ON c-NORMAL AND S-QUASINORMALLY EMBEDDED SUBGROUPS OF A FINITE GROUP

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Abstract. If P is a p-group for some prime p we shall write $\mathscr{M}(P)$ to denote the set of all maximal subgroups of P and $\mathscr{M}_d(P) = \{P_1, ..., P_d\}$ to denote any set of maximal subgroups of P such that $\bigcap_{i=1}^{d} P_i = \Phi(P)$ and d is as small as possible. In this paper, the structure of a finite group G under some assumptions on the c-normal or S-quasinormally embedded subgroups in $\mathscr{M}_d(P)$, for each prime p, and Sylow p-subgroups P of G is researched. Some known results are generalized.

Keywords: c-normal subgroup, S-quasinormally embedded subgroup, supersolvable groups.

1. Introduction

All groups considered in this paper are finite. Let G be a group and let $\mathscr{M}(G)$ be the set of all maximal subgroups of all Sylow subgroups of G. A interesting topic in group theory is to study the influence of the elements of $\mathscr{M}(G)$ on the structure of G. A typical result in this direction is due to Srinivasan [1]. He proved that G is supersolvable provided that every member of $\mathscr{M}(G)$ is normal in G. This result has been widely generalized.

A subgroup H of G is called s-quasinormal in G provided H permutes with all Sylow subgroups of G, i.e, $HP = PH$ for any Sylow subgroup P of G. This concept was introduced by Kegel in [2] and has been studied extensively by Deskins [3] and Schmidt [4]. More recently, Ballester-Bolinches and Pedraza-Aquilera [5] generalized s-quasinormal subgroups to s-quasinormally embedded subgroups. A subgroup H of G is said to be s-quasinormally embedded in G provided every Sylow subgroup of H is a Sylow subgroup of some s-quasinormal subgroup of G. In [5], Ballester-Bolinches and Pedraza-Aquilera showed that, if every subgroup in $\mathscr{M}(G)$ is s-quasinormally embedded in G, then G is supersolvable. Assad and Heliel [6] showed that G is p-nilpotent for the smallest prime p dividing $|G|$ if and only if all members of $\mathscr{M}(P)$ are s-quasinormally embedded in G, where P is a Sylow p-subgroup of G.

As another generalization of the normality, Wang [7] introduced the following concept: A subgroup H of G is called c-normal in G if there is a normal subgroup K such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the normal core of H in G. In [7], Wang showed that G is supersolvable if every member of $\mathscr{M}(G)$

is c-normal. Wang′s result has been generalized by some authors (see [8-12], etc). For example, Guo and Shum showed in [8] the following result. Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If every member of $\mathscr{M}(P)$ is c-normal, then G is p-nilpotent. The research on c-normal subgroups has formed a series, which is similar to the series of s-quasinormal subgroups. However, the two series are independent of each other. The aim of this article is to unify and improve the results of [1], [5], [7] and some of [8].

If P is a p-group for some prime p we shall write $\mathscr{M}(P)$ to denote the set of all maximal subgroups of P and $\mathscr{M}_d(P) = \{P_1, ..., P_d\}$ to denote any set of maximal subgroups of P such that $\bigcap_{i=1}^{d} P_i = \Phi(P)$ and d is as small as possible.

Such subset $\mathscr{M}_d(P)$ is not unique for a fixed P in general. We know that

$$|\mathscr{M}(P)| = \frac{p^d - 1}{p - 1}, |\mathscr{M}_d(P)| = d, \lim_{d \to \infty} \frac{p^d - 1}{(p-1)d} = \infty,$$

so $|\mathscr{M}(P)| \gg |\mathscr{M}_d(P)|$.

In this paper, we study the influence of the members of some fixed $\mathscr{M}_d(P)$ on the structure of group G. Our results are more general.

2. Basic definitions and preliminary results

In this section, we give some results that are needed in this paper.

