STUDY OF INTEGRAL TRANSFORMS ASSOCIATED WITH GENERALIZED BESSEL FUNCTION

N.U. Khan
M. Ghayasuddin*
Department of Applied Mathematics
Faculty of Engineering and Technology
Aligarh Muslim University
Aligarh-202002, India
nukhanmath@gmail.com
ghayas.maths@gmail.com

Waseem A. Khan
Sarvat Zia
Department of Mathematics
Integral University
Lucknow-226026, India
waseem08_khan@rediffmail.com
sarvatzia@gmail.com

Abstract. Integral transforms involving Bessel functions play a crucial role in problems related to many diverse field of mathematical physics. Due to the importance of such type of transforms, in this paper, we present (presumably) a new (potentially) useful integral transform involving the product of Whitaker and generalized Bessel functions, which is expressed in terms of Kampé de Fériet functions. Some more results as special cases of our main integral transform are also considered.
Keywords: Generalized Bessel function, Kampé de Fériet function, Whitaker function, Laplace Transform.

1. Introduction and definition

The generalized Bessel function $\omega^b_{\nu,c}(z)$ of the first kind is defined for $z \in C\backslash\{0\}$ and $b, c, \nu \in C$ with $\Re(\nu) > -1$ by the following series [12], (see also, eg., [4, p.10]), for recent works (see also [1,2,3] and [13, p.2]):

$$\omega^b_{\nu,c}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k c^k \ \left(\frac{z}{2}\right)^{\nu+2k}}{k! \ \Gamma(\nu + k + \frac{1+b}{2})}, \tag{1.1}$$

where C denotes the set of complex numbers, $\Gamma(z)$ is the familiar Gamma function (see, eg., [11, Section 1.1]), and $\omega^b_{\nu,c}(0) = 0$

It is well known that

$$\omega^1_{\nu,1}(z) = J_\nu(z), \tag{1.2}$$

∗. Corresponding author

where $J_\nu(z)$ is the Bessel function of first kind [6] and

$$\omega^1_{\nu,-1}(z) = I_\nu(z), \tag{1.3}$$

where $I_\nu(z)$ is the Modified Bessel function of first kind [6].

$\omega^b_{\nu,c}(z)$ also have the following relations with sine and cosine functions ([12], see also [13])

$$\omega^b_{-\frac{b}{2},c^2}(z) = \left(\frac{2}{z}\right)^{\frac{b}{2}} \frac{\cos cz}{\sqrt{\pi}}, \tag{1.4}$$

$$\omega^b_{-\frac{b}{2},-c^2}(z) = \left(\frac{2}{z}\right)^{\frac{b}{2}} \frac{\cosh cz}{\sqrt{\pi}}, \tag{1.5}$$

$$\omega^b_{1-\frac{b}{2},c^2}(z) = \left(\frac{2}{z}\right)^{\frac{b}{2}} \frac{\sin cz}{\sqrt{\pi}} \tag{1.6}$$

and

$$\omega^b_{1-\frac{b}{2},-c^2}(z) = \left(\frac{2}{z}\right)^{\frac{b}{2}} \frac{\sinh cz}{\sqrt{\pi}}. \tag{1.7}$$

The Whitaker function of second kind $W_{k,\mu}(z)$ is defined as [10, p,39, eq.(24)], (see also [7],[8] and [9])

$$W_{k,\mu}(z) = z^{\mu+\frac{1}{2}} \exp\left(-\frac{z}{2}\right) \Psi\left(\mu - k + \frac{1}{2}, 2\mu+1; z\right), \tag{1.8}$$

which have the following relations with other special functions

$$W_{\frac{\alpha}{2}+\frac{1}{2}+n,\frac{\alpha}{2}}(z) = (-1)^n n! e^{-\frac{z}{2}} z^{\frac{\alpha}{2}+\frac{1}{2}}\ L^\alpha_n(z), \tag{1.9}$$

where $L^\alpha_n(z)$ is a Laguerre polynomial [6],

$$W_{\frac{1}{4}+n,\frac{1}{4}}(z^2) = 2^{-n} e^{-\frac{z^2}{2}}\ \sqrt{z}\ H_n(z), \tag{1.10}$$

where $H_n(z)$ is a Hermite polynomial [6],

$$W_{0,\mu}(2z) = \sqrt{\frac{2z}{\pi}}\ K_\mu(z), \tag{1.11}$$

where $K_\mu(z)$ is the Modified Bessel function of second kind [6].

