n-EDGE-DISTANCE-BALANCED GRAPHS

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Abstract. Throughout this paper, we present a new class of graphs so-called n-edge-distance-balanced graphs inspired by the concept of edge-distance-balanced property initially introduced by Tavakoli et al. [Tavakoli M., Yousefi-Azari H., Ashrafi A.R., Note on edge distance-balanced graphs, Trans. Combin. 1 (1) (2012), 1-6]. Moreover, we propose some characteristic results to recognize 2-edge-distance-balanced graphs by using the lack of 2-connectivity in graphs. Some examples are provided in order to illustrate the obtained conclusions.

Keywords: graph theory, edge-distance-balanced, regularity.

1. Introduction and preliminaries

It is well known that the graph theory is a crucial tool to utilize the modeling of the phenomena and is extensively used in a series of investigations for a few decades. In order to use better and more beneficial from the graph theory, all graphs are usually classified based on their distinguishing quality. Recently, as a new tool using in optimization a class of graphs so-called distance-balanced graphs has been introduced by Jerebic et al. [5] and then studied in some recent papers, see ([1],[2],[4],[6]-[10]) and the reference therein.

Let G be a finite, undirected and connected graph with diameter d, and let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For $u, v \in V(G)$, we let $d(u, v) = d_G(u, v)$ denote the minimal path-length distance between u and v. For a pair of adjacent vertices u, v of G we denote

$$W_{uv}^{G} = \{x \in V(G) | d(x, u) < d(x, v)\}.$$

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Similarly, we can define W^G_{vu}. Also, consider the notion

$$ {}_uW^G_v = \{x \in V(G) | d(x,u) = d(x,v)\}. $$

which all the sets as above make a partition of $V(G)$ and moreover we have the following definition.

Definition 1.1 ([5]). *We say that G is distance-balanced whenever for an arbitrary pair of adjacent vertices u and v of G there exists a positive integer γ_{uv}, such that*

$$ |W^G_{uv}| = |W^G_{vu}| = \gamma_{uv}. $$

Very recently, the authors [3] generalized this concept and introduced n-distance-balanced graphs (denoted by n-DB) and characterized n-DB graphs for $n = 2, 3$. Furthermore, they studied the invariance of this property on the product of graphs.

In 2012, Tavakoli et al. [11] presented a new type of distance-balanced propery in the edge sense as follows.

Definition 1.2 ([11]). *Let G be a graph, $e = uv \in E(G)$, $m^G_u(e)$ denotes the number of edges lying closer to the vertex u than the vertex v, and $m^G_v(e)$ is defned analogously. Then G is called edge-distance-balanced (as we denote it by EDB), if $m^G_a(e) = m^G_b(e)$ holds for each edge $e = ab \in E(G)$.*

The purpose of this paper is to introduce a new class of graphs based on the distance-balanced property in edge sense. The structure of the paper is as follows. In the next section we define the notion of n-edge-distance-balanced graph and then present some results related to either regularity or n-locally regularity. We also create a series of n-locally regular graphs with diameter n, n-edge-distance-balanced property and without i-edge-distance-balanced property for $i = 1, 2, ..., n-1$. In Section 3, using the concept of 2-connectivity of graph we propose some results to characterize 2-edge-distance-balanced graphs by presenting some necessary and sufficient conditions.

2. EDB and n-EDB graphs and locally regularity

In this section, inspired by the concept of edge-distance-balanced graph, we initially define and introduce a class of graphs including these graphs which are called n-edge-distance-balanced graphs and then we conclude some results and examples related to this concept.

Definition 2.1. *A connected graph G is called n-edge-distance-balanced (n-EDB for short) if for each $a, b \in V(G)$ with $d(a,b) = n$ we have $|E^G_{a\underline{n}b}| = |E^G_{b\underline{n}a}|$ where*

$$ E^G_{a\underline{n}b} = \{e \in E(G) \mid e \text{ is closer to the vertex } a \text{ than the vertex } b\}. $$

Similarly, we can define $E^G_{b\underline{n}a}$. *Also, consider the notion*

$$_{a}\underset{\underline{n}}{E}{}^G_b = \{e \in E(G) \mid \textit{the distance of } e \textit{ to both vertices } a \textit{ and } b \textit{ is the same}\}.$$

Similar to the recent proposed concepts (DB, EDB and n-DB) one can observe that all three sets as above form a partition for $E(G)$. Moreover, 1-edge-distance-balanced property coincides to edge-distance-balanced property and so the collection of all n-EDB graphs are larger than the set of all EDB graphs. In the following, denoted by E^G_{ab} we mean $E^G_{a\underline{1}b}$. Otherwise specified, we consider all the notations without superscript G.

