ON DERIVED OF SOME POLYGROUPS AND GENERALIZED ALTERNATIVE POLYGROUPS

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Abstract. In this paper, we investigate the derived of some polygroups and we give some results on them. Also, we define generalized alternative polygroups which is the derived of generalized symmetric polygroups.
Keywords: Polygroups, derived polygroup, alternative polygroup.

1. Introduction

Hyperstructure theory was born in 1934 at the 8th congress of Scandinavian Mathematicians, when Marty [11] introduced the hypergroup notion as a generalization of groups and later, he proved its utility in solving some problems of groups, algebraic functions and rational fractions. Surveys of the theory can be found in the books of Corsini [3], Davvaz and Leoreanu-Fotea [6], Corsini and Leoreanu [4] and Vougiouklis [12]. One of the important classes of hypergroups is polygroups which their properties are close to groups. There are many mathematicians that interest to work on this class. The authors introduced the notion of derived of polygroups in [1], in this paper we investigate the derived of $\mathcal{A}[\mathcal{B}]$, the extension of $\mathcal{A}$ by $\mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$, are two polygroups and introduced by Comer [2]. Also, we define generalized alternative polygroup by the derived of generalized symmetric polygroups.

2. Preliminaries

In this section, first we recall some basic notions of polygroup theory and we give some results on the derived polygroups that we introduced in [1].

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Let H be a non-empty set and $\mathcal{P}^*(H)$ be the set of all non-empty subsets of H. Let $\cdot$ be a *hyperoperation* (or *join operation*) on H, that is, $\cdot$ is a function from $H \times H$ into $\mathcal{P}^*(H)$. If $(a, b) \in H \times H$, its image under $\cdot$ in $\mathcal{P}^*(H)$ is denoted by $a \cdot b$. The join operation is extended to subsets of H in a natural way, that is, for non-empty subsets A, B of H, $A \cdot B = \cup\{a \cdot b \mid a \in A, b \in B\}$. The notation $a \cdot A$ is used for $\{a\} \cdot A$ and $A \cdot a$ for $A \cdot \{a\}$. Generally, the singleton $\{a\}$ is identified with its member a. The structure $(H, \cdot)$ is called a *semihypergroup* if $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in H$, which means that

$$\bigcup_{u \in x \cdot y} u \cdot z = \bigcup_{v \in y \cdot z} x \cdot v,$$

A semihypergroup is a *hypergroup* if $a \cdot H = H \cdot a = H$ for all $a \in H$. A non-empty subset K of a hypergroup $(H, \cdot)$ is called a *subhypergroup* if it is a hypergroup. The subhypergroup K is called *invertible on the left (on the right)* if for all $(x, y) \in H^2$ from $x \in K \cdot y$ (*resp.* $x \in y \cdot K$), it follows that $y \in K \cdot x$ (*resp.* $y \in x \cdot K$). We say K is *invertible* if it is invertible to the left and to the right. An element e of H is called an *identity* if, for all $x \in H$, $x \in x \cdot e \cap e \cdot x$ and $a' \in H$ is called an *inverse* of a in H if $e \in a \cdot a' \cap a' \cdot a$. Suppose that $(H, \cdot)$ and $(H', \circ)$ are two semihypergroups. A function $f : H \longrightarrow H'$ is called a *homomorphism* if $f(a \cdot b) \subseteq f(a) \circ f(b)$ for all a and b in H. We say that f is a *good homomorphism* if for all a and b in H, $f(a \cdot b) = f(a) \circ f(b)$. If $(H, \cdot)$ is a hypergroup and $\rho \subseteq H \times H$ is an equivalence relation, then for all non-empty subsets A, B of H we set

$$A \overline{\overline{\rho}} B \Leftrightarrow a \rho b, \text{ for all } a \in A, \ b \in B.$$

The relation ρ is called *strongly regular on the left (on the right)* if $x \rho y \Rightarrow a \cdot x \overline{\overline{\rho}} a \cdot y$ ($x \rho y \Rightarrow x \cdot a \overline{\overline{\rho}} y \cdot a$, respectively), for all $(x, y, a) \in H^3$. Moreover, ρ is called *strongly regular* if it is strongly regular on the right and on the left.

