PRESERVING INJECTIVE PROPERTIES OF ACTS OVER MONOIDS UNDER LIMITS AND THEIR TRANSFER FROM COLIMITS TO THE COMPONENTS

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Abstract. This paper is devoted to the preservation of injective properties under limits and their transfer from colimits to the components. We prove that an injective property α is preserved under limits if and only if all acts satisfy property α. Besides we prove that an injective property α is transferred from colimits to their components if and only if all acts satisfy property α.
Keywords: Limit, colimit, product, coproduct, injective act.

1. Introduction and preliminaries

Throughout this paper, unless stated otherwise, S stands for a monoid and 1 denotes its identity element. A set A together with a mapping $A \times S \to A$, $(a, s) \rightsquigarrow as$, is called a right S-act or simply an act (and is denoted by A_S) if $a(st) = (as)t$ and $a1 = a$ for all $a \in A$, $s, t \in S$. For right S-acts A_S and B_S a mapping from A to B preserving S-action is called an act homomorphism. The class of right S-acts together with act homomorphisms as morphisms form a category denoted by **Act-**S. We mean by $A \sqcup B$ the disjoint union of sets A and B. The one element act is called zero act and is denoted by $\Theta_S = \{\theta\}$. A right S-act A_S is called decomposable provided that there exist subacts $B_S, C_S \subseteq A_S$ such that $A_S = B_S \cup C_S$ and $B_S \cap C_S = \emptyset$. In this case $A_S = B_S \cup C_S$ is called a decomposition of A_S. Otherwise A_S is called indecomposable. It is well-known that every S-act A_S has a unique decompo-

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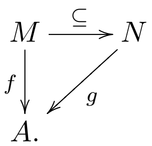
sition into indecomposable subacts. Indeed, indecomposable components of A_S are precisely the equivalence classes of the relation $\sim$ on A_S defined by $a \sim b$ if there exist $s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_n \in S,\ a_1, a_2, \ldots, a_n \in A_S$ such that

$$a = a_1 s_1,\ a_1 t_1 = a_2 s_2,\ a_2 t_2 = a_3 s_3, \ldots,\ a_n t_n = b$$

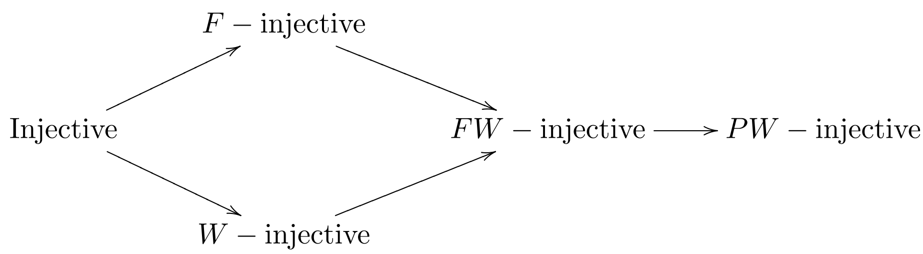
(see [6]). Therefore, elements $a, b \in A_S$ are in the same indecomposable component if and only if there exists a sequence of equalities of length n as above connecting a to b. We refer the reader to [1] and [5] for more details on the concepts mentioned in this paper.

A great deal of works has been done on the concept of injectivity relative to various classes of monomorphisms in the category of S-acts for a monoid S such as [2, 3, 4, 7, 8]. Hereby, this work mostly concentrates on the preservation of the notions C-injectivity and CC-injectivity under limits and colimits and their transfer from limits and colimits to the components in this category.