As the first result, in the following we present a characterization for the graphs with EDB and n-EDB properties inspired by the virtue of proof of similar result in [5]. We notice that the proof would be slightly different. For convenience of notation, we let for a vertex x of a connected graph G and $k \geq 0$, that

$$\begin{aligned} N_k(x) &= \{y \in V(G) \mid d(x,y) = k\}, \quad & N_k[x] &= \{y \in V(G) \mid d(x,y) \leq k\}, \\ M_k(x) &= \{e \in E(G) \mid d(x,e) = k\}, \quad & M_k[x] &= \{e \in E(G) \mid d(x,e) \leq k\}. \end{aligned}$$

and for $k = 1$, these symbols are used while the indices are removed. Here, denoted by $d(x,e)$ we mean the length of the shortest path between the vertex x and the edge e, i.e., the number of edges lying between the vertex x and the edge e in the shortest path. Following the notations as above we see that

$$|M_k(x)| \leq |N_k(x)| \leq |M_{k-1}(x)|, \quad \forall x \in V(G),$$

and also replacing $[x]$ by (x) we have the same conclusion.

Proposition 2.2. *A graph G of diameter d is EDB if and only if*

$$\sum_{k=0}^{d-1} |M_k(a) \setminus M_k[b]| = \sum_{k=0}^{d-1} |M_k(b) \setminus M_k[a]|, \tag{2.1}$$

or equivalently,

$$\sum_{k=0}^{d-1} |M_k(a) \cap M_{k+1}(b)| = \sum_{k=0}^{d-1} |M_k(b) \cap M_{k+1}(a)|.$$

holds for any edge $e = ab \in E(G)$.

Proof. Considering $W_k(ab) = \{e \in E(G) \mid d(e,a) = k, d(e,b) = k+1\}$, $k \geq 0$, we get

$$E_{ab} = \bigcup_{k=0}^{d-1} W_k(ab).$$

Using the fact that $W_k(ab) = M_k(a) \setminus M_k[b] = M_k(a) \cap M_{k+1}(b)$ we obtain that G is EDB if and only if one of the following equivalent equalities hold:

$$\sum_{k=0}^{d-1} |M_k(a) \setminus M_k[b]| = \sum_{k=0}^{d-1} |M_k(b) \setminus M_k[a]|,$$

$$\sum_{k=0}^{d-1} |M_k(a) \cap M_{k+1}(b)| = \sum_{k=0}^{d-1} |M_k(b) \cap M_{k+1}(a)|.$$

Corollary 2.3. *Let G be a regular graph of diameter d. Then G is EDB if and only if*

$$\sum_{k=1}^{d-1} |M_k(a) \setminus M_k[b]| = \sum_{k=1}^{d-1} |M_k(b) \setminus M_k[a]|$$

holds for any edge $e = ab \in E(G)$.

Corollary 2.4. *Suppose that G is a graph with diameter $d = 2$ and any circle C_5 (if there is) in G has no path ae_1ue_2b where $e_1, e_2 \notin E(C_5)$, $u \notin V(C_5)$ and $a, b \in V(C_5)$ with $d(a, b) = 2$ and $d(u, x) = 2$ for all $x \in V(C_5) - \{a, b\}$ (see Figure 1). Then G is an EDB graph if and only if G is regular.*

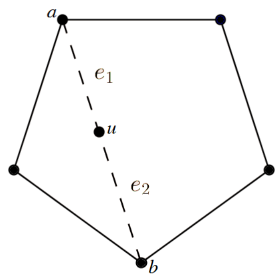

Figure 1: Circle C_5 in graph G not satisfying in hypotheses of Corollary 2.4.