2. Main transformation

We establish an integral of the form

$$\int_0^\infty t^{\lambda-\frac{1}{2}} e^{-pt}\, W_{k,\mu}(\alpha t)\omega^b_{\nu,c}(\beta t)\, dt$$

$$= \left(\frac{\beta}{2}\right)^\nu \alpha^{\mu+\frac{1}{2}} \frac{\Gamma(\lambda+\nu-\mu+1)}{\Gamma(\nu+\frac{1+b}{2})(p+\frac{\alpha}{2})^{\mu+\lambda+\nu+1}} \left\{ \frac{\Gamma(\mu+\lambda+\nu+1)}{\Gamma(\lambda+\nu-k+\frac{3}{2})}\right.$$

$$\times F^{2:2;2}_{2:1;1}\left[\begin{matrix} \frac{\mu+\lambda+\nu+1}{2}, \frac{\mu+\lambda+\nu+2}{2} : & \frac{\lambda+\nu-\mu+1}{2}, \frac{\lambda+\nu-\mu+2}{2} & ; & \frac{\mu-k+\frac{1}{2}}{2}, \frac{\mu-k+\frac{3}{2}}{2} & ; \\ & & & & & -c\left(\frac{2\beta}{2p+\alpha}\right)^2, \left(\frac{2p-\alpha}{2p+\alpha}\right)^2 \\ \frac{\lambda+\nu-k+\frac{3}{2}}{2}, \frac{\lambda+\nu-k+\frac{5}{2}}{2} : & \nu+\frac{1+b}{2} & ; & \frac{1}{2} & ; \end{matrix}\right]$$

$$+ \left(\frac{2p-\alpha}{2p+\alpha}\right) \frac{\Gamma(\mu+\lambda+\nu+2)\Gamma(\mu-k+\frac{3}{2})}{\Gamma(\mu-k+\frac{1}{2})\Gamma(\lambda+\nu-k+\frac{5}{2})}$$

(2.1)

$$\left.\times F^{2:2;2}_{2:1;1}\left[\begin{matrix} \frac{\mu+\lambda+\nu+2}{2}, \frac{\mu+\lambda+\nu+3}{2} : & \frac{\lambda+\nu-\mu+1}{2}, \frac{\lambda+\nu-\mu+2}{2} & ; & \frac{\mu-k+\frac{3}{2}}{2}, \frac{\mu-k+\frac{5}{2}}{2} & ; \\ & & & & & -c\left(\frac{2\beta}{2p+\alpha}\right)^2, \left(\frac{2p-\alpha}{2p+\alpha}\right)^2 \\ \frac{\lambda+\nu-k+\frac{5}{2}}{2}, \frac{\lambda+\nu-k+\frac{7}{2}}{2} : & \nu+\frac{1+b}{2} & ; & \frac{3}{2} & ; \end{matrix}\right]\right\},$$

where $F^{p:q:k}_{l:m;n}$ denotes the Kampé de Fériet function [10; p.63, Eq.(16)].

Proof. In order to derive our main result (2.1), we denote the left-hand side of (2.1) by I, expressing $\omega^b_{\nu,c}(\beta t)$ as a series with the help of (1.1) and integrating term by term with the help of the result [5; p.216(16)], we get

$$I = \left(\frac{\beta}{2}\right)^\nu \alpha^{\mu+\frac{1}{2}} \sum_{m=0}^\infty \frac{(-1)^m c^m \left(\frac{\beta}{2}\right)^{2m}}{m!\,\Gamma(\nu+m+\frac{1+b}{2})}$$

$$\frac{\Gamma(\mu+\lambda+\nu+2m+1)\Gamma(\lambda+\nu+2m-\mu+1)}{\Gamma(\lambda+\nu+2m-k+\frac{3}{2})(p+\frac{1}{2}\alpha)^{\mu+\lambda+\nu+2m+1}}$$

$$\times\, {}_2F_1\left[\begin{matrix} \mu+\lambda+\nu+2m+1, \quad \lambda+\nu+2m-\mu+1 & ; & \\ & & \frac{2p-\alpha}{2p+\alpha} \\ \lambda+\nu+2m-k+\frac{3}{2} & ; & \end{matrix}\right], \tag{2.2}$$

where $Re(\lambda+\nu+\frac{1}{2}\pm\mu) > -\frac{1}{2}$.