Recall that a right S-act A is called injective if for any S-act N, any subact M of N, and any homomorphism $f \in \mathrm{Hom}(M, A)$, there exists a homomorphism $g \in \mathrm{Hom}(N, A)$ making the following diagram commutative, i.e., $g\mid_M = f$,



In the diagram if M is finitely generated then A is called F-injective, if N is S and M is a (principal, finitely generated) right ideal of N then A is called W-injective (PW-injective, FW-injective), if M is cyclic then A is called C-injective and if both M and N are cyclic then A is called CC-injective. Indeed for a class $\mathfrak{M}$ of monomorphisms in **Act-**S, A is called $\mathfrak{M}$-injective if in the above diagram the inclusion mapping $\subseteq$, can be replaced by all monomorphisms in $\mathfrak{M}$. Note that in this paper we mean by injective properties all the mentioned properties. Visualizing relations between these notions we have the following strict implications:



2. Limits of injective properties

In this section we investigate conditions under which limit of a family of acts, satisfying an injective property, satisfies the same injective property. Recall that a category **A** is said to be complete (cocomplete) if for each small diagram in **A** there exists a limit (colimit). The next well-known theorem in Category Theory presents a crucial setting on limit and colimit preservation (see for example [1], p. 211):

Theorem 2.1. *For each category* **A** *the following conditions are equivalent:*

i) **A** *is complete,*

ii) **A** *has products and equalizers,*

iii) **A** *has products and finite intersections.*

In order to employ the proceeding theorem for a monoid S, we wish to consider the category $\overline{\mathbf{Act}} - S$ consisting of all right S-acts together with the empty set as an object for ensuring the existence of equalizer of parallel morphisms as $A_S \underset{g}{\overset{f}{\rightrightarrows}} B_S$.

Hereby, to reach our target we engage in the problem of preservation of injective properties under products, equalizers and their duals.

Injective properties for which acts satisfying them posses a zero element are transferred from products to their components. Indeed if $\{A_i \mid i \in I\}$ is a family of right S-acts for which each A_i, $i \in I$ has a zero element then A_i is a retract of $\prod_{i\in I} A_i$. Note that CC-injective acts do not have necessarily zero elements. Therefore the problem of transferring CC-injectivity from products to their components remains open.

Regarding the problem of injective properties preservation under products, an adaption of [5, Proposition 3.1.12] yields the following theorem.

Theorem 2.2. *All injective properties are preserved under products.*

In the next theorem we give conditions under which equalizers of parallel morphisms of acts satisfying injective properties are injective.

Proposition 2.3. *Suppose that α stands for an injective property. Equalizers of parallel morphisms of acts satisfying property α satisfy property α if and only if all right S-acts satisfy property α.*

Proof. We just need to prove the necessity part. Let A_S be a right S-act and let $E(A_S)$ be its injective envelope. Take the parallel morphisms

$$E(A_S) \underset{g}{\overset{f}{\rightrightarrows}} E(E(A_S) \coprod^{A_S} E(A_S)))$$

where

$$E(A_S) \coprod^{A_S} E(A_S) = \{(x,1) \mid x \in E(A_S) \backslash A_S\} \cup A_S \cup \{(x,2) \mid x \in E(A_S) \backslash A_S\},$$

with the operation of S defined by

$$(x,n)s = \begin{cases} (xs,n), & \text{if } xs \notin A_S \\ xs, & \text{otherwise} \end{cases}$$

for every $x \in E(A_S)\backslash A_S$, $s \in S$ and $n \in \{1,2\}$,

$$f(x) = \begin{cases} (x,1), & \text{if } x \in E(A_S)\backslash A_S \\ x, & \text{if } x \in A_S \end{cases}$$

and

$$g(x) = \begin{cases} (x,2), & \text{if } x \in E(A_S)\backslash A_S \\ x, & \text{if } x \in A_S. \end{cases}$$

Now it is clear that A_S together with the inclusion map is the equalizer of the parallel pair (f,g) and hence is injective. □

The next theorem is a result of Theorems 2.1, 2.2 and Proposition 2.3.

Theorem 2.4. *For a monoid S an injective property α is preserved under limits if and only if all right S-acts satisfy property α.*

3. Colimits of injective properties

In this section we provide conditions under which injective properties are preserved under colimits. To establish the goal we need to concentrate on coproducts and coequalizers of injective properties.