Proof. Suppose that G is regular. Since $d = 2$, for any $ab \in E(G)$ we have

$$\forall e = uv \in E(G) \text{ s.t. } d(e, a) = 0 \quad \to \quad d(e, b) = 1,$$

$$\forall e = uv \in E(G) \text{ s.t. } d(e, b) = 0 \quad \to \quad d(e, a) = 1.$$

Now, considering the fact that deg(a)=deg(b) we obtain that

$$|M_0(a) \setminus M_0[b]| = |M_0(b) \setminus M_0[a]|. \tag{2.2}$$

On the other hand, let

$$e_1 = uv \in E(G) \text{ s.t. } d(a,u) = 1, \ d(e_1,a) = 1, \ d(e_1,b) = 2,$$

where $p_w : vwb$ is the shortest path connecting v to b. This together with Figure 2 shows that for any edge $e_1 \in M(a) \setminus M[b]$, we easily find that $e_2 = vw \in M(b) \setminus M[a]$. Note that since e_1 and e_2 are located in C_5, then following the hypotheses, there exists no edge $v\hat{u}$ in the path $v\hat{u}b$. It means that in the way as above for any edge e_1, the edge e_2 is chosen, uniquely. Therefore,

$$|M(a) \setminus M[b]| \leq |M(b) \setminus M[a]|.$$

Similarly, the converse can be proved. Now, applying Proposition 2.2 and

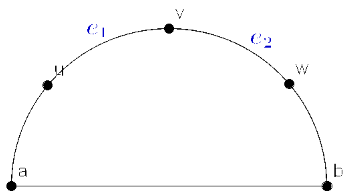

Figure 2: A section of graph G with edges ab, uv and vw.

relation (2.2) with equality

$$|M(a) \setminus M[b]| = |M(b) \setminus M[a]| \tag{2.3}$$

we conclude that G is an EDB graph. In order to prove the converse, i.e. equality (2.2), we only need to show the recent relation. To do this, one can easily see that conclusion (2.3) is only related to the fact that G has no subgraph C_5 with specified property and this completes the proof.

Remark 2.5. In Corollary 2.4, we notice that the mentioned condition for C_5 in G is a sufficient assumption to apply for characterization of all the EDB graphs with diameter 2. As illustration, Corollary 2.4 can be utilized for Peterson graph, Hoffman-Singleton graph, complete bipartite graph $K_{n,n}(n \neq 1)$, circles C_4 and C_5 and the following 3-regular EDB graph with $d = 2$ which also contains four circles C_5 not similar as depicted in Figure 1.

Remark 2.6. Following Corollary 2.4 and its proof, note that the vertex u generally can be adjacent to any other vertex in C_5 and so the hypotheses could be weakened, however, this additional condition on u makes the recent graph and similar ones included in the class of graphs compatible with conditions of Corollary 2.4.

Now, using the symbols $M_k[x]$ and $M_k(x)$ we develop our study to n-EDB graphs as follows.

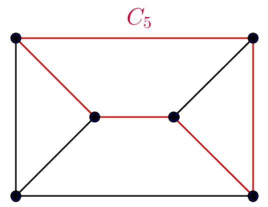

Figure 3: A 3-regular EDB graph with four circles C_5 not as form shown in Figure 1.

Proposition 2.7. *A graph G of diameter d is n-EDB if and only if*

$$\sum_{k=0}^{d-1} |M_k(a) \setminus M_k[b]| = \sum_{k=0}^{d-1} |M_k(b) \setminus M_k[a]|,$$

holds for any vertices $a, b \in V(G)$ with $d(a,b) = n$.

Proof. Suppose that $W_j^i(ab) = \{e \in E(G) \mid d(e,a) = j, d(e,b) = i + j\}$, for $i = 1, 2, ..., n$ and $0 \leq j \leq d-1$. Then we get

$$E_{a\underline{n}b} = \bigcup_{i=1}^{n} \bigcup_{j=0}^{d-1} W_j^i(ab).$$

On the other hand, since $M_k[a] \setminus M_k[b] = \bigcup_{i=1}^{n} \bigcup_{j=0}^{k} W_j^i(ab)$ and $M_k(a) \setminus M_k[b] = \bigcup_{i=1}^{n} W_k^i(ab)$, then G is n-EDB if and only if for all vertices $a, b \in V(G)$ with $d(a,b) = n$ we have

$$\sum_{k=0}^{d-1} |M_k(a) \setminus M_k[b]| = \sum_{k=0}^{d-1} |M_k(b) \setminus M_k[a]|,$$

which is as same as (2.1) in the previous proposition.

