A CHARACTERIZATION OF SOME PROJECTIVE SPECIAL LINEAR GROUPS

Ashraf Daneshkhah*
Department of Mathematics
Faculty of Science
Bu-Ali Sina University
Hamedan
Iran
adanesh@basu.ac.ir and daneshkhah.ashraf@gmail.com

Younes Jalilian
Department of Mathematics
Faculty of Science
Bu-Ali Sina University
Hamedan
Iran
y.jalilian91@basu.ac.ir

Abstract. In this paper, we show that projective special linear groups $S := L_3(q)$ with q less than 100 are uniquely determined by their orders and degree patterns of their prime graphs. Indeed, we prove that if G is a finite group whose order and degree pattern of its prime graph is the same as the order and the degree pattern of S, then G is isomorphic to S.
Keywords: Projective special linear groups; Prime graph; Degree pattern.

1. Introduction

For a positive integer n, the set of all primes dividing n is denoted by $\pi(n)$. Let G be a finite group. Set $\pi(G) := \pi(|G|)$, say, $\pi(G) = \{p_1, p_2, \ldots, p_k\}$. The *prime graph* $\Gamma(G)$ of G is a simple graph whose vertex set is $\pi(G)$ and two distinct primes p_i and p_j in $\pi(G)$ are adjacent if and only if there exists an element of order p_ip_j in G. If p_i is adjacent to p_j, we sometime write $p_i \sim p_j$. For $p_i \in \pi(G)$, the *degree* of p_i is the number $\deg_G(p_i) := |\{p_j \in \pi(G) \mid p_i \sim p_j\}|$. The *degree pattern* $\mathsf{D}(G)$ of G is the k-tuple $(\deg_G(p_1), \deg_G(p_2), \ldots, \deg_G(p_k))$ in which $p_1 < p_2 < \cdots < p_k$. A group G is said to be *OD-characterizable* if there exists exactly one isomorphic class of finite groups with the same order and degree pattern as G.

Darafsheh et al in [6] studied the quantitative structure of finite groups using their degree patterns and proved that if $|\pi((q^2 + q + 1)/d)| = 1$, where $d = (3, q - 1)$ and $q \geqslant 5$, then $L_3(q)$ is OD-characterizable. In [9], it is shown

∗. Corresponding author

that $L_3(25)$ is OD-characterizable. Also the authors in [8] proved that the simple group $L_3(2^n)$ with $n \in \{4, 5, 6, 7, 8, 10, 12\}$ is OD-characterizable. Note in passing that all finite simple groups whose orders are less than 10^8 are OD-characterizable, see [13]. In this paper, we show that $L_3(q)$, where q is a prime power less than 100, is uniquely determined by its order and the degree pattern of its prime graph, that is to say,

Theorem 1.1. *Let G be a finite group, and let q be a prime power less than 100. If $|G| = |L_3(q)|$ and $D(G) = D(L_3(q))$, then $G \cong L_3(q)$.*

In order to prove Theorem 1.1, by [6, 8, 9], we only need to show that $L_3(q)$ is OD-characterisable for $q \in \{11, 23, 29, 37, 47, 49, 53, 61, 67, 79, 81, 83\}$.

Throughout this article all groups are finite. The *spectrum* $\omega(G)$ of a group G is the set of orders of its elements, and $\mu(G)$ is the set of elements of $\omega(G)$ that are maximal with respect to divisibility relation. Let $\pi_i := \pi_i(G)$ be *connected components* of $\Gamma(G)$, for $i = 1, \ldots, t(G)$. When $|G|$ is even, we always assume that $2 \in \pi_1$. Let $|G| = m_1 \cdots m_{t(G)}$ with $\pi(m_i) = \pi_i$. Then each m_i is called an *order component* of G. If m_i is odd (even), then m_i is called an *odd (even) order component* of G. The *p-part* of a positive integer n is denoted by n_p, that is to say, $n_p = p^\alpha$ if $p^\alpha \mid n$ but $p^{\alpha+1} \nmid n$. All further definitions and notation are standard and can be found in [1, 2].

2. Preliminaries

In this section, we mention some useful results to be used in proof of Theorem 1.1.

Lemma 2.1 ([4, Theorem 1])**.** *Let G be a finite solvable group all of whose elements are of prime power order. Then $|\pi(G)| \leqslant 2$.*

Lemma 2.2. *Let G be a Frobenius group with kernel K and complement H. Then*

(*a*) *K is a nilpotent group.*

(*b*) *$|H|$ divides $|K| - 1$.*

(*c*) *Every subgroup of H of order pq with p and q primes (not necessarily distinct), is cyclic.*

(*d*) *Every Sylow subgroup of H of odd order is cyclic and a Sylow 2-subgroup of H is either cyclic, or a generalized quaternion group.*

(*e*) *If H is non-solvable group, then H has a subgroup of index at most 2 isomorphic to $SL(2, 5) \times M$, where M has cyclic Sylow p-subgroups and $(|M|, 30) = 1$.*

Proof. The proof follows from [2, Theorem 10.3.1] and [7, Theorem 18.6]. □

A group G is said to be a 2-*Frobenius* group if there exists a normal series $1 \trianglelefteq H \lhd K \trianglelefteq G$ such that K and G/H are Frobenius groups with kernel H and K/H, respectively.