On expanding ${}_2F_1$ in its defining series, separating this series into its even and odd terms, and then by arranging the resulting expression into Kampé de Fériet function [10; p.63, Eq.(16)], we get our required result (2.1). □

3. Special cases

(i) On taking $b = c = 1$ in (2.1) and then by using (1.2), we get

$$\int_0^\infty t^{\lambda-\frac{1}{2}} e^{-pt} W_{k,\mu}(\alpha t) J_\nu(\beta t)\, dt$$

$$= \left(\frac{\beta}{2}\right)^{\nu} \alpha^{\mu+\frac{1}{2}} \frac{\Gamma(\lambda+\nu-\mu+1)}{\Gamma(\nu+1)(p+\frac{\alpha}{2})^{\mu+\lambda+\nu+1}} \left\{ \frac{\Gamma(\mu+\lambda+\nu+1)}{\Gamma(\lambda+\nu-k+\frac{3}{2})} \right.$$

$$\times F^{2:2;2}_{2:1;1}\left[\begin{matrix} \frac{\mu+\lambda+\nu+1}{2}, \frac{\mu+\lambda+\nu+2}{2} : & \frac{\lambda+\nu-\mu+1}{2}, \frac{\lambda+\nu-\mu+2}{2} & ; & \frac{\mu-k+\frac{1}{2}}{2}, \frac{\mu-k+\frac{3}{2}}{2} & ; \\ \frac{\lambda+\nu-k+\frac{3}{2}}{2}, \frac{\lambda+\nu-k+\frac{5}{2}}{2} : & \nu+1 & ; & \frac{1}{2} & ; \end{matrix} -\left(\frac{2\beta}{2p+\alpha}\right)^2, \left(\frac{2p-\alpha}{2p+\alpha}\right)^2\right]$$

$$+ \left(\frac{2p-\alpha}{2p+\alpha}\right) \frac{\Gamma(\mu+\lambda+\nu+2)\Gamma(\mu-k+\frac{3}{2})}{\Gamma(\mu-k+\frac{1}{2})\Gamma(\lambda+\nu-k+\frac{5}{2})}$$

(3.1)

$$\left.\times F^{2:2;2}_{2:1;1}\left[\begin{matrix} \frac{\mu+\lambda+\nu+2}{2}, \frac{\mu+\lambda+\nu+3}{2} : & \frac{\lambda+\nu-\mu+1}{2}, \frac{\lambda+\nu-\mu+2}{2} & ; & \frac{\mu-k+\frac{3}{2}}{2}, \frac{\mu-k+\frac{5}{2}}{2} & ; \\ \frac{\lambda+\nu-k+\frac{5}{2}}{2}, \frac{\lambda+\nu-k+\frac{7}{2}}{2} : & \nu+1 & ; & \frac{3}{2} & ; \end{matrix} -\left(\frac{2\beta}{2p+\alpha}\right)^2, \left(\frac{2p-\alpha}{2p+\alpha}\right)^2\right]\right\}.$$

(ii) On taking $b = 1, c = -1$ in (2.1) and then by using (1.3) , we get

$$\int_0^{\infty} t^{\lambda-\frac{1}{2}} e^{-pt} W_{k,\mu}(\alpha t) I_{\nu}(\beta t)\, dt$$

$$= \left(\frac{\beta}{2}\right)^{\nu} \alpha^{\mu+\frac{1}{2}} \frac{\Gamma(\lambda+\nu-\mu+1)}{\Gamma(\nu+1)(p+\frac{\alpha}{2})^{\mu+\lambda+\nu+1}} \left\{ \frac{\Gamma(\mu+\lambda+\nu+1)}{\Gamma(\lambda+\nu-k+\frac{3}{2})} \right.$$

$$\times F^{2:2;2}_{2:1;1}\left[\begin{matrix} \frac{\mu+\lambda+\nu+1}{2}, \frac{\mu+\lambda+\nu+2}{2} : & \frac{\lambda+\nu-\mu+1}{2}, \frac{\lambda+\nu-\mu+2}{2} & ; & \frac{\mu-k+\frac{1}{2}}{2}, \frac{\mu-k+\frac{3}{2}}{2} & ; \\ \frac{\lambda+\nu-k+\frac{3}{2}}{2}, \frac{\lambda+\nu-k+\frac{5}{2}}{2} : & \nu+1 & ; & \frac{1}{2} & ; \end{matrix} \left(\frac{2\beta}{2p+\alpha}\right)^2, \left(\frac{2p-\alpha}{2p+\alpha}\right)^2\right]$$

$$+ \left(\frac{2p-\alpha}{2p+\alpha}\right) \frac{\Gamma(\mu+\lambda+\nu+2)\Gamma(\mu-k+\frac{3}{2})}{\Gamma(\mu-k+\frac{1}{2})\Gamma(\lambda+\nu-k+\frac{5}{2})}$$

