# ENTIRE FUNCTIONS SHARING TWO SMALLER ORDER ENTIRE FUNCTIONS WITH THEIR DIFFERENCE OPERATORS

**Guowei Zhang**
*School of Mathematics and Statistics, Anyang Normal University, Anyang, Henan 455000 P. R. China*
*herrzgw@foxmail.com*

**Abstract.** In this paper we mainly study the uniqueness of entire functions with finite order sharing two smaller order entire functions with their difference operators. Our results improve some recent theorems due to Liu and Mao, Zhang and Liao.
**Keywords:** Difference Nevanlinna theory, entire function, sharing value, uniqueness.

## 1. Introduction and main results

In this paper, a meromorphic function always means it is meromorphic in the whole complex plane. We assume that the reader is familiar with Nevanlinna's value distribution theory (see [5, 7, 13, 14]) and its associated standard notions, such as $T(r,f), m(r,f), N(r,f), \overline{N}(r,f), \cdots$. $S(r,f)$ denotes any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to \infty$ outside some exceptional set of finite measure, not necessarily the same at each occurrence. A meromorphic function $a(z)$ is said to be a small function with respect to $f(z)$ iff $T(r,a) = S(r,f)$.

Let $f$ and $g$ be two nonconstant meromorphic functions, and let $a$ be a value in the extended plane. We say that $f$ and $g$ share the value $a$ IM, provided that $f$ and $g$ have the same $a$-points ignoring multiplicities. Moreover, we say that $f$ and $g$ share the value $a$ CM, provided that $f$ and $g$ have the same $a$-points with the same multiplicities; see [13]. Suppose that $b$ is a meromorphic function. If $f-b$ and $g-b$ share 0 CM, we say that $f$ and $g$ share $b$ CM. If $f-b$ and $g-b$ share 0 IM, we say that $f$ and $g$ share $b$ IM. In addition, we denote by $\mu(f), \sigma(f)$ and $\lambda(f)$ the lower order of $f$, the order of $f$ and the exponent of convergence of zeros of $f$ respectively. If $\mu(f) = \sigma(f)$, we say that $f$ is of regular growth.

1977, Rubel and Yang [12] proved

**Theorem A.** *Let $f$ be a nonconstant entire function. If $f$ and $f'$ share two distinct finite values CM, then $f \equiv f'$.*

Later, Zheng and Wang [17] improved that the conclusion still holds if $f$ and $f'$ share two distinct non-infinite small functions $a(z)$ and $b(z)$ CM.

We define the difference operator $\Delta f = f(z+\eta) - f(z)$ as usual, where $\eta$ is a nonzero complex constant. Recently, Nevanlinna theory for the difference operator and the difference analogue of logarithmic derivative lemma have been established; see [3, 4]. These new theories bring about a lot of papers (for example [1, 2, 8, 15]) focusing on the uniqueness of meromorphic functions sharing some values with their difference operators (or shifts). It is well known that $\Delta f$ can be considered as the difference counterpart of $f'$ in certain sense. For example, Chen and Yi [2] considered the problem that $f$ and $\Delta f$ sharing some values CM under the hypothesis that $f$ has a finite Borel exceptional value. As an improvement, Zhang and Liao [16] proved the following result, which means the conclusion in Theorem A is still valid when $f'$ is replaced by $\Delta f$.

**Theorem B.** *Let $f(z)$ be a transcendental entire function of finite order and $a, b$ be two distinct constants. If $f$ and $\Delta f (\not\equiv 0)$ share $a, b$ CM, then $\Delta f \equiv f$. Furthermore, $f(z)$ must be of the following form $f(z) = 2^z h(z)$, where $h(z)$ is a periodic entire function with period 1.*

Regarding this theorem, a natural question is that what can be said if entire function $f$ with finite order shares some smaller order entire functions with $\Delta f$? This question has been studied in some recent papers [9, 10, 11]. Liu and Mao [11] obtained the following two results.