Lemma 2.3 ([4, Theorem 2]). *Let G be 2-Frobenius group of even order. Then*

(a) *$t(G) = 2$, $\pi_1(G) = \pi(H) \cup \pi(G/K)$ and $\pi_2(G) = \pi(K/H)$.*

(b) *G/K and K/H are cyclic, $|G/K| \mid |\mathsf{Aut}(K/H)|$, and $(|G/K|, |K/H|) = 1$.*

(c) *H is nilpotent and G is solvable group.*

Lemma 2.4 ([11]). *Let G be a finite group with $t(G) \geqslant 2$ and $2 \in \pi_1$. Then G is one of the following groups:*

(a) *G is a Frobenius or 2-Frobenuis group.*

(b) *G has a normal series $1 \trianglelefteq H \lhd K \trianglelefteq G$ such that H is nilpotent π_1-group, G/K is a π_1-group and K/H is non-Abelian finite simple group such that $|G/H| \mid |\mathsf{Aut}(K/H)|$. Moreover, any odd order component of G is also an odd order component of K/H.*

Lemma 2.5 ([5, Lemma 2.8]). *Let S be a finite non-abelian simple group, and let p be the largest prime divisor of $|S|$ with $|S|_p = p$. Then $p \nmid |\mathsf{Out}(S)|$.*

An *independent set* $\Delta(\Gamma)$ of a graph Γ is a set of vertices of Γ no two of which are adjacent. The *independence number* $\alpha(\Gamma)$ of Γ is the maximum cardinality of an independent set among all independent sets of Γ. For convenience, if G is a group, we set $\Delta(G) := \Delta(\Gamma(G))$ and $\alpha(G) := \alpha(\Gamma(G))$. Moreover, for a vertex $r \in \pi(G)$, let $\alpha(r, G)$ denote the maximal number of vertices in independent sets of $\Gamma(G)$ containing r.

Lemma 2.6 ([10, Theorem 1]). *Let G be a finite group with $\alpha(G) \geqslant 3$ and $\alpha(2, G) \geqslant 2$, and let K be the maximal normal solvable subgroup of G. Then the quotient group G/K is an almost simple group, that is, there exists a non-Abelian finite simple group S such that $S \leqslant G/K \leqslant \mathsf{Aut}(S)$.*

Lemma 2.7. *Let G be a finite group of even order with $\alpha(G) \geqslant 3$. Then G is non-solvable, and so it is not a 2-Frobenius group. If, moreover, $|G|_3 \neq 3$ or $|G|_5 \neq 5$, then G is not a Frobenius group.*

Proof. Suppose that $\alpha(G) \geqslant 3$. If G were solvable, then it would have a Hall $\{p, q, r\}$-subgroup T with $\{p, q, r\}$ an independent subset of $\Gamma(G)$. Then p, q and r are not pairwise adjacent in $\Gamma(G)$, and so each element of T is of prime power order. Since T is solvable, it follows from Lemma 2.1 that $|\pi(T)| \leqslant 2$, which is a contradiction. Therefore, G is non-solvable, and so by Lemma 2.3, G is not a 2-Frobenius. Let now G be a Frobenius group with complement H and kernel K. Since G is non-solvable, it follows from Lemma 2.2 that H has a normal subgroup H_0 with $|H : H_0| \leqslant 2$ such that $H_0 = SL(2, 5) \times Z$, where $(|Z|, 30) = 1$. Then $|H| = 2^a \cdot 3 \cdot 5 \cdot |Z|$ with $a = 3, 4$. Therefore, $|G|_3 = 3$ and $|G|_5 = 5$. □

Lemma 2.8. *Let $\Gamma(G)$ be the prime graph of a group G with exactly two vertices of degree 1. Then $\alpha(G) \geqslant 3$ if one of the following holds:*

(*a*) $|\pi(G)| = 6$ *and $\Gamma(G)$ has at least two vertices of degree 2;*

(*b*) $|\pi(G)| \geqslant 7$ *and $\Gamma(G)$ has at least two vertices of degree 3.*

Proof. Suppose that p_1 and p_2 are two distinct vertices of $\Gamma(G)$ with $\deg(p_i) = 1$, for $i = 1, 2$.

Let first p_1 be adjacent to p_2. If $|\pi(G)| = 6$, there are four vertices which are not adjacent to p_1 and p_2. Since in this case there are at least two vertices of degree 2, there exist two non-adjacent vertices p_3 and p_4 in $\Gamma(G)$. Therefore, $\{p_1, p_3, p_4\}$ is an independent set of $\Gamma(G)$ which implies that $\alpha(G) \geqslant 3$. Similarly, in the case where $|\pi(G)| \geqslant 7$, there are at least five vertices which are not adjacent to p_1 and p_2, and since we have at least two vertices of degree 3, we can find two non-adjacent vertices p_3 and p_4 in $\Gamma(G)$, and hence $\{p_1, p_3, p_4\}$ is an independent set of $\Gamma(G)$, consequently, $\alpha(G) \geqslant 3$.

Let now p_1 and p_2 be non-adjacent. Since $|\pi(G)| \geqslant 6$ and both p_1 and p_2 are of degree 1, there exists a vertex p_3 which is not adjacent to p_1 and p_2. Thus $\{p_1, p_2, p_3\}$ is an independent set of $\Gamma(G)$, and hence $\alpha(G) \geqslant 3$. □

3. Proof of main result

In this section, we prove Theorem 1.1. For convenience, in Table 1, we list the order, spectrum and degree pattern of $S := L_3(q)$, where $q \in \{11, 23, 29, 37, 47, 49$ $53, 61, 67, 79, 81, 83\}$. In order to determine the degree pattern of S as in the first column of Table 1, we use $\mu(S)$ (see [3, Theorem 9]):

$$\mu(S) = \left\{ \frac{q^2+q+1}{(3,q-1)}, \frac{q^2-1}{(3,q-1)}, q-1, \frac{p(q-1)}{(3,q-1)} \right\}.$$

Note that if $(3, q-1) = 1$, then $\mu(S) = \{q^2+q+1, q^2-1, p(q-1)\}$. We also note that the order of S is

$$|L_3(q)| = \frac{1}{(3,q-1)} q^3(q^2-1)(q^3-1).$$

In what follows, we assume that G is a finite group with $|G| = |S|$ and $\mathsf{D}(G) = \mathsf{D}(S)$, see Table 1 below.

