez-Velazquez, *On the Randić Index of Corona Product Graphs*, International Scholarly Research Network, 1 (2011), 1-100.

[15] I. G. Yero, J. A. Rodrguez-Velazquez and I. Gutman, *Estimating the higher-order Randić index*, Chemical Physics Letters, 489 (2010), 118-120.

[16] L. Zhong, *The harmonic index for graphs*, Appl. Math. Lett., 25 (2012), 561-566.

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