**Theorem C.** *Let $f(z)$ be a nonconstant entire function of finite order and $a(z)$ be an entire function of $\sigma(a) < \sigma(f)$. If $f$ and $\Delta f$ share $0, a(z)$ CM, then $\Delta f \equiv f$.*

**Theorem D.** *Let $f(z)$ be a nonconstant entire function of finite order and $a(z)(\not\equiv 0)$ be an entire function of $\sigma(a) < \sigma(f)$ and $\lambda(f-a) < \sigma(f)$. If $f$ and $\Delta f$ share $a(z)$ CM, then $\sigma(f) = 1$.*

In the present paper, we shall study the general case of Theorem C, that is, if $f$ and $\Delta f$ share two distinct smaller order entire functions $a(z), b(z)$ CM, does the conclusion still hold? In fact, we have

**Theorem 1.1.** *Let $f(z)$ be a nonconstant entire function of finite order and $a(z), b(z)$ be two entire functions with $a(z) \not\equiv b(z)$ and $\max\{\sigma(a), \sigma(b)\} < \sigma(f)$. If $f$ and $\Delta f$ share $a(z), b(z)$ CM, then $\Delta f \equiv f$.*

As to Theorem D, we get the following improvement.

**Theorem 1.2.** *Let $f(z)$ be a nonconstant entire function of finite order and $a(z)$ be an entire function with $\sigma(a) < \sigma(f)$ and $\lambda(f-a) < \sigma(f)$. If $f$ and $\Delta f$ share $a(z)$ CM, then $a(z) \equiv 0$ and $f(z)$ must be of the form $f(z) = e^{mz+n}$, where $m(\neq 0), n$ are two constants.*

## 2. Preliminary lemmas

**Lemma 2.1** ([7])**.** *Let $f(z)$ be a nonconstant meromorphic function. Then*

$$m\left(r, \frac{f'}{f}\right) = O(\log r), \quad r \to \infty,$$

*if $f$ is of finite order, and*

$$m\left(r, \frac{f'}{f}\right) = O(\log(rT(r, f))), \quad r \to \infty,$$

*possibly outside a set $E$ of $r$ with finite linear measure if $f(z)$ is of infinite order.*

**Lemma 2.2.** ([3])**.** *Let $f(z)$ be a meromorphic function in the complex plane with order $\sigma = \sigma(f) < \infty$, and let $c$ be a fixed nonzero complex constant. Then, for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = O(r^{\sigma-1+\varepsilon}).$$

**Remark 2.1** ([16, Remark 2.2])**.** The equation

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = O(r^{\sigma-1+\varepsilon}),$$

where $\sigma$ is the finite order of $f$ and $\varepsilon > 0$, implies

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r, f)$$

possibly outside a set of finite logarithmic measure.

**Lemma 2.3** ([3])**.** *Let $f(z)$ be a meromorphic function in the complex plane with order $\sigma = \sigma(f) < \infty$, and let $\eta$ be a fixed nonzero complex constant. Then, for each $\varepsilon > 0$, we have*

$$T(r, f(z+\eta)) = T(r, f(z)) + O(r^{\sigma-1+\varepsilon}) + O(\log r), i.e.,$$

$$T(r, f(z+\eta)) = T(r, f(z)) + S(r, f)$$

*possibly outside a set of finite logarithmic measure.*

It is evident that $S(r, f(z+\eta)) = S(r, f)$ from Lemma 2.3.

**Lemma 2.4** ([13])**.** *Let $f(z)$ be a non-constant meromorphic function in the complex plane and*

$$R(f) = \frac{P(f)}{Q(f)},$$

*where* $P(f) = \sum_{k=0}^{p} a_k f^k$ *and* $Q(f) = \sum_{j=0}^{q} b_j f^j$ *are two mutually prime polynomials in* $f(z)$*. If the coefficients* $a_k, b_j$ *are small functions of* $f(z)$ *and* $a_p \not\equiv 0, b_q \not\equiv 0$*, then*

$$T(r, R(f)) = \max\{p, q\} T(r, f) + S(r, f).$$

**Lemma 2.5** ([13])**.** *Let* $f_j(z) (j = 1, 2, \cdots, n)(n \geq 2)$ *be meromorphic functions and* $g_j(z) (j = 1, 2, \cdots, n)$ *be entire functions such that*

(1) $\sum_{j=1}^{n} f_j(z) \exp\{g_j(z)\} \equiv 0$*;*

(2) *when* $1 \leq j < k \leq n, g_j(z) - g_k(z)$ *is not constant;*

(3) *when* $1 \leq j \leq n, 1 \leq h < k \leq n$*,* $T(r, f_j) = o\{T(r, \exp\{g_h - g_k\})\}$*,* $(r \to \infty, r \notin E)$*, where* $E \subset (1, \infty)$ *is of finite linear measure or finite logarithmic measure. Then* $f_j(z) \equiv 0 \quad (j = 1, \cdots, n)$*.*