Lemma 2.1 ([5]). *Suppose that U is an s-quasinormally embedded subgroup of G and that K is a normal subgroup of G. Then:*

  (a) *U is s-quasinormally embedded in H whenever $U \leq H \leq G$;*

  (b) *UK is s-quasinormally embedded in G and UK/K is s-quasinormally embedded in G/K.*

Lemma 2.2 ([7]). *Let $X \leq H \leq G$ and $N \unlhd G$. Then:*

  (a) *If X is c-normal in G, then X is also c-normal in H;*

  (b) *If X is c-normal in G, then XN/N is c-normal in G/N.*

Lemma 2.3 ([3]). *If H is an s-quasinormal subgroup of the group G, then H/H_G is nilpotent.*

Lemma 2.4 ([4]). *For a nilpotent subgroup H of G, the following two statements are equivalent:*

  (a) *H is s-quasinormal in G;*

  (b) *The Sylow subgroups of H are s-quasinormal in G.*

Lemma 2.5 ([6])**.** *Let G be a group and let P_0 be a maximal subgroup of P. Then the following two statements are equivalent:*

  (a) P_0 *is normal in G;*

  (b) P_0 *is s-quasinormal in G.*

Lemma 2.6 ([13])**.** *If P is a Sylow p-subgroup of G and $N \trianglelefteq G$ such that $P \bigcap N \leq \Phi(P)$, then N is p-nilpotent.*

Lemma 2.7 ([14])**.** *A group G is superslovable if and only if there exists a subgroup of order dividing $|H|$ for every subgroup H of G.*

Lemma 2.8 ([15])**.** *Let p_1 be the minimal prime dividing $|G|$ and p_s the maximal prime dividing $|G|$. If G possesses two supersolvable subgroups H and K with $|G : H| = p_1$ and $|G : K| = p_s$, then G is superslovable.*

Lemma 2.9 ([16])**.** *Let H be a p-subgroup of G. Then, the following statements are equivalent:*
  (a) H *is s-quasinormal in G;*
  (b) $H \leq O_p(G)$, *and H is s-quasinormally embedded in G.*

3. Main results

Theorem 3.1. *Let p be the smallest prime dividing the order of a group G and P be a Sylow p-subgroup of G. Assume that every member of some fixed $\mathscr{M}_d(P)$ is either c-normal or s-quasinormally embedded in G. Then G is p-nilpotent.*

Proof. Assume that the result is not true and let G be a counterexample of minimal order. Let $\mathscr{M}_d(P) = \{P_1, ..., P_d\}$. By hypothesis, each P_i is either c-normal or s-quasinormally embedded in G. Without loss of generality, let I_1 be the subset of $\{1, ..., d\}$ such that every $P_i (i \in I_1)$ is c-normal in G and I_2 is the subset such that every $P_i (i \in I_2)$ is s-quasinormally embedded in G. We prove the theorem by the following claims:

   (1) $O_{p'}(G) = 1$.

   Set $N = O_{p'}(G)$. Consider the quotient group G/N. We know that PN/N is a Sylow p-subgroup of G/N, $N_{G/N}(PN/N) = N_G(P)N/N$ and $\mathscr{M}(PN/N) = \{P_1N/N, ..., P_mN/N\}$. Now, by Lemma 2.1 and Lemma 2.2, we see easily that G/N satisfies the condition. If $O_{p'}(G) > 1$, then $G/O_{p'}(G)$ is p-nilpotent and hence G itself is p-nilpotent, a contradiction. Thus claim (1) holds.

   (2) G/P_{iG} is p-nilpotent for all $i \in I_1$, where P_{iG} is the core of P_i in G.

   In this case, P_i is a c-normal subgroup of G. We know that there exists a normal subgroup K_i of G such that $G = P_iK_i$ and $P_i \cap K_i = P_{iG}$. Hence,

$$G/P_{iG} = P_i/P_{iG} \cdot K_i/P_{iG}, P_i \cap K_i = P_{iG}.$$

Therefore,
$$|K_i/P_{iG}|_p = |G : P_i|_p = |P : P_i| = p.$$
As p is the smallest prime dividing $|G|$, we know that K_i/P_{iG} is p-nilpotent by Burnside's theorem. Therefore, K_i/P_{iG} has a normal Hall p'-subgroup H/P_{iG}. We see that H/P_{iG} is also a normal Hall p'-subgroup of G/P_{iG} because K_i/P_{iG} is normal in G/P_{iG}. It follows that G/P_{iG} is p-nilpotent for all $i \in I_1$.

For every $P_i(i \in I_2)$, there exists an S-quasinormal subgroup H_i of G such that P_i is a Sylow p-subgroup of H_i.

(3) G/H_{iG} is p-nilpotent for all $i \in I_2$, where H_{iG} is the core of H_i in G.