(3.2)

$$\left.\times F^{2:2;2}_{2:1;1}\left[\begin{matrix} \frac{\mu+\lambda+\nu+2}{2}, \frac{\mu+\lambda+\nu+3}{2} : & \frac{\lambda+\nu-\mu+1}{2}, \frac{\lambda+\nu-\mu+2}{2} & ; & \frac{\mu-k+\frac{3}{2}}{2}, \frac{\mu-k+\frac{5}{2}}{2} & ; \\ \frac{\lambda+\nu-k+\frac{5}{2}}{2}, \frac{\lambda+\nu-k+\frac{7}{2}}{2} : & \nu+1 & ; & \frac{3}{2} & ; \end{matrix} \left(\frac{2\beta}{2p+\alpha}\right)^2, \left(\frac{2p-\alpha}{2p+\alpha}\right)^2\right]\right\},$$

where $I_{\nu}(\beta t)$ is the Modified Bessel function of first kind.
(iii) On taking $k = 0$ in (2.1) and then by using (1.11), we get

$$\int_0^{\infty} t^{\lambda} e^{-pt} K_{\mu}\left(\frac{\alpha}{2}t\right) \omega^{b}_{\nu,c}(\beta t)\, dt$$

$$= \sqrt{\pi}\left(\frac{\beta}{2}\right)^{\nu} \alpha^{\mu} \frac{\Gamma(\lambda+\nu-\mu+1)}{\Gamma(\nu+\frac{1+b}{2})(p+\frac{\alpha}{2})^{\mu+\lambda+\nu+1}} \left\{ \frac{\Gamma(\mu+\lambda+\nu+1)}{\Gamma(\lambda+\nu+\frac{3}{2})} \right.$$

$$\times F^{2:2;2}_{2:1;1}\left[\begin{matrix} \frac{\mu+\lambda+\nu+1}{2}, \frac{\mu+\lambda+\nu+2}{2} : & \frac{\lambda+\nu-\mu+1}{2}, \frac{\lambda+\nu-\mu+2}{2} & ; & \frac{\mu+\frac{1}{2}}{2}, \frac{\mu+\frac{3}{2}}{2} & ; \\ \frac{\lambda+\nu+\frac{3}{2}}{2}, \frac{\lambda+\nu+\frac{5}{2}}{2} : & \nu+\frac{1+b}{2} & ; & \frac{1}{2} & ; \end{matrix} -c\left(\frac{2\beta}{2p+\alpha}\right)^2, \left(\frac{2p-\alpha}{2p+\alpha}\right)^2\right]$$

$$+ \left(\frac{2p-\alpha}{2p+\alpha}\right) \frac{\Gamma(\mu+\lambda+\nu+2)\Gamma(\mu+\frac{3}{2})}{\Gamma(\mu+\frac{1}{2})\Gamma(\lambda+\nu+\frac{5}{2})}$$

(3.3)
$$\times F^{2:2;2}_{2:1;1}\left[\begin{array}{ccccc} \frac{\mu+\lambda+\nu+2}{2}, \frac{\mu+\lambda+\nu+3}{2} : & \frac{\lambda+\nu-\mu+1}{2}, \frac{\lambda+\nu-\mu+2}{2} & ; & \frac{\mu+\frac{3}{2}}{2}, \frac{\mu+\frac{5}{2}}{2} & ; \\ & & & & -c\left(\frac{2\beta}{2p+\alpha}\right)^2, \left(\frac{2p-\alpha}{2p+\alpha}\right)^2 \\ \frac{\lambda+\nu+\frac{5}{2}}{2}, \frac{\lambda+\nu+\frac{7}{2}}{2} : & \nu+\frac{1+b}{2} & ; & \frac{3}{2} & ; \end{array}\right]\Bigg\},$$

where $K_\nu(\frac{\alpha}{2}t)$ is the Modified Bessel function of second kind.

(iv) On taking $\nu=-\frac{b}{2}$, replacing c by $-c^2$ in (2.1) and then on using (1.5), we get

$$\int_0^\infty t^{\lambda-\frac{b}{2}-\frac{1}{2}}e^{-pt}W_{k,\mu}(\alpha t)\cosh(c\beta t)\,dt$$

$$=\alpha^{\mu+\frac{1}{2}}\frac{\Gamma(\lambda-\frac{b}{2}-\mu+1)}{(p+\frac{\alpha}{2})^{\mu+\lambda-\frac{b}{2}+1}}\Bigg\{\frac{\Gamma(\mu+\lambda-\frac{b}{2}+1)}{\Gamma(\lambda-\frac{b}{2}-k+\frac{3}{2})}$$