**Lemma 2.6** ([13, Theorem 2.11])**.** *Let* $f$ *be a transcendental meromorphic function in the complex plane such that* $\sigma(f) > 0$*. If* $f$ *has two distinct Borel exceptional values in the extended complex plane, then* $\mu(f) = \sigma(f)$ *and* $\sigma(f)$ *is a positive integer or* $\infty$*.*

**Lemma 2.7** ([6])**.** *Let* $\varphi(r)$ *be a nondecreasing, continuous function on* $\mathbb{R}^+$*, and let* $0 < \rho < \limsup_{r\to\infty} \frac{\log \varphi(r)}{\log r}$ *and* $H = \{r \in \mathbb{R}^+ : |\varphi(r)| \geq r^\rho\}$*. Then*

$$\overline{\log dens} H = \limsup_{r\to\infty} \frac{\int_{H\cup[1,r]} \frac{1}{t} dt}{\log r} > 0.$$

It's known that for two entire functions $f(z)$ and $a(z)$, $\sigma(f) > \sigma(a)$ can not guarantee $T(r, a) = o(T(r, f)) = S(r, f)$. However, we have the following result.

**Lemma 2.8.** *Let* $f(z)$ *and* $a(z)$ *be two entire functions, and* $\sigma(f) > \sigma(a)$*. Then* $T(r, a) = o(T(r, f))$ *holds for* $r \in H$*, which satisfies* $\overline{\log dens} H > 0$*.*

**Proof.** By the definition of the order entire function $f$, we know that

$$\limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} = \sigma(f).$$

Set $\sigma(f) > \rho_1 > \rho_2 > \sigma(a)$ and applying Lemma 2.7 to $T(r, f)$, we get $T(r, f) \geq r^{\rho_1}$ holds for $r \in H$, which satisfies $\overline{\log dens} H > 0$. Moreover, by the definition of order of entire function $a(z)$, we have $T(r, a) < r^{\rho_2}$. Thus, $T(r, a) < r^{\rho_2} < r^{\rho_1} \leq T(r, f)$ holds when $r \in H$. Then we complete the proof. □

**Remark 2.2.** We know the Nevanlinna characteristic function is non-decreasing function of $r$, then for any two entire functions $a(z)$ and $b(z)$, one of following three cases may occur: 1. $\lim_{r\to\infty} \frac{T(r,a(z))}{T(r,b(z))} = 0$, that is, $T(r, a(z)) = o(T(r, b(z)))$; 2. $\lim_{r\to\infty} \frac{T(r,a(z))}{T(r,b(z))} = \alpha$, $0 < \alpha < +\infty$; 3. $\lim_{r\to\infty} \frac{T(r,a(z))}{T(r,b(z))} = +\infty$, $T(r, b(z)) = o(T(r, a(z)))$.

## 3. Proof of Theorems

*Proof of Theorem 1.1.* Since $f$ is entire function with finite order and shares two nonequivalent smaller order entire functions $a(z), b(z)$ CM with $\Delta f$, then there exists two polynomial $p(z)$ and $q(z)$ such that

$$\Delta f - a(z) = [f(z) - a(z)]e^{p(z)} \tag{3.1}$$

and

$$\Delta f - b(z) = [f(z) - b(z)]e^{q(z)}. \tag{3.2}$$

Since $\max\{\sigma(a), \sigma(b)\} < \sigma(f)$ and $\sigma(\Delta f) \le \sigma(f)$, by (3.1) and (3.2) we have

$$\deg p(z) = \sigma(e^{p(z)}) \le \sigma(f), \deg q(z) = \sigma(e^{q(z)}) \le \sigma(f). \tag{3.3}$$

If $e^{p(z)} \equiv e^{q(z)}$, then we can deduce $f \equiv \Delta f$ easily by (3.1) and (3.2).