Remark 2.8. By the virtue of proof of Proposition 2.7 we see that graph G with diameter d is n-EDB if and only if

$$|M_{d-1}[a] \setminus M_{d-1}[b]| = |M_{d-1}[b] \setminus M_{d-1}[a]|,$$

for any vertices $a, b \in V(G)$ with $d(a,b) = n$.

Following Propositions 2.2 and 2.7 and in order to detect n-EDB graphs it only needs to establish (2.1) for any pair of vertices with distance n.

Definition 2.9 ([3])**.** *The graph G is called locally regular with respect to n (for short n-locally regular) if we have*

$$\forall a, b \in V(G), \;\; d(a,b) = n \quad \Longrightarrow \quad deg(a) = deg(b). \tag{2.4}$$

As the authors mentioned in [3], this kind of property is obviously weaker than the usual regularity. In the following we introduce a class of 2-locally regular graphs with 2-EDB property and diameter 2.

Suppose that $G := \vee_m K_n$ is a graph formed by joining complete graph K_n to $m-1$ copies of itself in a unique edge. Then G is 2-locally regular graph with diameter 2 and 2-EDB property (without EDB property), and has two central vertices. For example see the non-regular graphs shown in Figure 4. Similar to

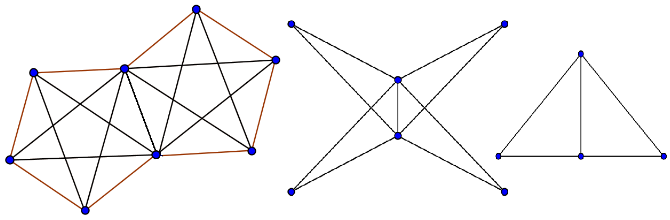

Figure 4: 2-locally regular graphs $\vee_2 K_5$, $\vee_4 K_3$ and $\vee_2 K_3$.

the technique as above, let $G := K_n^1 \vee_m K_n^2 \vee_m \cdots \vee_m K_n^k$ $(n \geq 4)$ be a chain of k number of K_n where each K_n has unique common edge to a copy of itself. Then G is a k-locally regular graph with diameter k, k-EDB property and without i-EDB property for $i = 1, 2, ..., k-1$. A 3-locally regular 3-EDB graph with diameter 3 and a 4-locally regular 4-EDB graph with diameter 4 are shown in Figures 5 and 6, respectively. The common edges in each one are indicated by the black bullets.

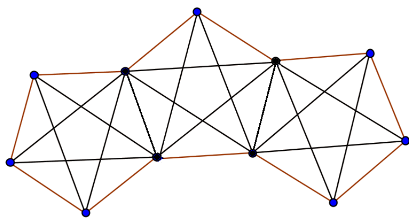

Figure 5: $K_5^1 \vee_1 K_5^2 \vee_1 K_5^3$.

Remark 2.10. We note that the structure of the graph given in definition as above can be modified and one can consider the chain in closed form, i.e., circular form, and obtain similar properties (for example see Figure 7).

3. 2-EDB graphs

In this section, inspired by the concept of graph joint in [3] and 2-connectivity we find necessary and sufficient conditions for 2-EDB graphs.

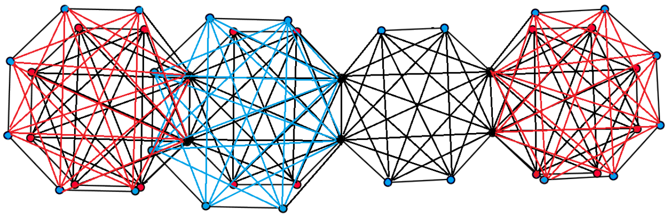

Figure 6: $K_8^1 \vee_3 K_8^2 \vee_3 K_8^3 \vee_3 K_8^4$ with three common edges and three plies of K_8 on each one.