In fact, As H_i is an S-quasinormal subgroup of G and P_i is a Sylow p-subgroup of H_i, it follows that H_i/H_{iG} is S-quasinormal in G/H_{iG}, and the Lemma of 2.3 asserts that H_i/H_{iG} is nilpotent. Hence, H_i/H_{iG} is an S-quasinormal nilpotent subgroup of G/H_{iG}. By Lemma 2.4, every Sylow subgroup of H_i/H_{iG} is S-quasinormal in G/H_{iG}. Since P_iH_i/H_{iG} is a Sylow p-subgroup of H_i/H_{iG}, it follows that P_iH_i/H_{iG} is S-quasinormal in G/H_{iG}. Thus, Lemma 2.5 indicates that P_iH_i/H_{iG} is normal in G/H_{iG}. Therefore, $P_iH_{iG} \trianglelefteq G$. Noting that P_i is a Sylow p-subgroup of H_i, we have $P_i \leq H_{iG}$. Therefore, $|G/H_{iG}|_p = p$. Now, as p is the smallest prime dividing $|G|$, by Burnside's theorem, we see that G/H_{iG} is p-nilpotent for each $i \in I_2$, which proves (3).

Let
$$N = (\bigcap_{i \in I_1} P_{iG}) \cap (\bigcap_{i \in I_2} H_{iG}).$$

(4) N is p-nilpotent.

First, as all P_{iG} and H_{iG} are normal in G, we get $N \trianglelefteq G$. Second, we consider the subgroup $P \cap N$. Recall that P_i is a Sylow p-subgroup of H_{iG} and $P_i \leq P$, so $P \cap H_{iG} \leq P_i$. Moreover, $P_i \leq P \cap H_{iG}$. We have $P \cap H_{iG} = P_i$. Therefore,
$$P \cap N = (\bigcap_{i \in I_1} P_{iG}) \cap (\bigcap_{i \in I_2} H_{iG} \cap P) = (\bigcap_{i \in I_1} P_{iG}) \cap (\bigcap_{i \in I_2} P_i) = \Phi(P).$$

Applying Lemma 2.6, we know that N is p-nilpotent.

(5) Final contradiction.

Now, N possesses a Hall p'-normal subgroup $N_{p'}$ such that $N = N_p N_{p'}$, where N_p is a Sylow p-subgroup of N. Then, $N_{p'}$ char $N \trianglelefteq G$, so $N_{p'}$ is normal in G, and hence, $N_{p'} \leq O_{p'}(G)$. It follows by $O_{p'}(G) = 1$ that $N_{p'} = 1$. Consequently, N is a normal p-subgroup of G, and so, $N = P \cap N = \Phi(P)$. Also, note that the class of p-nilpotent groups is a formation, by steps (2) and (3), we have G/N must be p-nilpotent. It follows that $G/\Phi(P)$ is p-nilpotent. Moreover, by III, 3.3 Hilfs-Satz in [17], $\Phi(P) \leq \Phi(G)$, so $G/\Phi(G)$ is p-nilpotent. It follows that G would be p-nilpotent, contrary to the choice of G.

The following corollaries are immediate from Theorem 3.1.

Corollary 3.2 ([8]). *Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If every member of $\mathscr{M}(P)$ is c-normal, then G is p-nilpotent.*

Corollary 3.3 ([6])**.** *Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If every member of $\mathscr{M}(P)$ is s-quasinormally embedded in G, then G is p-nilpotent.*

Theorem 3.4. *Let G be a group. If there exists a normal subgroup H of G such that G/H is supersolvable, and for each Sylow subgroup P of H, every member in some fixed $\mathscr{M}_d(P)$ is either c-normal or s-quasinormally embedded in G, then G is supersolvable.*

Proof. Suppose that the theorem is false so that there exists a counterexample G of minimal order. We shall finish the proof by the following claims.

(1) H is a q-group for some prime q.

By hypothesis and Theorem 3.1, we have that H has a Sylow tower of supersolvable type. Let q be the largest prime dividing $|H|$, and let Q be a Sylow q-subgroup of H. The property that H possesses an order Sylow tower property implies that Q is normal in H. Now, Q char H and $H \unlhd G$, so $Q \unlhd G$. Furthermore, $(G/Q)/(H/Q) \cong G/H$, and Lemmas 2.1 and 2.2 show that G/Q satisfies the condition of the theorem, by the choice of G, G/Q is supersolvable. Hence, $H = Q$ by the choice of H.