Theorem 2.1 (Theorem 31, [3]). *If $(H, \cdot)$ is a semihypergroup (hypergroup) and ρ is a strongly regular relation on H, then the quotient H/ρ is a semigroup (group) under the operation :*

$$\rho(x) \otimes \rho(y) = \rho(z), \text{ for all } z \in x \cdot y.$$

We denote $\rho(x)$ by $\bar{x}$ and instead of $\bar{x} \otimes \bar{y}$ we write $\bar{x}\bar{y}$.

For all $n > 1$, we define the relation β_n on a semihypergroup H, as follows :

$$a \, \beta_n \, b \Leftrightarrow \exists (x_1, \ldots, x_n) \in H^n \ : \ \{a, b\} \subseteq \textstyle\prod_{i=1}^n x_i,$$

and $\beta_H = \bigcup_{i=1}^n \beta_n$, where $\beta_1 = \{(x, x) \mid x \in H\}$ is the diagonal relation on H. This relation was introduced by Koskas [10] and studied mainly by Corsini [3]. Suppose that β_H^* is the transitive closure of β. The relation β_H^* is a strongly regular relation [3]. Also, we have :

Theorem 2.2 (Freni, [9]). *If H is hypergroup, then $\beta_H = \beta^*_H$.*

Note that, in general, for a semihypergroup may be $\beta_H \neq \beta^*_H$. The relation β^*_H is the least equivalence relation on a hypergroup H, such that the quotient H/β^*_H is a group. The *heart* ω_H of a hypergroup H is the set of all elements x of H, for which the equivalence class $\beta^*_H(x)$ is the identity of the group H/β^*_H.

A hypergroup P is called *polygroup* and is denoted by $\langle P, \cdot, e, ^{-1} \rangle$ if the following conditions hold :

(1) P has a scalar identity e (i.e., $e \cdot x = x \cdot e = x$, for every $x \in P$);

(2) every element x of P has a unique inverse x^{-1} in P;

(3) $x \in y \cdot z$ implies $y \in x \cdot z^{-1}$ and $z \in y^{-1} \cdot x$.

In the following we recall some properties of derived subpolygroups from [1].

Definition 2.3. Let P be a polygroup. We define

(1) $[x, y]_r = \{h \in P \mid x \cdot y \cap y \cdot x \cdot h \neq \emptyset\}$;

(2) $[x, y]_l = \{h \in P \mid x \cdot y \cap h \cdot y \cdot x \neq \emptyset\}$;

(3) $[x, y] = [x, y]_r \cup [x, y]_l$.

We call $[x, y]_r$, $[x, y]_l$ and $[x, y]$ *right commutator x and y, left commutator x and y* and *commutator x and y*, respectively. Also, we will denote $[P, P]_r$, $[P, P]_l$ and $[P, P]$ the set of all right commutators, left commutators and commutators, respectively.

A non-empty subset K of a polygroup $\langle P, \cdot, e, ^{-1} \rangle$ is a *subpolygroup* of P if $x, y \in K$ implies $x \cdot y \in K$, and $x \in K$ implies $x^{-1} \in K$.

Let X be a nonempty subset of a polygroup $\langle P, \cdot, e, ^{-1} \rangle$. Let $\{A_i |\ i \in J\}$ be the family of all subpolygroups of P in which contain X. Then, $\cap_{i\in J} A_i$ is called the subpolygroup generated by X. This subpolygroup is denoted by $< X >$ and we have $< X >= \cup\{x_1^{\varepsilon_1} \cdot \ldots \cdot x_k^{\varepsilon_k} \mid x_i \in X, k \in \mathbb{N}, \varepsilon_i \in \{-1, 1\}\}$. If $X = \{x_1, x_2, \ldots, x_n\}$, then the subpolygroup $< X >$ is denoted $< x_1, x_2, \ldots, x_n >$. In a special case $< [P, P]_r >$, $< [P, P]_l >$ and $< [P, P] >$ are shown by P'_r, P'_l and P', respectively.

Proposition 2.4. *Let $\langle P, \cdot, e, ^{-1} \rangle$ be a polygroup $(x, y) \in P^2$. Then,*

(1) $[x, y]_r = [x^{-1}, y^{-1}]_l$;

(2) $P' = P'_r = P'_l$;

(3) $x \in P' \Rightarrow x^{-1} \in P'$.