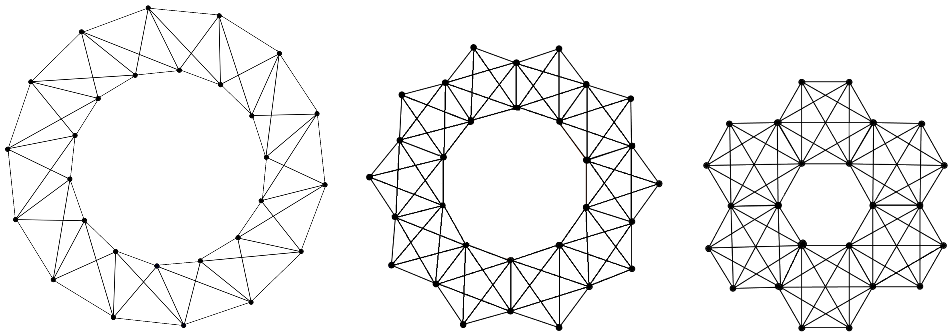

Figure 7: Closed forms of $K_4^1 \vee_1 K_4^2 \vee_1 \cdots \vee_1 K_4^{14}$, $K_5^1 \vee_1 K_5^2 \vee_1 \cdots \vee_1 K_5^{10}$ and $K_6^1 \vee_1 K_6^2 \vee_1 \cdots \vee_1 K_6^6$.

Definition 3.1 ([3]). *Let G be an arbitrary non-complete graph and $K_1 = \{u\}$ be an external vertex not belonging to $V(G)$. Then graph joint $G \vee K_1$ of graphs G and K_1 is a graph with*

$$\begin{cases} V(G \vee K_1) = V(G) \cup \{u\}, \\ E(G \vee K_1) = E(G) \cup \{uv \mid v \in V(G)\}. \end{cases}$$

Clearly, $G \vee K_1$ is connected and diam($G \vee K_1$)= 2. Moreover, G is connected if and only if $G \vee K_1$ is 2-connected, i.e., it remains connected whenever any arbitrary vertex is removed.

In the following we give a condition which will be needed in our next results.

(A) Suppose that G is an arbitrary connected graph and there is no induced subgraphs H_1 and H_2 shown in Figure 8 and obtained by removing finite number of vertices and their adjacent edges.

Theorem 3.2. *Suppose that G is a non-complete disconnected regular graph, then graph $G \vee K_1$ satisfying the condition (A) is 2-EDB and not 2-disconnected.*

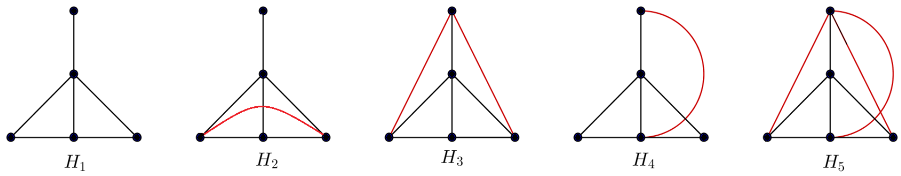

Figure 8: Impermissible subgraphs H_1 and H_2 together with some allowed subgraphs

Proof. Let G be a regular graph with valency k and $G \vee K_1$ be a graph constructed as above where $K_1 = \{u\}$ for an arbitrary fixed vertex $u \notin V(G)$. So u is adjacent to any vertex in $V(G)$. Assume $G_1, G_2, ..., G_n$ are all the connected components of G for some $n \geq 2$. All essentially different types of vertices a, b with $d(a, b) = 2$ in $G \vee K_1$ are either both from $V(G_i)$ or one from $V(G_i)$, the other from $V(G_j)$ for some $1 \leq i \neq j \leq n$. First, suppose that $a, b \in V(G_i)$ such that $d(a, b) = 2$ and $1 \leq i \leq n$. Then the fact that $\operatorname{diam}(G \vee K_1) = 2$ yields

$$E_{a\underline{2}b}^{G\vee K_1} = M_0^{(G_i)}(a) \cup M_1^{(G_i)}(a) \cup \{au\}. \tag{3.5}$$

where $M_j^{(G_i)}(a)$ means $M_j(a)$ limited to graph G_i. Based on (3.5) we observe that

$$|E_{a\underline{2}b}^{G\vee K_1}| = |M_0^{(G_i)}(a)| + |M_1^{(G_i)}(a)| + 1. \tag{3.6}$$