(2) Q is a Sylow q-subgroup of G.

Suppose that Q is not a Sylow q-subgroup of G. Let p be the smallest prime dividing $|G/Q|$ and r the largest prime dividing $|G/Q|$. By (1), G/Q is supersolvable. By Lemma 2.7, G/Q contains two subgroups M_1/Q and M_2/Q with $|G : M_1| = p$ and $|G : M_2| = r$. By Lemmas 2.1 and 2.2, $(M_i, Q)(i = 1, 2)$ satisfy the condition of the theorem. By the choice of G, M_1 and M_2 are supersolvable. Now, by Lemma 2.8, G would be supersolvable, which is a contradiction. Thus, (2) holds.

(3) $\Phi(Q) = 1$.

Otherwise, by Lemmas 2.1 and 2.2, $G/\Phi(Q)$ satisfies the hypothesis, applying induction, we have $G/\Phi(Q)$ is supersolvable. Furthermore, $\Phi(Q) \leq \Phi(G)$ by III, 3.3 Hilfs-satz in [17], so $G/\Phi(G)$ is supersolvable. It follows that G is supersolvable, which is a contradiction.

(4) Q is a minimal normal subgroup of G.

Let N be a minimal normal subgroup of G contained in Q. Clearly the quotient group $(G/N, Q/N)$ satisfies the condition, so G/N is supersolvable. As the class of supersolvable groups is a formation, N must be the unique minimal normal subgroup of G which is contained in Q and $N \nsubseteq (G)$. So there is a maximal subgroup M of G such that $G = NM$ and $N \cap M = 1$. Thus $Q = N(Q \cap M)$. As $G = QM$ and Q is normal abelian in G, we know that $Q \cap M$ is normal in G. If $Q \cap M > 1$, let N_1 be a minimal normal subgroup of G such that $N_1 \leq Q \cap M$, then $N_1 \leq Q$ and $N \neq N_1$, this is a contradiction. Hence $Q \cap M = 1$, which implies $Q = N$.

(5) Every $Q_i \in \mathscr{M}_d(Q) = \{Q_1, ..., Q_d\}$ is S-quasinormally embedded in G.

Assume that there is a Q_i in $\mathscr{M}_d(Q)$ such that Q_i is c-normal in G. By definition, there is a normal subgroup K_i of G such that $G = Q_i K_i$ and $Q_i \cap K_i = Q_{iG}$ is a normal subgroup of G. By (4), $Q_i \cap K_i = 1$ or Q. If $Q_i \cap K_i = Q$, then $Q_i = Q$, a contradiction. If $Q_i \cap K_i = 1$, then $Q = Q_i(Q \cap Ki)$. But then $Q \cap K_i$ is a normal subgroup of order q of G. So $Q = Q \cap K_i$ by (4). As the class of supersolvable groups is a formation, thus G is supersolvable, contrary to the choice of G.

(6) $Q_i(i = 1, 2, ..., d)$ are normal subgroups of G.

Lemmas 2.5 and 2.9 imply that $Q_i(i = 1, 2, ..., d)$ are normal subgroups of G.

(7) The final contradiction.

Now,
$$(G/Q_i)/(Q/Q_i) \cong G/Q,$$

by (1), G/Q is supersolvable. As Q/Q_i is cyclic of order q, it follows that G/Q_i is supersolvable. Set
$$N = \bigcap_{i=1}^{d} Q_i.$$

By the definition of $\mathscr{M}_d(Q)$,

$$\bigcap_{i=1}^{d} Q_i = \Phi(Q),$$

so $N = \Phi(Q)$. Now, by the class of supersolvable groups is a formation, $G/\Phi(Q)$ is supersolvable. It follows that $G/\Phi(G)$ is supersolvable, and hence, G is supersolvable. which is a final contradiction. The proof is now completed.

The following corollaries are immediate from Theorem 3.4.

Corollary 3.5 ([7]). *Let G be a group. If every member of $\mathscr{M}(G)$ is c-normal, then G is supersolvable.*

Corollary 3.6 ([5]). *Let G be a group. If every subgroup in $\mathscr{M}(G)$ is s-quasinormally embedded in G, then G is supersolvable.*

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