Corollary 2.5. *If $\langle P, \cdot, e, ^{-1} \rangle$ is a polygroup, then P' is a subpolygroup of P.*

From now on we call P' the *derived subpolygroup* of P.

Proposition 2.6. *Let $\langle P, \cdot, e, ^{-1} \rangle$ be a polygroup. Then, $P' = \{e\}$ if and only if P be an abelian group.*

A subpolygroup N of a polygroup $\langle P, \cdot, e, ^{-1} \rangle$ is *normal* in P if $x^{-1} \cdot N \cdot x \subseteq N$, for all $x \in P$.

Let K and N be subpolygroups of a polygroup P with N normal in P. Then,

(1) $Na = aN$, for all $a \in P$;

(2) $(Na)(Nb) = Nab$, for all $a, b \in P$;

(3) $Na = Nb$, for all $b \in Na$;

(4) $N \cap K$ is a normal subpolygroup of K;

(5) $NK = KN$, is a subpolygroup of P;

(6) N is a normal subpolygroup of NK.

Proposition 2.7. *If N is a normal subpolygroup of P, then $\langle P/N, \circ, N, -I \rangle$ is a polygroup, where $Nx \circ Ny = \{Nz | z \in xy\}$ and $(Nx)^{-I} = Nx^{-1}$.*

Proposition 2.8. *If N is a normal subpolygroup of P, then $(P/N)' = NP'/N$*

Proof. Suppose that $(x, y) \in P^2$. From the equations

$$[xN, yN] = xNyNx^{-1}Ny^{-1}N = \{zN \mid z \in [x, y]\}$$

and

$$NP'/N = \{yN \mid y \in NP'\} = \{yN \mid y \in nz, n \in N, z \in P'\} = \{zN \mid z \in P'\},$$

we obtain $(P/N)' =< [P/N, P/N] >= NP'/N$. □

3. On extension polygroups

In this section, we investigate extension polygroups and we give some new results on this class of polygroups.

Suppose that $\mathcal{A} = \langle A, \cdot, e, ^{-1} \rangle$ and $\mathcal{B} = \langle B, \cdot, e, ^{-1} \rangle$ are two polygroups whose elements have been renamed so that $A \cap B = \{e\}$. A new system $\mathcal{A}[\mathcal{B}] = \langle M, *, e, I \rangle$ called the extension of $\mathcal{A}$ by $\mathcal{B}$ is formed in the following way : Set $M = A \cup B$ and let $e^I = e, x^I = x^{-1}, x * e = e * x = x$ for all $x \in M$, and for all $x, y \in M - \{e\}$,

$$x * y = \begin{cases} x \cdot y, & \text{if } x, y \in A, \\ x, & \text{if } x \in B, y \in A \\ y, & \text{if } x \in A, y \in B \\ x \cdot y, & \text{if } x, y \in B, y \neq x^{-1} \\ x \cdot y \cup A, & \text{if } x, y \in B, y = x^{-1}. \end{cases}$$

In this case, $\mathcal{A}[\mathcal{B}]$ is a polygroup which is called the extension of $\mathcal{A}$ by $\mathcal{B}$ [2].

Remark 3.1. Notice that if $\mathcal{A} = \{e\}$, then $\mathcal{A}[\mathcal{B}] = \mathcal{B}$.

Lemma 3.2. *Let $\mathcal{A} = \langle A, \cdot, e, ^{-1} \rangle$ and $\mathcal{B} = \langle B, \cdot, e, ^{-1} \rangle$ be two polygroups, where $A \neq \{e\}$ and $\mathcal{A}[\mathcal{B}] = \langle M, *, e, I\rangle$ be a extension of $\mathcal{A}$ by $\mathcal{B}$. Then, for all $(x_1, x_2, \cdots, x_n) \in M^n$ we have*

(i) *If $x_i \in A$, for all $1 \leq i \leq n$, then $*\prod_{i=1}^{n} x_i \subseteq A$, where $*\prod_{i=1}^{n} x_i = x_1 * x_2 * \cdots * x_n$.*