Similarly, we have

$$|E_{b\underline{2}a}^{G\vee K_1}| = |M_0^{(G_i)}(b)| + |M_1^{(G_i)}(b)| + 1. \tag{3.7}$$

Now, if $e \in M_1^{(G_i)}(a)$, then $e \in E(G_i)$ and $d(e, a) = 1$. This together with the fact that G has at least two components implies that G has a subgraph as form of either H_1 or H_2 shown in Figure 8 which is a contradiction. So $M_1^{(G_i)}(a) = M_1^{(G_i)}(b) = \emptyset$. Applying this equality together with (3.6), (3.7) and regularity of G we conclude that $|E_{a\underline{2}b}^{G\vee K_1}| = |E_{b\underline{2}a}^{G\vee K_1}|$ and the claim is proven for the first case. For the second case, let us consider $a \in V(G_i)$ and $b \in V(G_j)$ arbitrarily, for some $1 \leq i \neq j \leq n$. Then we have the similar conclusions as follows.

$$\begin{aligned} E_{a\underline{2}b}^{G\vee K_1} &= M_0^{(G_i)}(a) \cup M_1^{(G_i)}(a) \cup \{au\}, \\ E_{b\underline{2}a}^{G\vee K_1} &= M_0^{(G_j)}(b) \cup M_1^{(G_j)}(b) \cup \{bu\}. \end{aligned} \tag{3.8}$$

Again the regularity of G together with (3.8) and the hypothesis (A) implies that $|E_{a\underline{2}b}^{G\vee K_1}| = |E_{b\underline{2}a}^{G\vee K_1}| = k + 1$ where k is the valency of G. Therefore, $G \vee K_1$ is 2-EDB and this completes the proof.

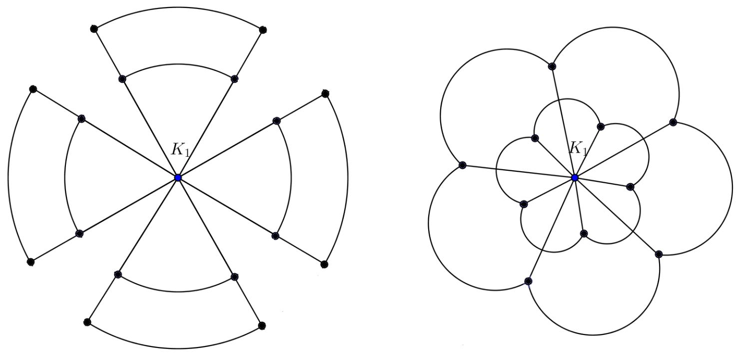

Figure 9: 2-EDB graphs $(C_4 \cup C_4 \cup C_4 \cup C_4) \vee K_1$ and $(C_5 \cup C_5) \vee K_1$

In Figure 9 two graphs satisfying the hypotheses of Theorem 3.2 are given. Now, we present the following result for the converse of the previous theorem which is more complicated and it needs some additional conditions.

Proposition 3.3. *Suppose that G is a connected 2-EDB graph with $|V(G)| < 2n+1$ for some integer $n \geq 3$ and G satisfies the condition (A). Moreover, assume that G is not 2-connected and $G - x$ has n nontrivial connected components for some $x \in V(G)$. Then $G \cong H \vee K_1$ for some regular graph H which is not connected and $K_1 = \{v\}$ for some $v \in V(G)$.*

Proof. Let G be a connected 2-EDB graph that is not 2-connected and u a cut vertex in G such that $G - u$ has at least three connected components in G. If we exclude the vertex u and the related edges, we obtain a subgraph H with connected components $H_1, H_2, ..., H_n$ for some $n \geq 3$. We prove that u is adjacent to every vertex in H_k for some $k \in \{1, 2, ..., n\}$. On the contrary, suppose that for arbitrary H_i there exists $a_i \in V(H_i)$ such that $d(a_i, u) = 2$. From the connectivity of H_i we can find $b_i \in V(H_i)$ with $d(b_i, u) = d(a_i, b_i) = 1$. Similarly, one can find such vertices for the other components with the properties as above. On the other hand, following the definition of 2-EDB we get