A monoid S is called *left reversible* if every two right ideals of S have a non-empty intersection, that is, $aS \cap bS \neq \emptyset$, for each $a, b \in S$.

It is known that coproducts of injective (F-injective, W-injective, FW-injective, PW-injective) acts are injective (F-injective, W-injective, FW-injective, PW-injective) if and only if S is left reversible.

The preservation of the above injective properties under coequalizers remains an open problem, though it is clear that preservation of such injective properties under quotient is a sufficient condition for the problem.

The case of C-injectivity. It can be routinely checked that any C-injective act contains a zero element and retracts of C-injective acts are C-injective. The next proposition is a straight forward result of the definition of C-injectivity and CC-injectivity.

Proposition 3.1. *Coproducts of C-injective (CC-injective) acts are C-injective (CC-injective).*

In light of the dual of Theorem 2.1 the following theorem is obtained.

Theorem 3.2. *Colimits of C-injective (CC-injective) acts are C-injective (CC-injective) if and only if coequalizer of any parallel morphisms as $A_S \overset{f}{\underset{g}{\rightrightarrows}} B_S$ is C-injective (CC-injective) for C-injective (CC-injective) acts A_S and B_S.*

Our next target is providing equivalent conditions on monoids for transferring C-injective and CC-injective properties from coproducts to their components. We need some ingredients to establish the results.

Proposition 3.3. *Let S be a left reversible monoid. Then a right S-act A_S is indecomposable if and only if for any $a, a' \in A_S$ there exist $s, s' \in S$ such that $as = a's'$.*

Proof. Let S be a left reversible monoid. Suppose that A_S is indecomposable and $a, a' \in A_S$. So there exists a sequence of equalities, connecting a to a', of the form

$$a = a_1s_1,\ a_1t_1 = a_2s_2,\ a_2t_2 = a_3s_3,\ \ldots\ a_nt_n = a',$$

for $a_i \in A_S,\ s_i, t_i \in S,\ 1 \leqslant i \leqslant n$. Left reversibility of S provides $u_1, u_2 \in S$ such that $s_1u_1 = t_1v_1$ and in consequence $au_1 = a_1s_1u_1 = a_1t_1v_1 = a_2s_2v_1$. Proceeding inductively, we get $u, v \in S$ providing $au = a_nt_nv = a'v$, as desired.

Conversely, it is obvious. □

Proposition 3.4. *For a monoid S all subacts of indecomposable right S-acts are indecomposable if and only if S is left reversible.*

Proof. Necessity. Let $a, b \in S$. Since S is indecomposable, our assumption implies that $aS \cup bS$ is indecomposable and therefore $aS \cap bS \neq \emptyset$.

Sufficiency. This is a straightforward application of the Proposition 3.3. □

Recall that for a nonempty set I, I^S is an $|I|$-cofree right S-act where fs for $f \in I^S,\ s \in S$ is defined by $fs(t) = f(st)$ for every $t \in S$. It should be mentioned that the 1-cofree object or terminal object in **Act-**S is the one element act which is indecomposable. The next proposition characterizes monoids over which non-zero cofree acts are decomposable.

Proposition 3.5. *For a monoid S the following are equivalent:*

i) all non-zero cofree S-acts are decomposable,

ii) there exists a decomposable cofree right S-act,

iii) S is left reversible.