Proposition 3.1. *If $|G| = |L_3(11)|$ and $D(G) = D(L_3(11))$, then $G \cong L_3(11)$.*

Proof. By Table 1, we have that $|G| = 2^4 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11^3 \cdot 19$ and $\mathsf{D}(G) = (3, 2, 3, 1, 2, 1)$. Then Lemma 2.8 implies that $\alpha(G) \geqslant 3$. Furthermore, $\alpha(2, G) \geqslant 2$ as $\deg(2) = 3$ and $|\pi(G)| = 6$. By Lemma 2.6, there is a non-Abelian finite simple group S such that $S \leqslant G/K \leqslant \mathsf{Aut}(S)$, where K is a maximal normal solvable subgroup of G.

Table 1: The order, spectrum and degree pattern of $L_3(q)$, for $q \in \{11, 23, 29, 37, 47, 49, 53, 61, 64, 67, 79, 81, 83\}$.

S	$\lvert S\rvert$	$\mu(S)$	$\mathsf{D}(S)$
$L_3(11)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11^3 \cdot 19$	$\{7 \cdot 19, 2^3 \cdot 3 \cdot 5, 2 \cdot 5 \cdot 11\}$	$(3, 2, 3, 1, 2, 1)$
$L_3(23)$	$2^5 \cdot 3 \cdot 7 \cdot 11^2 \cdot 23^3 \cdot 79$	$\{7 \cdot 79, 2^4 \cdot 3 \cdot 11, 2 \cdot 11 \cdot 23\}$	$(3, 2, 1, 3, 2, 1)$
$L_3(29)$	$2^5 \cdot 3 \cdot 5 \cdot 7^2 \cdot 13 \cdot 29^3 \cdot 67$	$\{13 \cdot 67, 2^3 \cdot 3 \cdot 5 \cdot 7, 2^2 \cdot 7 \cdot 29\}$	$(4, 3, 3, 4, 1, 2, 1)$
$L_3(37)$	$2^5 \cdot 3^4 \cdot 7 \cdot 19 \cdot 37^3 \cdot 67$	$\{7 \cdot 67, 2^3 \cdot 3 \cdot 19, 2^2 \cdot 3^2, 2^2 \cdot 3 \cdot 37\}$	$(3, 3, 1, 2, 2, 1)$
$L_3(47)$	$2^6 \cdot 3 \cdot 23^2 \cdot 37 \cdot 47^3 \cdot 61$	$\{37 \cdot 61, 2^5 \cdot 3 \cdot 23, 2 \cdot 23 \cdot 47\}$	$(3, 2, 3, 1, 2, 1)$
$L_3(49)$	$2^9 \cdot 3^2 \cdot 5^2 \cdot 7^6 \cdot 19 \cdot 43$	$\{19 \cdot 43, 2^5 \cdot 5^2, 2^4 \cdot 3, 2^4 \cdot 7\}$	$(3, 1, 1, 1, 1, 1)$
$L_3(53)$	$2^5 \cdot 3^3 \cdot 7 \cdot 13^2 \cdot 53^3 \cdot 409$	$\{7 \cdot 409, 2^3 \cdot 3^3 \cdot 13, 2^2 \cdot 13 \cdot 53\}$	$(3, 2, 1, 3, 2, 1)$
$L_3(61)$	$2^5 \cdot 3^2 \cdot 5^2 \cdot 13 \cdot 31 \cdot 61^3 \cdot 97$	$\{13 \cdot 97, 2^3 \cdot 5 \cdot 31, 2^2 \cdot 3 \cdot 5, 2^2 \cdot 5 \cdot 61\}$	$(4, 2, 4, 1, 2, 2, 1)$
$L_3(67)$	$2^4 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 17 \cdot 31 \cdot 67^3$	$\{7^2 \cdot 31, 2^3 \cdot 11 \cdot 17, 2 \cdot 3 \cdot 11, 2 \cdot 11 \cdot 67\}$	$(4, 2, 1, 4, 2, 1, 2)$
$L_3(79)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13^2 \cdot 43 \cdot 79^3$	$\{7^2 \cdot 43, 2^5 \cdot 5 \cdot 13, 2 \cdot 3 \cdot 13, 2 \cdot 13 \cdot 79\}$	$(4, 2, 2, 1, 4, 1, 2)$
$L_3(81)$	$2^9 \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13 \cdot 41 \cdot 73$	$\{7 \cdot 13 \cdot 73, 2^5 \cdot 5 \cdot 41, 2^4 \cdot 3 \cdot 5\}$	$(3, 2, 3, 2, 2, 2, 2)$
$L_3(83)$	$2^4 \cdot 3 \cdot 7 \cdot 19 \cdot 41^2 \cdot 83^3 \cdot 367$	$\{19 \cdot 367, 2^3 \cdot 3 \cdot 7 \cdot 41, 2 \cdot 41 \cdot 83\}$	$(4, 3, 3, 1, 4, 2, 1)$

We show that $\pi(K) \subseteq \{2, 3, 5, 11\}$. Assume the contrary. Then $19 \in \pi(K)$. We show that p is adjacent to 19, where $(p, a) \in \{(5, 1), (5, 2), (7, 1)\}$. If $p \in \pi(K)$, then K contains an Abelian Hall subgroup of order $p^a \cdot 19$, and so p is adjacent to 19. If $p \notin \pi(K)$, then by Frattini argument $G = K\mathbf{N}_G(P)$, where P is a Sylow 19-subgroup of K. Thus $\mathbf{N}_G(P)$ contains an element of order p, say x. So $P\langle x\rangle$ is a cyclic subgroup of G of order $p \cdot 19$ concluding that p is adjacent to 19. Therefore, both 5 and 7 are adjacent to 19, and hence 19 is of degree at least 2, which is a contradiction. Similarly, we can show that $7 \notin \pi(K)$. Hence $\pi(K) \subseteq \{2, 3, 5, 11\}$.

We now prove that S is isomorphic to $L_3(11)$. Note by Lemma 2.5 that $19 \notin \pi(\mathsf{Out}(S))$. Then $19 \notin \pi(K) \cup \pi(\mathsf{Out}(S))$, and so $19 \in \pi(S)$. Now by [12, Table 1], S is isomorphic to one of the simple groups J_1 and $L_3(11)$.