By solving $f$ and $\Delta f$ from (3.1) and (3.2), we get

$$f(z) = \frac{b(z) - a(z) + a(z)e^{p(z)} - b(z)e^{q(z)}}{e^{p(z)} - e^{q(z)}} \tag{3.4}$$

and

$$\Delta f = \frac{b(z)e^{p(z)} - a(z)e^{q(z)} + (a(z) - b(z))e^{p(z)+q(z)}}{e^{p(z)} - e^{q(z)}}. \tag{3.5}$$

If $e^{p(z)} \not\equiv e^{q(z)}$ and $p(z), q(z)$ are two distinct constants, then by (3.4) we have $\sigma(f) \le \max\{\sigma(a), \sigma(b)\}$, which contradicts the assumption. If $e^{p(z)} \not\equiv e^{q(z)}$ and only one of $p(z), q(z)$ is constant. Without loss of generality, we assume $q(z)$ is a constant $q$. From (3.4), if $z_0$ is a zero of $e^{p(z)} - e^q$, then $a(z_0) - b(z_0) = 0$ or $e^{p(z_0)} = 1$. We claim $e^q = 1$. Otherwise, all the zeros of $e^{p(z)} - e^q$ must be zeros of $a(z) - b(z)$, that is,

$$N\left(r, \frac{1}{e^{p(z)} - e^q}\right) \le N\left(r, \frac{1}{a(z) - b(z)}\right). \tag{3.6}$$

By the second main theorem of Nevanlinna theory, we get

$$\begin{aligned} T(r, e^{p(z)}) &\le N\left(r, \frac{1}{e^{p(z)} - e^q}\right) + S(r, e^{p(z)}) \le N\left(r, \frac{1}{a(z) - b(z)}\right) + S(r, e^{p(z)}) \\ &\le 2\max\{T(r, a(z)), T(r, b(z))\} + S(r, e^{p(z)}). \end{aligned} \tag{3.7}$$

Note that $q(z) = q$ is a constant, combining (3.4) and (3.7) we have $\sigma(f) < \max\{\sigma(a(z)), \sigma(b(z))\}$, which contradicts the assumption. Thus, $e^q = 1$, then by (3.2), we have $f \equiv \Delta f$. In the following, we just need to consider the remaining case that $e^{p(z)} \not\equiv e^{q(z)}$ and $p(z), q(z)$ both are nonconstant polynomials. If one of $a(z), b(z)$ is zero, it reduces to Theorem C. Thus, we assume $a(z), b(z)$ are both nonzero entire functions.

Firstly, we set

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$$

and

$$q(z) = b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0,$$

where $a_n (\neq 0), a_{n-1}, \cdots, a_0$ and $b_m (\neq 0), b_{m-1}, \cdots, b_0$ are constants. We claim

$$m = n \quad \text{and} \quad |a_n| = |b_m|. \tag{3.8}$$

Set $g(z) := f(z) - a(z)$, then (3.1) becomes

$$\Delta g + a(z+\eta) - 2a(z) = g(z) e^{p(z)}. \tag{3.9}$$

By differentiating (3.9) and eliminating $e^{p(z)}$, we obtain

$$A(z) g(z) + B(z) + [2a(z) - a(z+\eta)] \frac{g'(z)}{g(z)} = 0, \tag{3.10}$$

where

$$A(z) := \frac{g'(z+\eta)}{g(z)} - \frac{g'(z) g(z+\eta)}{g(z)^2} - p'(z) \frac{g(z+\eta)}{g(z)} + p'(z),$$

$$B(z) := a'(z+\eta) - 2a'(z) - a(z+\eta) p'(z) + 2a(z) p'(z).$$

If $A(z) \equiv 0$, then (3.10) becomes

$$B(z) + [2a(z) - a(z+\eta)] \frac{g'(z)}{g(z)} = 0. \tag{3.11}$$

Solving it, we get the solutions

$$g(z) = c[a(z+\eta) - 2a(z)] e^{-p(z)}, \tag{3.12}$$

where $c$ is a nonzero constant. Substituting (3.12) into (3.9) we get

$$\frac{a(z+2\eta) - 2a(z+\eta)}{a(z+\eta) - 2a(z)} e^{-p(z+\eta)} = e^{-p(z)} + \frac{c-1}{c}. \tag{3.13}$$

If $c \neq 1$, then by (3.13) we get

$$N\left(r, \frac{1}{e^{-p(z)} - \frac{c-1}{c}}\right) \leq N\left(r, \frac{1}{a(z+2\eta) - 2a(z+\eta)}\right). \tag{3.14}$$