Proof. $i \implies ii$ is clear. $ii \implies iii$. By way of contradiction suppose that $aS \cap bS = \emptyset$ for some $a, b \in S$. Let X^S be a decomposable $|X|$-cofree act and $f, g \in X^S$. Let $h \in X^S$ be given by

$$h(x) = \begin{cases} f(x), & \text{if } x \in aS, \\ g(x), & \text{otherwise.} \end{cases}$$

So we get the sequence $f = f.1$, $fa = ha$, $hb = gb$, $g.1 = g$, which implies that f and g are in the same indecomposable component. Therefore X^S is indecomposable a contradiction.

iii $\Longrightarrow$ *i*. Let S be a left reversible monoid and X^S be a non-zero cofree S-act. Take constant functions $f = c_{x_1}$ and $g = c_{x_2}$ in X^S for different elements x_1 and x_2 in X. Then f and g are zero elements of X^S (note that zero elements of X^S are the same constant functions). If f and g are in the same indecomposable component, in light of Proposition 3.3, there exist $a, b \in S$ such that $fa = gb$ and by this we obtain $f = g$, a contradiction. □

Proposition 3.6. *Let S be a monoid. The C-injective property is transferred from coproducts to their components if and only if S is not left reversible or contains a left zero.*

Proof. *Necessity.* Let S be a left reversible monoid for which C-injective property is transferred from coproducts to their components. Since S is left reversible using Proposition 3.5 the cofree right $S-$act S^S is decomposable. Suppose $\coprod_{i\in I} Q_i$ is its unique decomposition into indecomposable acts. Let $\mathrm{id}_S \in Q_{i_0}$, for some $i_0 \in I$. Our assumption necessitates the existence of a zero element in Q_{i_0} namely f. Then in account of Proposition 3.4 there exists $s, t \in S$ such that $\mathrm{id}_S t = fs = f$. So $\mathrm{id}_S t$ is a zero element and hence for each $x \in S, \mathrm{id}_S t(x) = \mathrm{id}_S(t)$, which implies $tx = t$, for each $x \in S$. So t is a left zero in S, as desired.

Sufficiency: Suppose that $\{Q_i \,|\, i \in I\}$ is a family of right S-acts for which $Q_S = \coprod_{i\in I} Q_i$ is C-injective. If S contains a left zero then all right S-acts have zeros element and hence regarding [7, Proposition 8] Q_i is injective for any $i \in I$. If S is not left reversible suppose that $aS \cap bS = \emptyset$ for some $a, b \in S$. For $i \in I$ and $q \in Q_i$ consider the homomorphism $f : aS \to Q_S$ given by $f(as) = qas$ for each $s \in S$. Taking aS as a subact of the Rees factor act S/bS, f can be extended to a morphism from S/bS and hence Q_i contains a zero element. Thus Q_i is a retract of the C-injective act Q_S and is consequently C-injective. □

Note that the above proposition is a generalized version of Proposition 8 in [7].

Considering the fact that homomorphic images of indecomposable acts are indecomposable we have the next proposition.

Proposition 3.7. *Let S be a monoid. The CC-injective property is transferred from coproducts to their components.*

Proposition 3.8. *Let S be a monoid and α be an injective property. The following are equivalent:*

i) For any parallel morphisms as $A_S \overset{f}{\underset{g}{\rightrightarrows}} B_S$, if their coequalizer satisfies property α then A_S and B_S satisfy property α, *ii*) *All right S-acts satisfy property α.*

Proof. Clearly we just need to prove the necessity part. Let A_S be a right S-act. Take the parallel morphisms as $A_S \overset{\subseteq}{\underset{\subseteq}{\rightrightarrows}} E(A_S)$ whereas $E(A_S)$ is injective envelope of A_S and $\subseteq$ is the inclusion homomorphism from A_S to $E(A_S)$. Therefore their coequalizer is $E(A_S)/\rho$ together with the canonical homomorphism $\pi_\rho : E(A_S) \to E(A_S)/\rho$ where ρ is the right congruence on A_S generated by all the pairs (a, a), $a \in A$ which yields $\rho = \Delta_{E(A_S)}$ and hence $E(A_S)/\rho$ is isomorphic to $E(A_S)$. Since $E(A_S)$ is injective then it satisfies property α and by assumption A_S satisfies property α. □

The next theorem is a result of the above proposition and Theorem 2.1.

Theorem 3.9. *For a monoid S an injective property α transferred from colimits to their components if and only if all right S-acts satisfy property α.*

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