If S were isomorphic to J_1, then $|S|$ would be $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$, and since $\mathsf{Out}(S) = 1$, we must have $|K| = 2 \cdot 5 \cdot 11^2$. Let $P \in \mathsf{Syl}_{11}(K)$, and let $r \in \{7, 19\}$. By Frattini argument, $G = K\mathbf{N}_G(P)$, and so $\mathbf{N}_G(P)$ contains an element of order r, say x. Since P is normal in K and $P \cap \langle x\rangle = 1$, $L := P\langle x\rangle$ is a subgroup of K of order $r \cdot 11^2$. Since also L is Abelian, it has an element of order $r \cdot 11$. This shows that both 7 and 19 are adjacent to 11. Note that the degree of 11 is two, and 7 and 19 are of degree one. Thus 2 can not be adjacent to none of 7, 11 and 19. Since $|\pi(G)| = 6$, the degree of 2 is at most 2, which is a contradiction.

Therefore, S is isomorphic to $L_3(11)$, and hence $L_3(11) \leqslant G/K \leqslant \mathsf{Aut}(L_3(11))$. Note that $|G| = |L_3(11)|$. Thus $K = 1$, and hence G is isomorphic to $L_3(11)$. □

Proposition 3.2. *If* $|G| = |L_3(23)|$ *and* $D(G) = D(L_3(23))$, *then* $G \cong L_3(23)$.

Proof. According to Table 1, we have that $|G| = 2^5 \cdot 3 \cdot 7 \cdot 11^2 \cdot 23^3 \cdot 79$ and $\mathsf{D}(G) = (3, 2, 1, 3, 2, 1)$. Then by Lemma 2.8, we conclude that $\alpha(G) \geqslant 3$. Since $\deg(2) = 3$ and $|\pi(G)| = 6$, $\alpha(2, G) \geqslant 2$. Therefore, by Lemma 2.6, there is a non-Abelian finite simple group S such that $S \leqslant G/K \leqslant \mathsf{Aut}(S)$, where K is a maximal normal solvable subgroup of G.

We claim that $79 \notin \pi(K)$. Assume the contrary. We show that p is adjacent to 79, where $(p, a) \in \{(7, 1), (11, 1), (11, 2)\}$. If $p \in \pi(K)$, then K contains an abelain Hall subgroup of order $p^a \cdot 79$ which implies that p is adjacent to 79. If $p \notin \pi(K)$, then by Frattini argument $G = K\mathbf{N}_G(P)$, where P is a Sylow 79-subgroup of K, and so $\mathbf{N}_G(P)$ contains an element x of order p. Note that $P\langle x\rangle$ is a cyclic subgroup of G of order $p \cdot 79$. Then p is adjacent to 79. Hence, both 7 and 11 are adjacent to 79, and consequently, degree of 79 is at least 2, which is a contradiction. Similarly, we can show that $7 \notin \pi(K)$. Therefore $\pi(K) \subseteq \{2, 3, 7, 11, 23\}$, by Lemma 2.5, we have that $79 \notin \pi(\mathsf{Out}(S))$. Then $79 \notin \pi(K) \cup \pi(\mathsf{Out}(S))$, and so $79 \in \pi(S)$. Therefore by [12, Table 1], S is isomorphic to $L_3(23)$. Since $|G| = |L_3(23)|$, we must have $K = 1$, and hence G is isomorphic to $L_3(23)$. □

Proposition 3.3. *If* $|G| = |L_3(29)|$ *and* $\mathsf{D}(G) = \mathsf{D}(L_3(29))$*, then* $G \cong L_3(29)$.

Proof. It follows from Table 1 that $|G| = 2^5 \cdot 3 \cdot 5 \cdot 7^2 \cdot 13 \cdot 29^3 \cdot 67$ and $\mathsf{D}(G) = (4, 3, 3, 4, 1, 2, 1)$. Then Lemma 2.8 implies that $\alpha(G) \geqslant 3$. Furthermore, $\alpha(2, G) \geqslant 2$ as $\deg(2) = 4$ and $|\pi(G)| = 7$. Therefore, Lemma 2.6 implies that there is a non-Abelian finite simple group S such that $S \leqslant G/K \leqslant \mathsf{Aut}(S)$, where K is a maximal normal solvable subgroup of G.

We show that $67 \notin \pi(K)$. Assume the contrary. Then K has element of order 67. We show that p is adjacent to 67 for all $p \in \{5, 13\}$. If $p \in \pi(K)$, then we consider a cyclic Hall subgroup of order $p \cdot 67$ of K, and so p and 73 are adjacent. If $p \notin \pi(K)$, then we apply Frattini argument and have that $G = K\mathbf{N}_G(P)$, where P is a Sylow 67-subgroup of K. Thus $\mathbf{N}_G(P)$ contains an element x of order p. Now $P\langle x\rangle$ is a cyclic subgroup of G of order $p \cdot 67$ which again implies that p and 67 are adjacent. Therefore, both 5 and 13 are adjacent to 67, and hence the degree of 67 must be at least 2, which is a contradiction. Then $\pi(K) \subseteq \{2, 3, 5, 7, 13, 29\}$. By Lemma 2.5, $67 \notin \pi(\mathsf{Out}(S))$, then $67 \notin \pi(K) \cup \pi(\mathsf{Out}(S))$, and so $67 \in \pi(S)$. Using [12, Table 1] we observe that S is isomorphic to $L_3(29)$. Since $|G| = |L_3(29)|$, we conclude that $K = 1$, and hence G is isomorphic to $L_3(29)$. □

Proposition 3.4. *If* $|G| = |L_3(37)|$ *and* $\mathsf{D}(G) = \mathsf{D}(L_3(37))$*, then* $G \cong L_3(37)$.