$$\times F^{2:2;2}_{2:1;1}\left[\begin{array}{ccccc} \frac{\mu+\lambda-\frac{b}{2}+1}{2}, \frac{\mu+\lambda-\frac{b}{2}+2}{2} : & \frac{\lambda-\frac{b}{2}-\mu+1}{2}, \frac{\lambda-\frac{b}{2}-\mu+2}{2} & ; & \frac{\mu-k+\frac{1}{2}}{2}, \frac{\mu-k+\frac{3}{2}}{2} & ; \\ & & & & \left(\frac{2c\beta}{2p+\alpha}\right)^2, \left(\frac{2p-\alpha}{2p+\alpha}\right)^2 \\ \frac{\lambda-\frac{b}{2}-k+\frac{3}{2}}{2}, \frac{\lambda-\frac{b}{2}-k+\frac{5}{2}}{2} : & \frac{1}{2} & ; & \frac{1}{2} & ; \end{array}\right]$$

$$+\left(\frac{2p-\alpha}{2p+\alpha}\right)\frac{\Gamma(\mu+\lambda-\frac{b}{2}+2)\Gamma(\mu-k+\frac{3}{2})}{\Gamma(\mu-k+\frac{1}{2})\Gamma(\lambda-\frac{b}{2}-k+\frac{5}{2})}$$

(3.4)
$$\times F^{2:2;2}_{2:1;1}\left[\begin{array}{ccccc} \frac{\mu+\lambda-\frac{b}{2}+2}{2}, \frac{\mu+\lambda-\frac{b}{2}+3}{2} : & \frac{\lambda-\frac{b}{2}-\mu+1}{2}, \frac{\lambda-\frac{b}{2}-\mu+2}{2} & ; & \frac{\mu-k+\frac{3}{2}}{2}, \frac{\mu-k+\frac{5}{2}}{2} & ; \\ & & & & \left(\frac{2c\beta}{2p+\alpha}\right)^2, \left(\frac{2p-\alpha}{2p+\alpha}\right)^2 \\ \frac{\lambda-\frac{b}{2}-k+\frac{5}{2}}{2}, \frac{\lambda-\frac{b}{2}-k+\frac{7}{2}}{2} : & \frac{1}{2} & ; & \frac{3}{2} & ; \end{array}\right]\Bigg\}.$$

(v) On taking $\nu=1-\frac{b}{2}$, replacing c by c^2 in (2.1) and then on using (1.6), we get

$$\int_0^\infty t^{\lambda-\frac{b}{2}-\frac{1}{2}}e^{-pt}W_{k,\mu}(\alpha t)\sin(c\beta t)\,dt$$

$$=\alpha^{\mu+\frac{1}{2}}\beta\frac{\Gamma(\lambda-\frac{b}{2}-\mu+2)}{(p+\frac{\alpha}{2})^{\mu+\lambda-\frac{b}{2}+2}}\Bigg\{\frac{\Gamma(\mu+\lambda-\frac{b}{2}+2)}{\Gamma(\lambda-\frac{b}{2}-k+\frac{5}{2})}$$

$$\times F^{2:2;2}_{2:1;1}\left[\begin{array}{ccccc} \frac{\mu+\lambda-\frac{b}{2}+2}{2}, \frac{\mu+\lambda-\frac{b}{2}+3}{2} : & \frac{\lambda-\frac{b}{2}-\mu+2}{2}, \frac{\lambda-\frac{b}{2}-\mu+3}{2} & ; & \frac{\mu-k+\frac{1}{2}}{2}, \frac{\mu-k+\frac{3}{2}}{2} & ; \\ & & & & -\left(\frac{2c\beta}{2p+\alpha}\right)^2, \left(\frac{2p-\alpha}{2p+\alpha}\right)^2 \\ \frac{\lambda-\frac{b}{2}-k+\frac{5}{2}}{2}, \frac{\lambda-\frac{b}{2}-k+\frac{7}{2}}{2} : & \frac{3}{2} & ; & \frac{1}{2} & ; \end{array}\right]$$

$$+\left(\frac{2p-\alpha}{2p+\alpha}\right)\frac{\Gamma(\mu+\lambda-\frac{b}{2}+3)\Gamma(\mu-k+\frac{3}{2})}{\Gamma(\mu-k+\frac{1}{2})\Gamma(\lambda-\frac{b}{2}-k+\frac{7}{2})}$$