(ii) *If there exists j such that $x_j \in B$, then $*\prod_{i=1}^{n} x_i \subseteq B$ or $A \subseteq *\prod_{i=1}^{n} x_i$.*

(iii) *If $A \subseteq *\prod_{i=1}^{n} x_i$, then there exists j such that for all $k \geq j$, $x_k = x_{k-1}^{-1}$ or $x_k^{-1} \in \cdot\prod_{i=1}^{n} x_i$.*

(iv) *If $A \subseteq *\prod_{i=1}^{n} x_i$, then $*\prod_{i=1}^{n} x_i = \cdot\prod_{j=1}^{m} t_j \cup A$ and $e \in \cdot\prod_{j=1}^{m} t_j$, where $t_j \in \{x_1, x_2, \cdots, x_n\} \cap B \subseteq M$, for all $1 \leq j \leq m$.*

Proof. The part (i) is obvious. (ii) Suppose that there exists j, such that $x_j \in B$. It is easy to see that $*\prod_{i=1}^{n} x_i = *\prod_{j=1}^{m} y_j$, where $y_j \in \{x_1, x_2, \cdots, x_n\} \cap B$. Hence $*\prod_{i=1}^{n} x_i = *\prod_{j=1}^{m} y_j \subseteq B$ or $A \subseteq *\prod_{i=1}^{n} x_i = *\prod_{j=1}^{m} y_j$. (iii) Suppose that (iii) is not true therefore $x_k \neq x_{k-1}^{-1}$ and $x_k^{-1} \notin \cdot\prod_{i=1}^{n} x_i$, for all $1 \leq k \leq n$. Thus $*\prod_{i=1}^{n} x_i = \cdot\prod_{i=1}^{n} x_i \subseteq B$, which is a contradiction. (iv) It is easy to see that $*\prod_{i=1}^{n} x_i = *\prod_{i=1}^{m} t_i$, where $t_i \in \{x_1, x_2, \cdots, x_m\} \cap B$. According to (iii) there exists j such that for all $k \geq j$, $t_k = t_{k-1}^{-1}$ or $t_k^{-1} \in \cdot\prod_{i=1}^{m} t_i$. Suppose that j is the smallest number with the mentioned condition hence $*\prod_{i=1}^{m} t_i = (\cdot\prod_{i=1}^{j-1} t_i) * t_j * (*\prod_{i=j+1}^{m} t_i) = (\cdot\prod_{i=1}^{j} t_i \cup A) * (*\prod_{i=j+1}^{m} t_i) = (\cdot\prod_{i=1}^{j} t_i) * (*\prod_{i=j+1}^{m} t_i) \cup *\prod_{i=j+1}^{m} t_i = \cdot\prod_{i=1}^{m} t_i \cup A$. Moreover $e \in \cdot\prod_{i=1}^{m} t_i$. □

Example 3.3. Let $\mathcal{A}$ and $\mathcal{B}$ be the following polygroups

$\cdot$	1	2
1	1	2
2	2	$\{1, 2\}$

$\cdot$	1	3	4
1	1	3	4
3	3	$\{1, 4\}$	$\{3, 4\}$
4	4	$\{3, 4\}$	$\{1, 3\}$

The hyperoperation extension $\mathcal{A}$ by $\mathcal{B}$, $\mathcal{A}[\mathcal{B}] = M$ as follows :

$*$	1	2	3	4
1	1	2	3	4
2	2	$\{1, 2\}$	3	4
3	3	3	$\{1, 2, 4\}$	$\{3, 4\}$
4	4	4	$\{3, 4\}$	$\{1, 2, 3\}$

Notice that if $(x_1, x_2, ..., x_n) \in M^n$ and $A \subseteq *\prod_{i=1}^{n} x_i$ we have $*\prod_{i=1}^{n} x_i = \{\{e, a\}, \{e, a, b\}, \{e, a, c\}, \{e, a, b, c\}\}$. Hence $*\prod_{i=1}^{n} x_i = e \cdot e \cup A$, or $c \cdot c \cup A$, or $b \cdot b \cup A$ and or $b \cdot b \cdot c \cup A$, where $e \in e \cdot e \cap c \cdot c \cap b \cdot b \cap b \cdot b \cdot c$.