Proof. Note by Table 1 that $|G| = 2^5 \cdot 3^4 \cdot 7 \cdot 19 \cdot 37^3 \cdot 67$ and $\mathsf{D}(G) = (3, 3, 1, 2, 2, 1)$. Then by Lemma 2.8, we must have $\alpha(G) \geqslant 3$. Furthermore, $\alpha(2, G) \geqslant 2$ since $\deg(2) = 3$ and $|\pi(G)| = 6$. By Lemma 2.6, there is a non-Abelian finite simple group S such that $S \leqslant G/K \leqslant \mathsf{Aut}(S)$, where K is a maximal normal solvable subgroup of G. We show that $67 \notin \pi(K)$. Assume the contrary. $67 \in \pi(K)$. We prove that p would be adjacent to 67, where $p \in \{7, 19\}$. If $p \in \pi(K)$, then K contains a cyclic Hall subgroup of order $p \cdot 67$, and so p is adjacent to 67. If $p \notin \pi(K)$, then it follows from Frattini argument that $G = K\mathbf{N}_G(P)$, where P is a Sylow 67-subgroup of K, and so $\mathbf{N}_G(P)$ has an element x of order p. Thus $P\langle x\rangle$ is a cyclic subgroup of G of order $p \cdot 67$. Therefore both 7 and 19 are adjacent to 67 which contradicts the fact that the degree of 67 is 1. Therefore, $67 \notin \pi(K)$, and hence $\pi(K) \subseteq \{2, 3, 7, 19, 37\}$. Now we prove that $S \cong L_3(37)$.

By Lemma 2.5, $67 \notin \pi(\mathsf{Out}(S))$, then $67 \notin \pi(K) \cup \pi(\mathsf{Out}(S))$, and so $67 \in \pi(S)$. Therefore by [12, Table 1], S is isomorphic to $L_3(37)$ as claimed. Since now $|G| = |L_3(37)|$, we must have $K = 1$, and hence G is isomorphic to $L_3(37)$. □

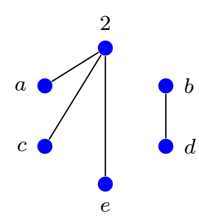
Proposition 3.5. *If* $|G| = |L_3(47)|$ *and* $\mathsf{D}(G) = \mathsf{D}(L_3(47))$, *then* $G \cong L_3(47)$.

Proof. By Table 1, we have that $|G| = 2^6 \cdot 3 \cdot 23^2 \cdot 37 \cdot 47^3 \cdot 61$ and $\mathsf{D}(G) = (3, 2, 3, 1, 2, 1)$. It follows from Lemma 2.8 that $\alpha(G) \geqslant 3$. Note that $\deg(2) = 3$ and $|\pi(G)| = 6$. Then $\alpha(2, G) \geqslant 2$, and so by Lemma 2.6, there is a non-Abelian finite simple group S such that $S \leqslant G/K \leqslant \mathsf{Aut}(S)$, where K is a maximal normal solvable subgroup of G. We claim that $61 \notin \pi(K)$. Assume the contrary. Then $19 \in \pi(K)$. We show that p is adjacent to 61, where $(p, a) \in \{(23, 1), (23, 2), (37, 1)\}$. If $p \in \pi(K)$, then K contains an abelain Hall subgroup of order $p^a \cdot 61$, and so p is adjacent to 61. If $p \notin \pi(K)$, then by Frattini argument $G = K\mathbf{N}_G(P)$, where P is a Sylow 61-subgroup of K, and so $\mathbf{N}_G(P)$ contains an element x of order p. Note that $P\langle x\rangle$ is a cyclic subgroup of G of order $p \cdot 61$. Then p is adjacent to 61. Therefore, 61 is adjacent to both 23 and 37 in $\Gamma(G)$, and hence the degree of 61 is at least 2, which is a contradiction. Hence $61 \notin \pi(K)$. Therefore, $\pi(K) \subseteq \{2, 3, 23, 37, 47\}$, and hence $61 \notin \pi(\mathsf{Out}(S))$ by Lemma 2.5. Then $61 \notin \pi(K) \cup \pi(\mathsf{Out}(S))$, and so $61 \in \pi(S)$. Now by [12, Table 1], S is isomorphic to $L_3(47)$, and so $L_3(47) \leqslant G/K \leqslant \mathsf{Aut}(L_3(47))$. Note that $|G| = |L_3(47)|$. Then $K = 1$, and hence G is isomorphic to $L_3(47)$. □

Proposition 3.6. *If* $|G| = |L_3(49)|$ *and* $\mathsf{D}(G) = \mathsf{D}(L_3(49))$, *then* $G \cong L_3(49)$.

Proof. According to Table 1, $|G| = 2^9 \cdot 3^2 \cdot 5^2 \cdot 7^6 \cdot 19 \cdot 43$ and $\mathsf{D}(G) = (3, 1, 1, 1, 1, 1)$. Then we observe that $\Gamma(G)$ is the graph as in Figure 1 in which $\{a, b, c, d, e\} = \{3, 5, 7, 19, 43\}$. We also observe that $t(G) = 2$ and $\{a, b, e\}$ is an

Figure 1: Possibilities for the prime graph of G in Proposition 3.6.



independent set. Thus $\alpha(G) \geqslant 3$. It is also easily seen that $\alpha(2, G) \geqslant 2$. Then by Lemma 2.7, G is neither Frobenius, nor 2-Frobenius, and so Lemma 2.4 implies that G has a normal series $1 \trianglelefteq H \triangleleft K \trianglelefteq G$ such that K/H is a non-Abelian finite simple group. Since $|K/H|$ divides $|K|$, it divides $|G|$, and so by [12, Table 1], the factor group K/H is isomorphic to one of the simple groups S as in the first column of Table 2 below.