(3.5)
$$\times F^{2:2;2}_{2:1;1}\left[\begin{array}{ccccc} \frac{\mu+\lambda-\frac{b}{2}+3}{2}, \frac{\mu+\lambda-\frac{b}{2}+4}{2} : & \frac{\lambda-\frac{b}{2}-\mu+2}{2}, \frac{\lambda-\frac{b}{2}-\mu+3}{2} & ; & \frac{\mu-k+\frac{1}{2}}{2}, \frac{\mu-k+\frac{3}{2}}{2} & ; \\ & & & & -\left(\frac{2c\beta}{2p+\alpha}\right)^2, \left(\frac{2p-\alpha}{2p+\alpha}\right)^2 \\ \frac{\lambda-\frac{b}{2}-k+\frac{7}{2}}{2}, \frac{\lambda-\frac{b}{2}-k+\frac{9}{2}}{2} : & \frac{3}{2} & ; & \frac{3}{2} & ; \end{array}\right]\Bigg\}.$$

(vi) On taking $\nu=1-\frac{b}{2}$, replacing c by $-c^2$ in (2.1) and then by using (1.7), we get

$$\int_0^\infty t^{\lambda-\frac{b}{2}-\frac{1}{2}}e^{-pt}W_{k,\mu}(\alpha t)\sinh(c\beta t)\,dt$$

$$= \alpha^{\mu+\frac{1}{2}}\beta \frac{\Gamma(\lambda-\frac{b}{2}-\mu+2)}{(p+\frac{\alpha}{2})^{\mu+\lambda-\frac{b}{2}+2}} \left\{ \frac{\Gamma(\mu+\lambda-\frac{b}{2}+2)}{\Gamma(\lambda-\frac{b}{2}-k+\frac{5}{2})} \right.$$

$$\times F^{2:2;2}_{2:1;1}\left[\begin{array}{llll} \frac{\mu+\lambda-\frac{b}{2}+2}{2}, \frac{\mu+\lambda-\frac{b}{2}+3}{2}: & \frac{\lambda-\frac{b}{2}-\mu+2}{2}, \frac{\lambda-\frac{b}{2}-\mu+3}{2} & ; \frac{\mu-k+\frac{1}{2}}{2}, \frac{\mu-k+\frac{3}{2}}{2} & ; \\ & & & \left(\frac{2c\beta}{2p+\alpha}\right)^2, \left(\frac{2p-\alpha}{2p+\alpha}\right)^2 \\ \frac{\lambda-\frac{b}{2}-k+\frac{5}{2}}{2}, \frac{\lambda-\frac{b}{2}-k+\frac{7}{2}}{2}: & \frac{3}{2} & ; \frac{1}{2} & ; \end{array}\right]$$

$$+\left(\frac{2p-\alpha}{2p+\alpha}\right) \frac{\Gamma(\mu+\lambda-\frac{b}{2}+3)\Gamma(\mu-k+\frac{3}{2})}{\Gamma(\mu-k+\frac{1}{2})\Gamma(\lambda-\frac{b}{2}-k+\frac{7}{2})}$$

$$\times F^{2:2;2}_{2:1;1}\left[\begin{array}{llll} \frac{\mu+\lambda-\frac{b}{2}+3}{2}, \frac{\mu+\lambda-\frac{b}{2}+4}{2}: & \frac{\lambda-\frac{b}{2}-\mu+2}{2}, \frac{\lambda-\frac{b}{2}-\mu+3}{2} & ; \frac{\mu-k+\frac{1}{2}}{2}, \frac{\mu-k+\frac{3}{2}}{2} & ; \\ & & & \left(\frac{2c\beta}{2p+\alpha}\right)^2, \left(\frac{2p-\alpha}{2p+\alpha}\right)^2 \\ \frac{\lambda-\frac{b}{2}-k+\frac{7}{2}}{2}, \frac{\lambda-\frac{b}{2}-k+\frac{9}{2}}{2}: & \frac{3}{2} & ; \frac{3}{2} & ; \end{array}\right]\left.\vphantom{\frac{1}{1}}\right\}. \tag{3.6}$$