Theorem 3.4. *Let* $\mathcal{A} = \langle A, \cdot, e, ^{-1} \rangle$ *and* $\mathcal{B} = \langle B, \cdot, e, ^{-1} \rangle$ *be two polygroups and* $\mathcal{A}[\mathcal{B}]$ *be an extension of* $\mathcal{A}$ *by* $\mathcal{B}$*. Then,*

$$\mathcal{A}[\mathcal{B}]/\beta^*_{\mathcal{A}[\mathcal{B}]} \cong \mathcal{B}/\beta^*_{\mathcal{B}}.$$

Proof. Suppose that $\varphi : \mathcal{A}[\mathcal{B}]/\beta^*_{\mathcal{A}[\mathcal{B}]} \longrightarrow \mathcal{B}/\beta^*_{\mathcal{B}}$, such that

$$\varphi([x]) = \begin{cases} \bar{e}, & \text{if } x \in A \\ \bar{x}, & \text{if } x \in B - A, \end{cases}$$

where $[x] = \beta^*_{\mathcal{A}[\mathcal{B}]}(x)$ and $\bar{x} = \beta^*_{\mathcal{B}}(x)$. First of all, we prove φ is well defined map. Suppose that $x, y \in M$ and $[x] = [y]$, so there exists $(z_1, z_2, \cdots, z_n) \in M^n$ such that $x, y \in * \prod_{i=1}^n z_i$. By Lemma 3.2, we have the following cases:

Case 1. If $z_i \in A$, for all $1 \leq i \leq n$, then $x, y \in A$ and so $\varphi([x]) = \varphi([y]) = \bar{e}$.

Case 2. If there exists j such that $z_j \in B$ and $* \prod_{i=1}^n z_i \subseteq B$, we have $* \prod_{i=1}^n z_i = * \prod_{j=1}^m t_j$, where $t_j \in \{z_1, z_2, \cdots, z_n\} \cap B$ and hence $x, y \in \cdot \prod_{j=1}^m t_j$ and so $\bar{x} = \bar{y}$ which means that $\varphi([x]) = \varphi([y])$.

Case 3. If there exists j, $z_j \in B$ and $A \subseteq * \prod_{i=1}^n z_i$. In this case, we have $* \prod_{i=1}^n z_i = \cdot \prod_{j=1}^m t_j \cup A$, where $t_j \in \{z_1, z_2, \cdots, z_n\} \cap B$ and $e \in \cdot \prod_{j=1}^m t_j$. Since $\{x, y, e\} \subseteq * \prod_{i=1}^n z_i$, one of the following statements happen :

(i) $\{x, y\} \subseteq \cdot \prod_{j=1}^m t_j$;

(ii) $\{x, y\} \subseteq A$;

(iii) $x \in \cdot \prod_{j=1}^m t_j$ and $y \in A$;

(iv) $y \in \cdot \prod_{j=1}^m t_j$ and $x \in A$.

In all of the above cases, we have $\varphi([x]) = \varphi([y])$. Therefore, φ is well defined. Suppose that $x, y \in M$. In order to prove that φ is a good homomorphism we need to consider the following steps :

(1) $(x, y) \in A^2$, $\varphi([x] \otimes [y]) = \varphi([z])$, where $z \in x * y = x \cdot y$. So, $z \in A$ and hence $\varphi([z]) = \bar{e} = \bar{e}\bar{e} = \varphi([x])\varphi([y])$.

(2) $(x, y) \in B^2$, $\varphi([x] \otimes [y]) = \varphi([z])$, where $z \in x * y = x \cdot y$ or $x * y = x \cdot y \cup A$ and $y = x^{-1}$. If $z \in x * y = x \cdot y$, then $\varphi([z]) = \bar{z} = \bar{x} \cdot \bar{y} = \varphi([x])\varphi([y])$. If $z \in x * y = x \cdot y \cup A$ and $y = x^{-1}$, then $z \in A$ or $z \in x \cdot y$, if $z \in A$ we have $\varphi([z]) = \bar{e}$ and $\varphi([x])\varphi([y]) = \varphi([x])\varphi([x^{-1}]) = \bar{x}\overline{x^{-1}} = \bar{x}\bar{x}^{-1} = \bar{e}$. Consequently, $\varphi([x] \otimes [y]) = \varphi([x])\varphi([y])$. If $z \in x \cdot y = x \cdot x^{-1}$, then $\bar{z} = \bar{x} \cdot \bar{y} = \bar{x} \cdot \bar{x}^{-1} = \bar{e}$. Thus, $\bar{e} = \varphi([z]) = \varphi([x])\varphi([y])$.