Table 2: Non-Abelian finite simple groups S whose order divides $|L_3(49)|$

S	$\|S\|$	$\|\mathsf{Out}(S)\|$	Primes in $\pi(H)$
$L_2(4)$	$2^2 \cdot 3 \cdot 5$	2	7, 19, 43
$L_2(9)$	$2^3 \cdot 3^2 \cdot 5$	4	7, 19, 43
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	7, 19, 43
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	7, 19, 43
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	7, 19, 43
$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	4	7, 19, 43
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	12	7, 19, 43
$L_4(2)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	7, 19, 43
$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2	7, 19, 43
$L_2(19)$	$2^2 \cdot 3^2 \cdot 5 \cdot 19$	2	2, 7, 43
$L_3(7)$	$2^5 \cdot 3^2 \cdot 7^3 \cdot 19$	6	2, 7, 43
$U_3(7)$	$2^7 \cdot 3 \cdot 7^3 \cdot 43$	2	2, 5, 7
$L_2(7^3)$	$2^3 \cdot 3^2 \cdot 7^3 \cdot 19 \cdot 43$	6	2, 5, 7
$L_3(49)$	$2^9 \cdot 3^2 \cdot 5^2 \cdot 7^6 \cdot 19 \cdot 43$	12	-

If K/H is isomorphic to one of the groups S listed in the first column of Table 2 except $L_3(49)$, then $\pi(H)$ consists of three primes as in the forth column of the same table. Since H is nilpotent, it follows that the prime graph of G has a triangle, which is a contradiction. Therefore, $K/H \cong L_3(49)$. As $L_3(49) \leqslant G/H \leqslant \mathsf{Aut}(L_3(49))$ and $|G| = |L_3(49)|$, we conclude that $|H| = 1$, and hence G is isomorphic to $L_3(49)$. □

Proposition 3.7. *If* $|G| = |L_3(53)|$ *and* $D(G) = D(L_3(53))$, *then* $G \cong L_3(53)$.

Proof. By Table 1, $|G| = 2^5 \cdot 3^3 \cdot 7 \cdot 13^2 \cdot 53^3 \cdot 409$ and $\mathsf{D}(G) = (3, 2, 1, 3, 2, 1)$. Now by applying Lemma 2.8, we must have $\alpha(G) \geqslant 3$. Furthermore, $\alpha(2, G) \geqslant 2$ since $\deg(2) = 3$ and $|\pi(G)| = 6$. So by Lemma 2.6, there is a non-Abelian finite simple group S such that $S \leqslant G/K \leqslant \mathsf{Aut}(S)$, where K is a maximal normal solvable subgroup of G. We claim that $\pi(K)$ does not contain 409. Assume the contrary. Then $409 \in \pi(K)$. We show that p is adjacent to 409, where $(p, a) \in \{(7, 1), (13, 1), (13, 2)\}$. If $p \in \pi(K)$, then K contains an abelain Hall subgroup of order $p^a \cdot 409$, so p and 409 are adjacent. If $p \notin \pi(K)$, then we apply Frattini argument and have that $G = K\mathbf{N}_G(P)$, where P is a Sylow 409-subgroup of K, and so $\mathbf{N}_G(P)$ has an element x of order p. Now $P\langle x\rangle$ is a cyclic subgroup of G of order $p \cdot 409$ concluding that p and 409 are adjacent. Thus both 7 and 13 are adjacent to 409 in $\Gamma(G)$, and hence the degree of 409 is at least 2, which is a contradiction. Therefore, $409 \notin \pi(K)$, and hence it follows from Lemma 2.5 that $409 \notin \pi(\mathsf{Out}(S))$. Then $409 \notin \pi(K) \cup \pi(\mathsf{Out}(S))$, and so $409 \in \pi(S)$. Therefore by [12, Table 1], S is isomorphic to $L_3(53)$ and $L_3(53) \leqslant G/K \leqslant \mathsf{Aut}(L_3(53))$. Moreover, since $|G| = |L_3(53)|$, it follows that $K = 1$, and hence $G \cong L_3(53)$. □

Proposition 3.8. *If* $|G| = |L_3(61)|$ *and* $D(G) = D(L_3(61))$, *then* $G \cong L_3(61)$.

Table 3: Non-Abelian finite simple groups S whose order divides $|L_3(67)|$

S	$\|S\|$	$\|\mathsf{Out}(S)\|$
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3
$L_2(17)$	$2^4 \cdot 3^2 \cdot 17$	2
$L_2(67)$	$2^2 \cdot 3 \cdot 11 \cdot 17 \cdot 67$	2
$L_3(67)$	$2^4 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 17 \cdot 31 \cdot 67^3$	6

Proof. It follows from Table 1 that $|G| = 2^5 \cdot 3^2 \cdot 5^2 \cdot 13 \cdot 31 \cdot 61^3 \cdot 97$ and $\mathsf{D}(G) = (4, 2, 4, 1, 2, 2, 1)$. Then by Lemma 2.8, $\alpha(G) \geqslant 3$. Moreover, $\alpha(2, G) \geqslant 2$ as $\deg(2) = 4$ and $|\pi(G)| = 7$. By Lemma 2.6, there is a non-Abelian finite simple group S such that $S \leqslant G/K \leqslant \mathsf{Aut}(S)$, where K is a maximal normal solvable subgroup of G. We show that $97 \notin \pi(K)$. Assume the contrary. Then $97 \in \pi(K)$. Let $p \in \{13, 31\}$. If $p \in \pi(K)$, then K contains a cyclic Hall subgroup of order $p\cdot 97$, and so p is adjacent to 97. If $p \notin \pi(K)$, then by Frattini argument $G = K\mathbf{N}_G(P)$, where P is a Sylow 97-subgroup of K, and so $\mathbf{N}_G(P)$ has an element x of order p. Now $P\langle x\rangle$ is a cyclic subgroup of G of order $p \cdot 97$ which implies that p is adjacent to 97. Therefore, 97 is adjacent to 13 and 31 which is a contraiction as 97 is of degree 1. Thus $97 \notin \pi(K)$. Now by Lemma 2.5, $97 \notin \pi(\mathsf{Out}(S))$, then $97 \notin \pi(K) \cup \pi(\mathsf{Out}(S))$, and so $97 \in \pi(S)$. Therefore by [12, Table 1], S is isomorphic to $L_3(61)$ and $L_3(61) \leqslant G/K \leqslant \mathsf{Aut}(L_3(61))$. Moreover, since $|G| = |L_3(61)|$, we have that $K = 1$, and hence G is isomorphic to $L_3(61)$. □