(vii) On taking $k = \frac{\delta}{2} + \frac{1}{2} + q,\ \mu = \frac{\delta}{2}$ in (2.1) and then by using (1.9), we obtain

$$\int_0^\infty t^{\lambda+\frac{\delta}{2}+\frac{1}{2}-\frac{1}{2}} e^{-(p+\frac{\alpha}{2})t} L_q^\delta(\alpha t)\omega^b_{\nu,c}(\beta t)\ dt$$

$$= \frac{1}{(-1)^q q!}\left(\frac{\beta}{2}\right)^\nu \frac{\Gamma(\lambda+\nu-\frac{\delta}{2}+1)}{\Gamma(\nu+\frac{1+b}{2})(p+\frac{\alpha}{2})^{\frac{\delta}{2}+\lambda+\nu+1}} \left\{ \frac{\Gamma(\frac{\delta}{2}+\lambda+\nu+1)}{\Gamma(\lambda+\nu-\frac{\delta}{2}-q+1)} \right.$$

$$\times F^{2:2;2}_{2:1;1}\left[\begin{array}{llll} \frac{\frac{\delta}{2}+\lambda+\nu+1}{2}, \frac{\frac{\delta}{2}+\lambda+\nu+2}{2}: & \frac{\lambda+\nu-\frac{\delta}{2}+1}{2}, \frac{\lambda+\nu-\frac{\delta}{2}+2}{2} & ; \frac{-q}{2}, \frac{-q+1}{2} & ; \\ & & & -c\left(\frac{2\beta}{2p+\alpha}\right)^2, \left(\frac{2p-\alpha}{2p+\alpha}\right)^2 \\ \frac{\lambda+\nu-\frac{\delta}{2}-q+1}{2}, \frac{\lambda+\nu-\frac{\delta}{2}-q+2}{2}: & \nu+\frac{1+b}{2} & ; \frac{1}{2} & ; \end{array}\right]$$

$$+\left(\frac{2p-\alpha}{2p+\alpha}\right) \frac{\Gamma(\frac{\delta}{2}+\lambda+\nu+2)\Gamma(-q+1)}{\Gamma(-q)\Gamma(\lambda+\nu-\frac{\delta}{2}-q+2)}$$

$$\times F^{2:2;2}_{2:1;1}\left[\begin{array}{llll} \frac{\frac{\delta}{2}+\lambda+\nu+2}{2}, \frac{\frac{\delta}{2}+\lambda+\nu+3}{2}: & \frac{\lambda+\nu-\frac{\delta}{2}+1}{2}, \frac{\lambda+\nu-\frac{\delta}{2}+2}{2} & ; \frac{-q+1}{2}, \frac{-q+2}{2} & ; \\ & & & -c\left(\frac{2\beta}{2p+\alpha}\right)^2, \left(\frac{2p-\alpha}{2p+\alpha}\right)^2 \\ \frac{\lambda+\nu-\frac{\delta}{2}-q+2}{2}, \frac{\lambda+\nu-\frac{\delta}{2}-q+3}{2}: & \nu+\frac{1+b}{2} & ; \frac{3}{2} & ; \end{array}\right]\left.\vphantom{\frac{1}{1}}\right\}, \tag{3.7}$$

where $L_q^\delta(\alpha t)$ is the Laguerre polynomial.

(viii) On taking $k = \frac{1}{4} + q, \mu = \frac{1}{4}$ in (2.1) and then by using (1.10), we get

$$\int_0^\infty t^{\lambda+\frac{1}{4}-\frac{1}{2}} e^{-(p+\frac{\alpha}{2})t} H_q(\sqrt{\alpha}t)\omega^b_{\nu,c}(\beta t)\ dt$$

$$= 2^q\left(\frac{\beta}{2}\right)^\nu \sqrt{\alpha}\frac{\Gamma(\lambda+\nu+\frac{3}{4})}{\Gamma(\nu+\frac{1+b}{2})(p+\frac{\alpha}{2})^{\lambda+\nu+\frac{5}{4}}} \left\{ \frac{\Gamma(\lambda+\nu+\frac{5}{4})}{\Gamma(\lambda+\nu-q+\frac{5}{4})} \right.$$

$$\times F^{2:2;2}_{2:1;1}\left[\begin{array}{llll} \frac{\lambda+\nu+\frac{5}{4}}{2}, \frac{\lambda+\nu+\frac{9}{4}}{2}: & \frac{\lambda+\nu+\frac{3}{4}}{2}, \frac{\lambda+\nu+\frac{7}{4}}{2} & ; \frac{-q+\frac{1}{2}}{2}, \frac{-q+\frac{3}{2}}{2} & ; \\ & & & -c\left(\frac{2\beta}{2p+\alpha}\right)^2, \left(\frac{2p-\alpha}{2p+\alpha}\right)^2 \\ \frac{\lambda+\nu-q+\frac{5}{4}}{2}, \frac{\lambda+\nu-q+\frac{9}{4}}{2}: & \nu+\frac{1+b}{2} & ; \frac{1}{2} & ; \end{array}\right]$$

$$+\left(\frac{2p-\alpha}{2p+\alpha}\right)\frac{\Gamma(\lambda+\nu+\frac{9}{4})\Gamma(-q+\frac{3}{2})}{\Gamma(-q+\frac{1}{2})\Gamma(\lambda+\nu-q+\frac{9}{4})}$$