(3) $x \in A$ and $y \in B$, $\varphi([x] \otimes [y]) = \varphi([z])$, where $z \in x * y = y$. So, $z = y$ and hence $\varphi([x] \otimes [y]) = \varphi([y]) = \bar{y} = \bar{e} \cdot \bar{y} = \varphi([x])\varphi([y])$.

(4) $x \in B$ and $y \in A$, this step is similar to (3).

It is easy to see that φ is one to one and onto and the proof is completed. □

Corollary 3.5. *Let $\mathcal{A} = \langle A, \cdot, e, ^{-1} \rangle$ and $\mathcal{B} = \langle B, \cdot, e, ^{-1} \rangle$ be two polygroups and $\mathcal{A}[\mathcal{B}]$ be an extension of $\mathcal{A}$ by $\mathcal{B}$. Then, $\mathcal{A}[\mathcal{B}]$ is special* (*in the sense of Definition* 4.8) *if and only if $\mathcal{B}$ is special.*

4. Derived of extension polygroups and generalized symetric polygroups

In this section first we investigate the derived of $\mathcal{A}[\mathcal{B}]$, the extension of the $\mathcal{A}$ by $\mathcal{B}$, and then we define generalized alternative polygroup by the derived of generalized symmetric polygroups. Also, we give some properties of alternative polygroups.

Proposition 4.1. *Let $\mathcal{A} = \langle A, \cdot, e, ^{-1} \rangle$ and $\mathcal{B} = \langle B, \cdot, e, ^{-1} \rangle$ be two polygroups and $\mathcal{A}[\mathcal{B}] = \langle M, *, e, I \rangle$ be a extension of $\mathcal{A}$ by $\mathcal{B}$. Then, for all $x, y \in M$,*

$$[x,e]_M = [e,x]_M = \begin{cases} [e,x]_A, & \textit{if } \ x \in A \\ [e,x]_B \cup A, & \textit{if } \ x \in B. \end{cases}$$

$$[x,y]_M = \begin{cases} [x,y]_A, & \textit{if } \ x,y \in A \\ [x,y]_B \cup A, & \textit{if } \ x,y \in B, y \neq x^{-1}, xy \cap yx \neq \emptyset \ \textit{or} \ x,y \in B, y = x^{-1} \\ [x,y]_B, & \textit{if } \ x,y \in B, y \neq x^{-1}, xy \cap yx = \emptyset \\ [x,e]_B \cup A, & \textit{if } \ x \in B, y \in A \\ [e,y]_B \cup A, & \textit{if } \ x \in A, y \in B. \end{cases}$$

Proof. Suppose that $(x,y) \in M^2$. If $(x,y) \in B^2$, then

$$\begin{aligned} [x,y]_M &= (x * y) * (y * x)^{-1} \\ &= \begin{cases} (x \cdot y) * (y \cdot x)^{-1}, & \text{if } \ x \neq y^{-1} \\ (x \cdot y \cup A) * (y \cdot x \cup A)^{-1}, & \text{if } \ x = y^{-1} \end{cases} \\ &= \begin{cases} (x \cdot y) * (y \cdot x)^{-1}, & \text{if } \ x \neq y^{-1} \\ (x \cdot y) * (y \cdot x)^{-1} \cup x \cdot y \cup (y \cdot x)^{-1} \cup A, & \text{if } \ x = y^{-1}. \end{cases} \end{aligned}$$

Since $x = y^{-1}$, we conclude that $e \in x \cdot y$ and so $x \cdot y \cup (y \cdot x)^{-1} \subseteq (x \cdot y) \cdot (y \cdot x)^{-1}$. Hence, we have

$$[x,y]_M = \begin{cases} [x,y]_B \cup A, & \text{if } \ x,y \in B, x \neq y^{-1}, xy \cap yx \neq \emptyset \ \text{ or } x,y \in B, x = y^{-1}, \\ [x,y]_B, & \text{if } \ x,y \in B, x \neq y^{-1}, xy \cap yx = \emptyset. \end{cases}$$