Proposition 3.9. *If $|G| = |L_3(67)|$ and $D(G) = D(L_3(67))$, then $G \cong L_3(67)$.*

Proof. By Table 1, we have $|G| = 2^4 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 17 \cdot 31 \cdot 67^3$ and $\mathsf{D}(G) = (4, 2, 1, 4, 2, 1, 2)$. It follows from Lemma 2.8 that $\alpha(G) \geqslant 3$. Note that $\deg(2) = 4$ and $|\pi(G)| = 7$. Then $\alpha(2, G) \geqslant 2$. It follows from Lemma 2.6 that there is a non-Abelian finite simple group S such that $S \leqslant G/K \leqslant \mathsf{Aut}(S)$, where K is a maximal normal solvable subgroup of G. Since $|S|$ divides $|G/K|$, so does $|G|$, and so by [12, Table 1], S isomorphic to one of groups in the first column of Table 3.

If S is isomorphic to one of the groups $L_2(7)$, $L_2(8)$, $L_2(17)$ and $L_2(67)$, then $7, 11, 31 \in \pi(K)$. As K is solvable, we can consider a Hall $\{7, 31\}$-subgroup $K_1 := P_7P_{31}$ and a Hall $\{11, 31\}$-subgroup $K_2 := P_{11}P_{31}$. Then $|K_1| = 7^2{\cdot}31$ and $|K_2| = 11^2 \cdot 31$, and consequently K_i, for $i = 1, 2$, is Abelian which implies that 31 is adjacent to both 7 and 11, and so $\deg(37) \geqslant 2$, which is a contradiction. Thus $S \cong L_3(67)$. Since $L_3(67) \leqslant G/K \leqslant \mathsf{Aut}(L_3(67))$ and $|G| = |L_3(67)|$, we conclude that $|K| = 1$, and hence G isomorphic to $L_3(67)$. □

Proposition 3.10. *If $|G| = |L_3(79)|$ and $D(G) = D(L_3(79))$, then $G \cong L_3(79)$.*

Table 4: Non-Abelian finite simple groups S whose order divides $|L_3(79)|$.

S	$\|S\|$	$\|\mathsf{Out}(S)\|$
$L_2(4)$	$2^2 \cdot 3 \cdot 5$	2
$L_2(9)$	$2^3 \cdot 3^2 \cdot 5$	4
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	12
$L_4(2)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2
$L_2(13)$	$2^2 \cdot 3 \cdot 7 \cdot 13$	2
$Sz(8)$	$2^6 \cdot 5 \cdot 7 \cdot 13$	3
$L_2(64)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	6
$L_2(79)$	$2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 79$	2
$L_3(79)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13^2 \cdot 43 \cdot 79^3$	6

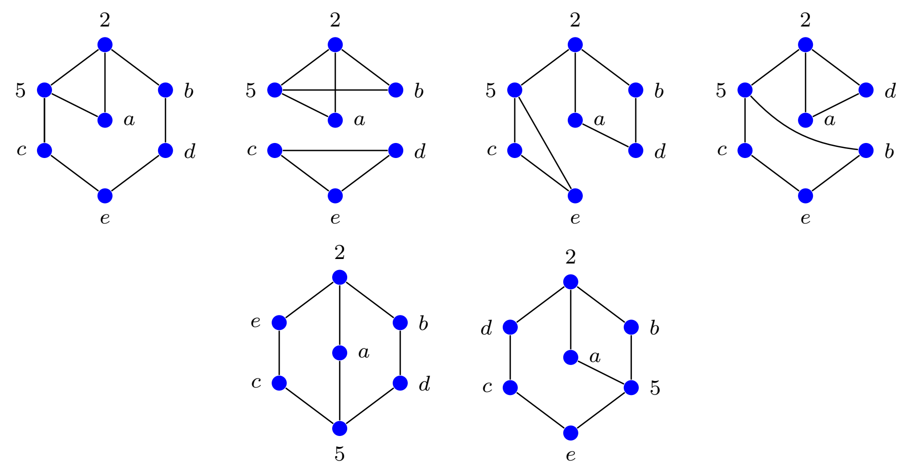
Proof. By Table 1, $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 13^2 \cdot 43 \cdot 79^3$ and $\mathsf{D}(G) = (4, 2, 2, 1, 4, 1, 2)$. Then by applying Lemma 2.8, we conclude that $\alpha(G) \geqslant 3$. Furthermore, $\alpha(2, G) \geqslant 2$ since $\deg(2) = 4$ and $|\pi(G)| = 7$. By Lemma 2.6, there is a non-Abelian finite simple group S such that $S \leqslant G/K \leqslant \mathsf{Aut}(S)$, where K is a maximal normal solvable subgroup of G. We show that $\pi(K)$ does not contain 43. Assume the contrary. Then $43 \in \pi(K)$. Let $(p, a) \in \{(5, 1), (13, 1), (13, 2)\}$. If $p \in \pi(K)$, then K contains an abelian Hall subgroup of order $p \cdot 43$, and so p and 43 are adjacent. If $p \notin \pi(K)$, then by Frattini argument $G = K\mathbf{N}_G(P)$, where P is a Sylow 43-subgroup of K. This shows that $\mathbf{N}_G(P)$ contains an element x of order p, and so $P\langle x\rangle$ is a cyclic subgroup of G of order $p \cdot 43$ concluding that p is adjacent to 43. Therefore, 5 and 13 are adjacent to 43, and so 43 has degree at least 2, which is a contradiction. Therefore, $43 \notin \pi(K)$. We prove that $S \cong L_3(79)$. Since $|S|$ divides G/K, so does $|G|$, and so by [12, Table 1], S isomorphic to one of groups in Table 4 below.