By the first and second main theorem of Nevanlinna theory and Lemma 2.3, we have

$$\begin{aligned} T(r,e^{p(z)}) = T(r,e^{-p(z)}) + O(1) &\leq N\left(r, \frac{1}{e^{-p(z)} - \frac{c-1}{c}}\right) + S(r,e^{p(z)}) \\ &\leq N\left(r, \frac{1}{a(z+2\eta) - 2a(z+\eta)}\right) + S(r,e^{p(z)}) \\ &\leq 3T(r,a(z)) + S(r,a(z)) + S(r,e^{p(z)}) \end{aligned} \tag{3.15}$$

Thus, we have

$$\deg p(z) = \sigma(e^{p(z)}) \leq \sigma(a). \tag{3.16}$$

Combining (3.12) and (3.16) we have $\sigma(g) \leq \sigma(a)$. Since $\sigma(f) > \sigma(a)$ and $g(z) = f(z) - a(z)$, we have $\sigma(g) = \sigma(f)$. Thus, we obtain a contradiction.

Then $c = 1$. By (3.13), we have $\Delta g \equiv 0$, that is, $\Delta f \equiv \Delta a$. Since $|\eta|$ can be sufficiently small and $f(z), a(z)$ are entire functions, by the definition of derivatives, we have $f'(z) = a'(z)$. Recalling $\sigma(f) = \sigma(f'), \sigma(a) = \sigma(a')$ and $\sigma(f) > \sigma(a)$, we obtain a contradiction again.

Thus, we have $A(z) \not\equiv 0$. By Lemma 2.8 and Remark 2.2, we declare in advance that the following equalities and inequalities concerning $S(r,f)$ hold for $r \in H$, which satisfies $\overline{\log dens} H > 0$. By Lemma 2.1 and Lemma 2.2, we obtain

$$m(r,A) = S(r,f). \tag{3.17}$$

By the definition of $A(z)$, we see

$$N(r,A) \leq \overline{N}\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g}\right). \tag{3.18}$$

By (3.10), (3.18), $\sigma(f) > \sigma(a)$ and the first main theorem of Nevanlinna theory, we have

$$\begin{aligned} m(r,g) \leq m\left(r, \frac{1}{A}\right) + S(r,f) &\leq T(r,A) + S(r,f) = N(r,A) + S(r,f) \\ &\leq \overline{N}\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g}\right) + S(r,f). \end{aligned} \tag{3.19}$$

Since $g(z)$ is entire function, $N(r,g) = 0$. So, we have

$$m(r,g) = T(r,g) = T\left(r, \frac{1}{g}\right) + O(1) = m\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g}\right) + O(1). \tag{3.20}$$

Thus, combining (3.19) and (3.20) we obtain

$$m\left(r, \frac{1}{g}\right) \leq \overline{N}\left(r, \frac{1}{g}\right) + S(r,f). \tag{3.21}$$

Rewrite (3.9) as

$$\frac{g(z+\eta)}{g(z)}+\frac{a(z+\eta)-2a(z)}{g(z)}=e^{p(z)}+1. \tag{3.22}$$

By (3.22), we get

$$m(r,e^{p(z)})\le m\left(r,\frac{1}{g}\right)+S(r,f). \tag{3.23}$$

Since $e^{q(z)}$ is entire function, by the second main theorem of Nevanlinna theory, we can easily get

$$\overline{N}\left(r,\frac{1}{e^{q(z)}-1}\right)=m(r,e^{q(z)})+S(r,e^{q(z)}). \tag{3.24}$$

By (3.4), we deduce that

$$\begin{aligned}\overline{N}\left(r,\frac{1}{f-a}\right)&\le\overline{N}\left(r,\frac{1}{e^{q(z)}-1}\right)+\overline{N}\left(r,\frac{1}{a(z)-b(z)}\right)\\&\le\overline{N}\left(r,\frac{1}{e^{q(z)}-1}\right)+S(r,f).\end{aligned} \tag{3.25}$$

Combining (3.21), (3.23), (3.24) and (3.25), we see

$$m(r,e^{p(z)})\le m(r,e^{q(z)})+S(r,e^{q(z)})+S(r,f). \tag{3.26}$$

Similarly, we have

$$m(r,e^{q(z)})\le m(r,e^{p(z)})+S(r,e^{p(z)})+S(r,f). \tag{3.27}$$

From (3.1) and Lemma 2.3, 2.4 we have

$$T(r,e^{p(z)})\le T(r,f)+S(r,f). \tag{3.28}$$

By (3.4) we obtain

$$T(r,f)\le\max\{T(r,e^{p(z)}),T(r,e^{q(z)})\}+S(r,f). \tag{3.29}$$

Without loss of generality, suppose $\max\{T(r,e^{p(z)}),T(r,e^{q(z)})\}=T(r,e^{p(z)})$. Thus, we have