It is not difficult to see that the other cases also hold. □

Corollary 4.2. *Let $\mathcal{A} = \langle A, \cdot, e, ^{-1} \rangle$ and $\mathcal{B} = \langle B, \cdot, e, ^{-1} \rangle$ be two polygroups and $\mathcal{A}[\mathcal{B}] = \langle M, *, e, I \rangle$ be an extension of $\mathcal{A}$ by $\mathcal{B}$. Then, $\mathcal{A} \cup \mathcal{B}' \subseteq (\mathcal{A}[\mathcal{B}])'$.*

In [7], Davvaz studied some aspects of polygroups. By using the concept of generalized permutation, he defined permutation polygroups and some concepts related to it. He obtained a generalization of Cayley's theorem, too. In the following, we recall the definition.

Definition 4.3. Let Ω be a non-empty set. A map $f:\Omega \longrightarrow P^*(\Omega)$ is called a generalized permutation on Ω if

$$\bigcup_{\omega\in\Omega} f(\omega) = f(\Omega) = \Omega,$$

where $P^*(\Omega)$ is the set of all the non-empty subsets of Ω. We write $f = \binom{x}{f(x)}$ for the generalized permutation f. Denote M_Ω the set of all the generalized permutations on Ω.

Definition 4.4. Let $M = \langle P, \cdot, e, ^{-1} \rangle$ be a polygroup and Ω be a non-empty set. A map $f:\Omega\times P \longrightarrow P^*(\Omega)$ is called an action P on Ω if the following axioms hold :

(1) $f(\omega, e) = \{\omega\} = \omega$, for all $\omega\in\Omega$;

(2) $f(f(\omega, g), h) = \bigcup_{\omega\in g\cdot h} f(\omega,\alpha)$, for all $g, h\in P$ and $\omega\in\Omega$;

(3) $\bigcup_{\omega\in\Omega} f(\omega, g) = \Omega$, for all $g\in P$;

(4) For all $g\in P$, $\alpha\in f(\beta, g) \Longrightarrow \beta\in f(\alpha, g^{-1})$.

From the second condition, we obtain $\bigcup_{\omega_0\in f(\omega,g)} f(\omega_0, h) = \bigcup_{\alpha\in g\cdot h} f(\omega,\alpha)$. For $\omega\in\Omega$, we write $\omega^g := f(\omega, g)$. In this case, we say that P is a permutation polygroup on a set Ω and it is said that P acts on Ω.

Theorem 4.5 (Generalization of Cayley's Theorem)**.** *Let P be a polygroup acting on a nonempty finite set Ω. Then, there is a subset of M_Ω which is a polygroup under the induced action of P and is isomorphism to P.*

Proposition 4.6 ([7])**.** *Let* $S_\Omega=\{\begin{pmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|}\\ \alpha_1^g & \alpha_2^g & \cdots & \alpha_{|\Omega|}^g\end{pmatrix} | g\in P\}$. *Then,* $\langle S_\Omega, \circ, i, ^{-I}\rangle$ *is a polygroup, where*

$$\begin{pmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|}\\ \alpha_1^g & \alpha_2^g & \cdots & \alpha_{|\Omega|}^g\end{pmatrix} \circ \begin{pmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|}\\ \alpha_1^h & \alpha_2^h & \cdots & \alpha_{|\Omega|}^h\end{pmatrix}$$
$$= \{\begin{pmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|}\\ \alpha_1^f & \alpha_2^f & \cdots & \alpha_{|\Omega|}^f\end{pmatrix} | f\in g\cdot h\}$$

and

$$\begin{pmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|}\\ \alpha_1^g & \alpha_2^g & \cdots & \alpha_{|\Omega|}^g\end{pmatrix}^{-I} = \begin{pmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|}\\ \alpha_1^{g^{-1}} & \alpha_2^{g^{-1}} & \cdots & \alpha_{|\Omega|}^{g^{-1}}\end{pmatrix}.$$

Example 4.7. Let $\Omega = \{1, 2, 3, \cdots, n\}$, $n \in \mathbb{N}$, $P = S_n$(the symmetric group of order n). Then, $f : \Omega \times P \longrightarrow P^*(\Omega)$, such that $f(k, \sigma) = \{\sigma(k)\}$ is an action, which we call it trivial action P on Ω.