$$\begin{cases} E^G_{u\underline{2}a_i} \supseteq \{b_i u\} \bigcup \{e_j \mid \text{there exists path } p_{c_j} := (u \cdots e_j \cdots c_j) \text{ with vertices in } H_j, c_j \in V(H_j)\} \\ \qquad \bigcup \{e_k \mid \text{there exists path } p_{c_k} := (u \cdots e_k \cdots c_k) \text{ with vertices in } H_k, c_k \in V(H_k)\} \\ \qquad \Longrightarrow 1 + |V(H_j)| + |V(H_k)| \leq |E^G_{u\underline{2}a_i}|, \\ E^G_{a_i\underline{2}u} \subseteq E(H_i) \implies |E^G_{a_i\underline{2}u}| \leq \dfrac{|V(H_i)|(|V(H_i)|-1)}{2}, \end{cases}$$

which together with the fact that G is 2-EDB implies that

$$1+|V(H_j)|+|V(H_k)| \leq \frac{|V(H_i)|(|V(H_i)|-1)}{2}. \tag{3.9}$$

Following the process as above, one can obtain similar conclusions for compo-

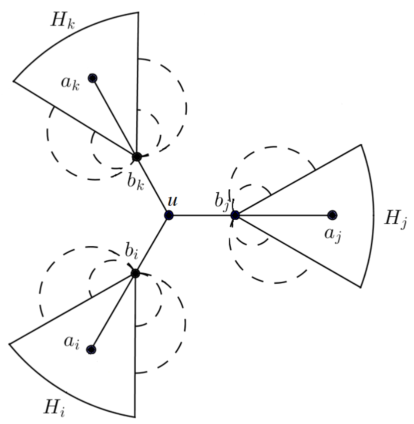

Figure 10: Cut vertex u with some conncected components of $G-u$.

nents H_j, H_k and the others as same as (3.9). Therefore,

$$\begin{aligned}(n-1)A+n &\leq \frac{1}{2}\Big(|V(H_1)|(|V(H_1)|-1)+|V(H_2)|(|V(H_2)|-1)\\ &\qquad +\cdots+|V(H_n)|(|V(H_n)|-1)\Big)\\ &\leq \frac{1}{2}(A^2-A)\end{aligned}$$

where $A := |V(G)|-1$. Hence $|V(G)| \geq 2n+1$ which is a contradiction.

Hence, u is adjacent to every vertex in H_k for at least one $k \in \{1,2,...,n\}$. Assume that u is adjacent to every vertex in $V(H_1)$. We prove that the induced subgraph H_1 is regular. Let us take an arbitrary $w \in V(H) \setminus V(H_1)$ adjacent to u. Obviously, $d(v,w)=2$ for every $v \in V(H_1)$. On the other hand, for vertex $x \in V(H_1)$ if $e \in E^G_{w\underline{2}x} \cap E(H_1)$ then

$$d(e,w)=d(e,u)+d(u,w)=2<d(x,e),$$

which is a contradiction followed by the fact that

$$d(x,e) \leq d(x,u)+d(u,e)=2.$$

Hence we get $E^G_{w\underline{2}x} \subseteq E(G) \setminus E(H_1)$. If we choose $e \in E^G_{w\underline{2}x}$, then by the recent inclusion we have

$$d(x,e)=d(x,u)+d(u,e)=1+d(u,e)=d(y,e)$$

for any vertex $y \in V(H_1)$. This implies that $|E^G_{w\underline{2}x}| = |E^G_{w\underline{2}y}|$. Next, since $E^G_{x\underline{2}w} \subseteq E(H_1)$ then

$$E^G_{x\underline{2}w} = M_0^{(H_1)}(x) \cup M_1^{(G)}(x) \cup \{ux\}.$$

Thus, by the fact that G is 2-EDB we get

$$\begin{aligned}|E^G_{x\underline{2}w}| &= |M_0^{(H_1)}(x)| + |M_1^{(G)}(x)| + 1,\\ |E^G_{x\underline{2}w}| &= |E^G_{w\underline{2}x}| = |E^G_{w\underline{2}y}| = |E^G_{y\underline{2}w}|,\end{aligned}$$