(3.8)

$$\times F^{2:2;2}_{2:1;1}\left[\begin{matrix}\frac{\lambda+\nu+\frac{9}{4}}{2},\frac{\lambda+\nu+\frac{13}{4}}{2}: & \frac{\lambda+\nu+\frac{3}{4}}{2},\frac{\lambda+\nu+\frac{7}{4}}{2} & ; & \frac{-q+\frac{3}{2}}{2},\frac{-q+\frac{5}{2}}{2} & ; \\ & & & & & -c\left(\frac{2\beta}{2p+\alpha}\right)^2,\left(\frac{2p-\alpha}{2p+\alpha}\right)^2 \\ \frac{\lambda+\nu-q+\frac{9}{4}}{2},\frac{\lambda+\nu-q+\frac{13}{4}}{2}: & \nu+\frac{1+b}{2} & ; & \frac{3}{2} & ; \end{matrix}\right]\Bigg\},$$

where $H_q(\sqrt{\alpha}t)$ is the Hermite polynomial.

(ix) On taking $\nu=-\frac{b}{2}$, replacing c by c^2 in (2.1) and then by using (1.4), we get

$$\int_0^\infty t^{\lambda-\frac{b}{2}-\frac{1}{2}}e^{-pt}W_{k,\mu}(\alpha t)\cos(c\beta t)\,dt$$

$$=\alpha^{\mu+\frac{1}{2}}\frac{\Gamma(\lambda-\frac{b}{2}-\mu+1)}{(p+\frac{\alpha}{2})^{\mu+\lambda-\frac{b}{2}+1}}\left\{\frac{\Gamma(\mu+\lambda-\frac{b}{2}+1)}{\Gamma(\lambda-\frac{b}{2}-k+\frac{3}{2})}\right.$$

$$\times F^{2:2;2}_{2:1;1}\left[\begin{matrix}\frac{\mu+\lambda-\frac{b}{2}+1}{2},\frac{\mu+\lambda-\frac{b}{2}+2}{2}: & \frac{\lambda-\frac{b}{2}-\mu+1}{2},\frac{\lambda-\frac{b}{2}-\mu+2}{2} & ; & \frac{\mu-k+\frac{1}{2}}{2},\frac{\mu-k+\frac{3}{2}}{2} & ; \\ & & & & & -\left(\frac{2c\beta}{2p+\alpha}\right)^2,\left(\frac{2p-\alpha}{2p+\alpha}\right)^2 \\ \frac{\lambda-\frac{b}{2}-k+\frac{3}{2}}{2},\frac{\lambda-\frac{b}{2}-k+\frac{5}{2}}{2}: & \frac{1}{2} & ; & \frac{1}{2} & ; \end{matrix}\right]$$

$$+\left(\frac{2p-\alpha}{2p+\alpha}\right)\frac{\Gamma(\mu+\lambda-\frac{b}{2}+2)\Gamma(\mu-k+\frac{3}{2})}{\Gamma(\mu-k+\frac{1}{2})\Gamma(\lambda-\frac{b}{2}-k+\frac{5}{2})}$$

(3.9)

$$\times F^{2:2;2}_{2:1;1}\left[\begin{matrix}\frac{\mu+\lambda-\frac{b}{2}+2}{2},\frac{\mu+\lambda-\frac{b}{2}+3}{2}: & \frac{\lambda-\frac{b}{2}-\mu+1}{2},\frac{\lambda-\frac{b}{2}-\mu+2}{2} & ; & \frac{\mu-k+\frac{3}{2}}{2},\frac{\mu-k+\frac{5}{2}}{2} & ; \\ & & & & & -\left(\frac{2c\beta}{2p+\alpha}\right)^2,\left(\frac{2p-\alpha}{2p+\alpha}\right)^2 \\ \frac{\lambda-\frac{b}{2}-k+\frac{5}{2}}{2},\frac{\lambda-\frac{b}{2}-k+\frac{7}{2}}{2}: & \frac{1}{2} & ; & \frac{3}{2} & ; \end{matrix}\right]\Bigg\}.$$

Remark. In a similar way, for some parametric replacement of b, c, ν, k and μ, we can easily establish some other integral transforms involving the product of Laguerre polynomial, Hermite polynomial and Modified Bessel function of second kind with Bessel function of first kind, Modified Bessel function of first kind, sine function and cosine function.

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Accepted: 16.09.2016