If S is isomorphic to a simple group listed in the first column of Table 4 except $L_3(79)$, then $43 \in K$, which is a contradiction. Therefore, $S \cong L_3(79)$ and $L_3(79) \leqslant G/K \leqslant \mathsf{Aut}(L_3(79))$. Moreover, since $|G| = |L_3(79)|$, it follows that $|K| = 1$, and hence $G \cong L_3(79)$. □

Proposition 3.11. *If* $|G| = |L_3(81)|$ *and* $D(G) = D(L_3(81))$, *then* $G \cong L_3(81)$.

Proof. According to Table 1, $|G| = 2^9 \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13 \cdot 41 \cdot 73$ and $\mathsf{D}(G) = (3, 2, 3, 2, 2, 2, 2)$. Then the only possible graphs for $\Gamma(G)$ are as in Figure 2. In each case, we observe that $\Delta = \{a, b, c\}$ forms an independent set of $\Gamma(G)$, and so $\alpha(G) \geqslant 3$. Note that $\deg(2) = 3$ and $|\pi(G)| = 7$. Then $\alpha(2, G) \geqslant 2$, and so by Lemma 2.6, there is a non-Abelian finite simple group S such that $S \leqslant G/K \leqslant \mathsf{Aut}(S)$, where K is a maximal normal solvable subgroup of G. We show that $73 \notin \pi(K)$. Assume the contrary. Then $73 \in \pi(K)$. Suppose

Figure 2: Possibilities for the prime graph of G in Proposition 3.11.



$p \in \{7, 13, 41\}$. If $p \in \pi(K)$, then K has a cyclic Hall subgroup of order $p \cdot 73$, and so p is adjacent to 73, for all $p \in \{7, 13, 41\}$. If $p \notin \pi(K)$, then by Frattini argument $G = K\mathbf{N}_G(P)$, where P is a Sylow 73-subgroup of K. Hence $\mathbf{N}_G(P)$ contains an element x of order p. Now $P\langle x\rangle$ is a cyclic subgroup of order $p \cdot 73$. This implies that p and 73 are adjacent. Therefore, 73 is adjacent to p, for all $p \in \{7, 13, 41\}$, which is a contradiction as the degree of 73 is 2. Therefore, $73 \notin \pi(K)$. We now prove that $S \cong L_3(81)$. By Lemma 2.5, we must have $73 \notin \pi(\mathsf{Out}(S))$, and so $73 \notin \pi(K) \cup \pi(\mathsf{Out}(S))$. This implies that $73 \in \pi(S)$. Now by [12, Table 1], S is isomorphic to one of the groups in Table 5 below. If S is isomorphic to one of the groups $U_3(9)$, $L_2(3^6)$ and $G_2(9)$, then $41 \in \pi(K)$, which is a contradiction. Therefore $S \cong L_3(81)$, and since $L_3(81) \leqslant G/K \leqslant \mathsf{Aut}(L_3(81))$ and $|G| = |L_3(81)|$, it follows that $|K| = 1$, and hence G is isomorphic to $L_3(81)$. □

Table 5: Non-Abelian finite simple groups S whose order divides $|L_3(81)|$.

S	$\|S\|$	$\|\mathsf{Out}(S)\|$
$U_3(9)$	$2^5 \cdot 3^6 \cdot 5^2 \cdot 73$	2
$L_2(3^6)$	$2^3 \cdot 3^6 \cdot 5 \cdot 7 \cdot 13 \cdot 73$	9
$G_2(9)$	$2^8 \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13 \cdot 73$	6
$L_3(81)$	$2^9 \cdot 3^{12} \cdot 5^2 \cdot 7 \cdot 13 \cdot 41 \cdot 73$	8

Proposition 3.12. *If* $|G| = |L_3(83)|$ *and* $\mathsf{D}(G) = \mathsf{D}(L_3(83))$, *then* $G \cong L_3(83)$.

Proof. According to Table 1, $|G| = 2^4 \cdot 3 \cdot 7 \cdot 19 \cdot 41^2 \cdot 83^3 \cdot 367$ and $\mathsf{D}(G) = (4, 3, 3, 1, 4, 2, 1)$. By Lemma 2.8, we have that $\alpha(G) \geqslant 3$. Furthermore, $\alpha(2, G) \geqslant 2$ as $\deg(2) = 4$ and $|\pi(G)| = 7$. By Lemma 2.6, there is a non-Abelian finite simple group S such that $S \leqslant G/K \leqslant \mathsf{Aut}(S)$, where K is a maximal normal solvable subgroup of G. We show that $\pi(K)$ does not contain 367. Assume the contrary. Then $367 \in \pi(K)$. Suppose $p \in \{7, 19\}$. If $p \in \pi(K)$, then K contains a cyclic Hall subgroup of order $p \cdot 367$, and so p is adjacent to 367. If $p \notin \pi(K)$, then by Frattini argument $G = K\mathbf{N}_G(P)$, where P is a Sylow 367-subgroup of K, and so $\mathbf{N}_G(P)$ has an element x of order p. Note that $P\langle x\rangle$ is a cyclic subgroup of G of order $p \cdot 367$. Then p is adjacent to 367. Therefore, both 7 and 19 and 367 are adjacent in $\Gamma(G)$, which is a contradiction. We prove that $S \cong L_3(83)$. By Lemma 2.5, $367 \notin \pi(\mathsf{Out}(S))$, then $367 \notin \pi(K) \cup \pi(\mathsf{Out}(S))$, and so $367 \in \pi(S)$. Therefore by [12, Table 1], S is isomorphic to $L_3(83)$, and so $L_3(83) \leqslant G/K \leqslant \mathsf{Aut}(L_3(83))$. Moreover, since $|G| = |L_3(83)|$, it follows that $|K| = 1$, and hence G is isomorphic to $L_3(83)$. □

Proof of Theorem 1.1. The proof of Theorem 1.1 follows immediately from [6, 8, 9] and Propositions 3.1-3.12.

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