$$T(r,e^{p(z)})=(1+o(1))T(r,f),\quad S(r,e^{p(z)})=S(r,f). \tag{3.30}$$

From (3.2) and Lemma 2.3, 2.4, we know that

$$T(r,e^{q(z)})\le T(r,f)+S(r,f). \tag{3.31}$$

Thus, $S(r, e^{q(z)}) = S(r, f)$. Combining (3.26) and (3.27), we have

$$m(r, e^{p(z)}) \leq m(r, e^{q(z)}) + S(r, f) \leq m(r, e^{p(z)}) + S(r, f) \tag{3.32}$$

that is,

$$T(r, e^{q(z)}) = T(r, e^{p(z)}) + S(r, f). \tag{3.33}$$

Combining (3.30) and (3.33), we deduce

$$S(r, f) = S(r, e^{p(z)}) = S(r, e^{q(z)}) \tag{3.34}$$

and

$$m(r, e^{p(z)}) = m(r, e^{q(z)}) + S(r, e^{q(z)}) = m(r, e^{q(z)}) + S(r, e^{p(z)}). \tag{3.35}$$

Noting the fact that $m(r, e^{a_n z^n}) = \frac{|a_n| r^n}{\pi}$, we have, for all $r \in (0, +\infty)$,

$$m(r, e^{p(z)}) = \frac{|a_n| r^n}{\pi}(1 + o(1)), \quad m(r, e^{q(z)}) = \frac{|b_m| r^m}{\pi}(1 + o(1)). \tag{3.36}$$

Thus, from (3.35) and (3.36) we proved (3.8).

For simplicity, denote $\overline{a} := a(z+\eta), a := a(z)$, so do $\overline{b}, \overline{p}, \overline{q}$. By (3.4) we have

$$\Delta f = \frac{\overline{b} - \overline{a} + \overline{a} e^{\overline{p}} - \overline{b} e^{\overline{q}}}{e^{\overline{p}} - e^{\overline{q}}} - \frac{b - a + a e^p - b e^q}{e^p - e^q}. \tag{3.37}$$

Combining the above identity with (3.5), we obtain that

$$C_1 e^p + C_2 e^q + C_3 e^{2p} + C_4 e^{2q} + C_5 e^{2p+q} + C_6 e^{p+2q} + C_7 e^{p+q} = 0, \tag{3.38}$$

where

$$\begin{aligned}
C_1 &:= \overline{b} - \overline{a} + (a - b) e^{\Delta p};\\
C_2 &:= \overline{a} - \overline{b} + (b - a) e^{\Delta q};\\
C_3 &:= (\overline{a} - a - b) e^{\Delta p};\\
C_4 &:= (\overline{b} - a - b) e^{\Delta q};\\
C_5 &:= (b - a) e^{\Delta p};\\
C_6 &:= (a - b) e^{\Delta q};\\
C_7 &:= (a + b - \overline{a}) e^{\Delta p} + (a + b - \overline{b}) e^{\Delta q}.
\end{aligned}$$

Our key point is applying Lemma 2.5 to (3.38). In order to do this, we need to verify (3.38) satisfies the conditions of this lemma. We need two steps.

Step 1, we shall show the polynomials $p, q, 2p, 2q, 2p+q, p+2q, p+q$, which means $g_j(z)$ in Lemma 2.5, have the same degree $n$ as the polynomial $p$. Since $m = n$ and $|a_n| = |b_m|$, we just need to show that the degree of $p + q$ is $n$.

Suppose that the degree of $p+q$ is $k<n$, then it must have $p(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_0$ and $q(z)=-a_nz^n+b_{n-1}z^{n-1}+\cdots+b_0$. Substituting them into (3.38), we deduce $C_3e^{A_3}e^{4a_nz^n}+(C_1e^{A_1}+C_5e^{A_5})e^{3a_nz^n}+C_7e^{A_7}e^{2a_nz^n}+(C_2e^{A_2}+C_6e^{A_6})e^{a_nz^n}+C_4e^{A_4}=0$, where $A_j$ $(j=1,2,\cdots,7)$ are polynomials with degree at most $n-1$. Denote $H:=e^{a_nz^n}$, we have

$$\begin{aligned}&C_3e^{A_3}H^4+(C_1e^{A_1}+C_5e^{A_5})H^3+C_7e^{A_7}H^2\\&+(C_2e^{A_2}+C_6e^{A_6})H+C_4e^{A_4}=0.\end{aligned}\tag{3.39}$$

Applying Lemma 2.5 to (3.39), we get $C_3\equiv C_4\equiv 0$, that is, $\overline{a}\equiv\overline{b}$. For arbitrary $z$, let $\eta\to 0$ we have $a(z)\equiv b(z)$ by the continuous of functions $a(z)$ and $b(z)$. This is impossible. Therefore, $\deg(p+q)=n$.