Definition 4.8. A polygroup P is called special if $P/\beta^* \cong S_n$.

Example 4.9. Let $P = \{1, 2, 3, 4, 5, 6, 7\}$. Consider the polygroup $\langle P, \bullet, 1, ^{-1} \rangle$, where $\bullet$ is defined on P as follows :

$\bullet$	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	$\{1,2\}$	3	4	5	6	7
3	3	3	$\{1,2\}$	7	6	5	4
4	4	4	6	$\{1,2\}$	7	3	5
5	5	5	7	6	$\{1,2\}$	4	3
6	6	6	4	5	3	7	$\{1,2\}$
7	7	7	5	3	4	$\{1,2\}$	6

It is easy to see that P is a special polygroup.

Definition 4.10. Let P be a special polygroup and f be an action of P on $\Omega = \{1, 2, 3, \cdots, n\}$, we call S_Ω *generalized symmetric polygroup* and $S'_\Omega = A_\Omega$ is called the *generalized alternative polygroup.*

Example 4.11. Let P be the special polygroup of Example 4.9 and $\Omega = \{1, 2, 3, \cdots, 7\}$. Consider the action $k \cdot g = k \bullet g$. In this case $S_\Omega = P$ and $A_\Omega = P'$, where $P' = \langle 1, 2, 6, 7\rangle = \{1, 2, 6, 7\} \trianglelefteq P$.

Lemma 4.12. $A_\Omega = \{\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^g & \alpha_2^g & \cdots & \alpha_{|\Omega|}^g \end{pmatrix} |g \in P'\}$.

Proof. We have

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^g & \alpha_2^g & \cdots & \alpha_{|\Omega|}^g \end{pmatrix} \circ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^h & \alpha_2^h & \cdots & \alpha_{|\Omega|}^h \end{pmatrix} \circ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^g & \alpha_2^g & \cdots & \alpha_{|\Omega|}^g \end{pmatrix}^{-I}$$
$$\circ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^h & \alpha_2^h & \cdots & \alpha_{|\Omega|}^h \end{pmatrix}^{-I} = \{\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^f & \alpha_2^f & \cdots & \alpha_{|\Omega|}^f \end{pmatrix} |f \in g \cdot h \cdot g^{-1} \cdot h^{-1}\}.$$

It is easy to see that $A_\Omega = \{\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^g & \alpha_2^g & \cdots & \alpha_{|\Omega|}^g \end{pmatrix} |g \in P'\}$ □

Theorem 4.13. $A_\Omega \trianglelefteq S_\Omega$.

Proof. Suppose that
$x = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^a & \alpha_2^a & \cdots & \alpha_{|\Omega|}^a \end{pmatrix} \in S_\Omega$ and $y = \begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_{|\Omega|} \\ \beta_1^b & \beta_2^b & \cdots & \beta_{|\Omega|}^b \end{pmatrix} \in A_\Omega$, where $a \in$

P and $b \in P'$. We have $x^{-1}yx = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^{a^{-1}} & \alpha_2^{a^{-1}} & \cdots & \alpha_{|\Omega|}^{a^{-1}} \end{pmatrix} \circ \begin{pmatrix} \beta_1 & \beta_2 & \cdots & \beta_{|\Omega|} \\ \beta_1^{b} & \beta_2^{b} & \cdots & \beta_{|\Omega|}^{b} \end{pmatrix} \circ$ $\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^{a} & \alpha_2^{a} & \cdots & \alpha_{|\Omega|}^{a} \end{pmatrix} = \{ \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{|\Omega|} \\ \alpha_1^{g} & \alpha_2^{g} & \cdots & \alpha_{|\Omega|}^{g} \end{pmatrix} | g \in a^{-1}ba \}$.

Since $a^{-1}ba \subseteq bb^{-1}a^{-1}ba \subseteq P'$, hence $x^{-1}yx \subseteq A_\Omega$. Therefore, $x^{-1}A_\Omega x \subseteq A_\Omega$. □

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