and so

$$|M_0^{(H_1)}(x)| + |M_1^{(G)}(x)| + 1 = |M_0^{(H_1)}(y)| + |M_1^{(G)}(y)| + 1$$

$$\Longrightarrow \deg(x) + |M_1^{(G)}(x)| = \deg(y) + |M_1^{(G)}(y)|.$$

Now, using the assumption (A) and the fact that $u \in V(G) \setminus V(H_1)$ we conclude that $M_1^{(G)}(x) \cap E(H_1) = \emptyset$ and $M_1^{(G)}(x) = \{uv \mid v \in V(H_1) \setminus \{x\}\}$ for any vertex $x \in V(H_1)$. It means that $\deg(x) = \deg(y)$ and so H_1 is regular.

Suppose that H_1 has a valency k. We show that u is adjacent to every vertex in $V(H)$. Suppose that this statement is not true. Then without loss of generality, we can assume that there is $v_2 \in V(H_2)$ such that $d(v_2, u) = 2$. Hence one can find $v_1 \in V(H_2)$ such that $d(v_1, u) = d(v_1, v_2) = 1$. As we know that $|M_0^{(G)}(w)| = k + 1$ for arbitrary $w \in V(H_1)$, then using the hypothesis (A) we conclude

$$\begin{aligned}&E^G_{w\underline{2}v_1} = M_0^{(G)}(w) \cup M_1^{(H_1)}(w) \implies |E^G_{w\underline{2}v_1}| = |M_1^{(H_1)}(w)| + k + 1 = k + 1\\ &E^G_{u\underline{2}v_2} \supseteq E(H_1) \cup \{uw \mid w \in V(H_1)\} \implies |E^G_{u\underline{2}v_2}| \geq |E(H_1)| + |V(H_1)|\end{aligned}$$

Now we define $D = \bigcup_{i=1}^{d} (W_i^0(v_1u) \cup W_{i-1}^1(v_1u))$ where W_i^0 and W_i^1 are given in the previous section. We easily observe that $E^G_{v_2\underline{2}u} \subseteq D \subseteq E^G_{v_1\underline{2}w}$ which means that

$$\begin{aligned}|E(H_1)| + k + 1 \leq |E(H_1)| + |V(H_1)| &\leq |E^G_{u\underline{2}v_2}| = |E^G_{v_2\underline{2}u}| \leq |E^G_{v_1\underline{2}w}|\\ &= |E^G_{w\underline{2}v_1}| = k + 1,\end{aligned}$$

which implies that H_1 is trivial and $V(H_1) = \{w\}$. This is a contradiction. So v is adjacent to every vertex in $V(H)$. Finally, we prove that graph H is regular with valency k. In order to do this, it suffices to show that H_2 is regular with valency k. Let us consider an arbitrary $v \in V(H_2)$ and $w \in V(H_1)$ which has already been found that $d(v, w) = 2$. Therefore,

$$\begin{aligned}E^G_{v\underline{2}w} &= M_0^{(G)}(v) \cup M_1^{(H_1)}(v) \cup \{uv\},\\ E^G_{w\underline{2}v} &= M_0^{(G)}(w) \cup M_1^{(H_1)}(w) \cup \{uw\}.\end{aligned}$$

Hence we get

$$|M_0^{(G)}(v)| + |M_1^{(H_1)}(v)| = |M_0^{(G)}(w)| + |M_1^{(H_1)}(w)|. \tag{3.10}$$

On the other hand, using the property (A) we have

$$|M_1^{(H_1)}(v)| = |M_1^{(H_1)}(w)| = 0.$$

We notice that

$$M_1^{(G)}(v) = M_1^{(H_1)}(v) \cup \{xu \mid x \in V(H_i),\ i = 1, 2, ..., n\} \setminus \{uv\},$$
$$M_1^{(G)}(w) = M_1^{(H_1)}(w) \cup \{xu \mid x \in V(H_i),\ i = 1, 2, ..., n\} \setminus \{wv\},$$

Now, relation (3.10) implies that

$$\deg(v) = |M_0^{(G)}(v)| = |M_0^{(G)}(w)| = k + 1 \quad \text{for all} \quad v \in V(H_2).$$

Hence, H is regular with valency k and the consequence follows.

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