Step 2, we shall show that $p,q,2p,2q,p-q,p+q,2p+q,p+2q,\cdots$, which mean $g_j-g_k$ in Lemma 2.5, are polynomials with degree $n$. From step 1 and $m=n,|a_n|=|b_m|$, we just need to consider polynomial $p-q$. Suppose that the degree of $p-q$ is $k<n$, it must be $p(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_0$ and $q(z)=a_nz^n+b_{n-1}z^{n-1}+\cdots+b_0$. Denote $r(z):=p(z)-q(z)$, then $\deg r(z)=k<n$. Substituting $r(z)$ into (3.38), we get

$$C_1e^re^q+C_2e^q+C_3e^{2r}e^{2q}+C_4e^{2q}+C_5e^{2r}e^{3q}+C_6e^re^{3q}+C_7e^re^{2q}=0,\tag{3.40}$$

that is,

$$(C_1e^r+C_2)e^q+(C_4+C_7e^r+C_3e^{2r})e^{2q}+(C_5e^{2r}+C_6e^r)e^{3q}=0.\tag{3.41}$$

By Lemma 2.5 and (3.41), we obtain

$$C_1e^r+C_2\equiv C_5e^{2r}+C_6e^r\equiv 0.\tag{3.42}$$

Since $a(z)\not\equiv b(z)$, we obtain $e^{r(z)}\equiv 1$, i.e., $e^{p(z)}\equiv e^{q(z)}$, which is a contradiction. So, $\deg(p-q)=n$.

Finally, applying Lemma 2.5 to (3.38), we see

$$C_j\equiv 0,\ (j=1,2,\cdots,7),\tag{3.43}$$

which means $a(z)\equiv b(z)$. It obviously contradicts the assumption. Thus, we complete the proof.

*Proof of Theorem 1.2.* Suppose that $a(z)\not\equiv 0$, then by Theorem D, we have $\sigma(f)=1$, then $\sigma(a)<\sigma(f)=1$. By the assumption and as in the proof of Theorem 1.1, there exists a polynomial $p(z)$ such that

$$\Delta f-a(z)=[f(z)-a(z)]e^{p(z)}\tag{3.44}$$

and $\deg p(z)=\sigma(e^{p(z)})\le\sigma(f)$. Since $\lambda(f-a)<\sigma(f)$ and the fact $\sigma(f-a)=\sigma(f)=1$, by Lemma 2.6 and Hadamard factorization theorem, there exists an entire function $H(z)$ and a polynomial $h(z)$ such that

$$f(z)-a(z)=H(z)e^{h(z)}\tag{3.45}$$

and $\sigma(H) = \lambda(H) = \lambda(f-a) < \sigma(f) = 1$. Thus, we have $\deg h = \sigma(f) = 1$. We set $h(z) := sz + t$, where $s(\neq 0), t$ are constants. By (3.45) we can obtain

$$e^{h(z)}[H(z+\eta)e^{s\eta} - H(z)] + a(z+\eta) - 2a(z) = H(z)e^{p(z)+h(z)}. \tag{3.46}$$

If $a(z+\eta) - 2a(z) \not\equiv 0$, from (3.46) and $\sigma(H) < 1, \sigma(a(z)) < 1$, we have

$$N\left(r, \frac{1}{e^{h(z)} + \frac{a(z+\eta)-2a(z)}{w(z)}}\right) = N\left(r, \frac{1}{H(z)}\right) \leq T(r, H(z)) = S(r, e^{h(z)}), \tag{3.47}$$

where $w(z) := H(z+\eta)e^{s\eta} - H(z)$. Moreover, by the second main theorem of Nevanlinna theory to small functions [13, Theorem 1.36], we have

$$(1-\varepsilon)T(r, e^{h(z)}) \leq N\left(r, \frac{1}{e^{h(z)} - \frac{a(z+\eta)-2a(z)}{w(z)}}\right) + S(r, e^{h(z)}), \tag{3.48}$$

where $\varepsilon$ is any positive constant. From (3.47), we know it is impossible.

If $a(z+\eta) - 2a(z) \equiv 0$, i.e., $a(z+\eta) \equiv 2a(z)$. Suppose $a(z)$ is nonconstant function, since $a(z)$ is entire and $\sigma(a) < 1$, we can set $z_0$ is a zero of it, then $z_0 + \eta, z_0 + 2\eta, z_0 + 3\eta, \cdots$ are zeros of $a(z)$. Thus,

$$T(r, a(z)) \geq N\left(r, \frac{1}{a(z)}\right) \geq cr, \tag{3.49}$$

where $c$ is a positive constant. By (3.49), we get $\sigma(a(z)) \geq 1$, this is also impossible. Therefore, $a(z)$ is a constant. We get $a = 0$ immediately, which contradicts the assumption in the beginning.

So $a(z) \equiv 0$. Similar as the arguments in case 2, proof of Theorem 1.3 in [16], we get $f(z)$ must be the form of $f(z) = e^{mz+n}$, where $m(\neq 0), n$ are constants.

**Acknowledgements.** The author wishes to express his thanks to the referees' valuable suggestions and comments. This work was supported by the NSFC (no. 11426035, 11301008) and the key scientific research project for higher education institutions of Henan Province, China (no. 18A110002).

## References

[1] Z.X. Chen, C.C. Yang, *On entire solutions of certain type of differential-difference equations*, Taiwanese J. Math., 18 (2014), 677-685.

[2] Z.X. Chen, H.X. Yi, *On sharing values of meromorphic functions and their differences*, Res. Math., 63 (2013), 557-565.

[3] Y. M. Chiang, S. J. Feng, *On the Nevanlinna Characteristic of $f(z+\eta)$ and difference equations in the complex plane*, Ramanujan J., 16 (2008), 105-129.

[4] R.G. Halburd, R.J. Korhonen, *Difference analogue lemma on the logarithmic derivative with application to difference equations*, J. Math. Anal. Appl., 314 (2006), 477-487.

[5] W. Hayman, *Meromorphic functions*, Clarendon Press, Oxford, 1964.

[6] K. Ishizaki, K. Tohge, *On the complex oscillation of some linear differential equations*, J. Math. Anal. Appl., 206 (1997), 503-517.

[7] I. Laine, *Nevanlinna theory and complex differential equations*, Walter de Gruyter, Berlin, 1993.

[8] I. Laine, C.C. Yang, *Clunie theorems for difference and q-difference polynomials*, J. London Math. Soc., 76 (2007), 556-566.

[9] X.M. Li, X. Yang, H.X. Yi, *Entire functions sharing an entire function of smaller order with their shifts*, Proc. Japan Acad., 89, Ser. A (2013), 34-39.

[10] X.M. Li, H.X. Yi, *Entire functions sharing an entire function of smaller order with their difference operators*, Acta. Math. Sin. English Series, 30 (2014), 481-498.

[11] H.F. Liu, Z.Q. Mao, *On the uniqueness problems of entire functions and their difference operators*, Taiwanese. J. Math., 19 (2015), 907-917.

[12] L.A. Rubel and C.-C. Yang, *Values shared by an entire function and its derivative, in Complex analysis* (Proc. Conf., Univ. Kentucky, Lexington, Ky., 1976), 101C103. Lecture Notes in Math., 599, Springer, Berlin, 1977.

[13] C.C. Yang, H.X. Yi, *Uniqueness theory of meromorphic functions*, Kluwer Academic Publishers, Dordrecht, 2003.

[14] L. Yang, *Value distribution theory*, Springer-Verlag, Berlin, 1993.

[15] J. Zhang, L.W. Liao, *On entire solutions of a certain type of nonlinear differential and difference equations*, Taiwanese J. Math., 15 (2011), 2145-2157.

[16] J. Zhang, L.W. Liao, *Entire functions sharing some values with their difference operators,* Sci. China Math., 57 (2014), 2143-2152.

[17] J.H. Zheng, S.P. Wang, *On unicity properities of meromorphic functions and their derivatives* (Chinese), Advances in Mathematics, 21 (1992), 334-341.

Accepted: 22.09.2016