WEAK CLOSURE OPERATIONS WITH SPECIAL TYPES IN LOWER BCK-SEMILATTICES

Dapeng Yu

Hashem Bordbar[1]

Faculty of Mathematics
Statistics and Computer Science
Shahid Bahonar University
Kerman
Iran
e-mail: bordbar.amirh@gmail.com

Mohammad Mehdi Zahedi

Department of Mathematics
Graduate University of Advanced Technology
Mahan-Kerman
Iran
e-mail: zahedi_mm@kgut.ac.ir

Young Bae Jun

Department of Mathematics Education (and RINS)
Gyeongsang National University
Jinju 52828
Korea
e-mail: skywine@gmail.com

Abstract. The notions of (strong) quasi prime mapping on the set of all ideals, t-type weak closure operation, and tender (resp., naive, sheer, feeble tender) weak closure operation are introduced, and their relations and properties are investigated. Conditions for a weak closure operation to be of t-type are provided. Given a weak closure operation, conditions for the new weak closure operation to be of t-type and to be a naive (sheer, feeble tender) weak closure operation are considered. We show that the new weak closure operation is the smallest tender weak closure operation containing the given weak closure operation.

Keywords: (strong) quasi prime mapping, t-type weak closure operation, naive (sheer, tender, feeble tender) weak closure operation.

2010 Mathematics Subject Classification: 06F35, 03G25.

---

[1]Corresponding author

1. Introduction

In [4], Bordbar et al. introduced a weak closure operation, which is more general form than closure operation, on ideals of BCK-algebras. Bordbar and Zahedi [2], [3] studied a finite type closure operations and semi-prime closure operations on BCK-algebras. Regarding weak closure operation "*cl*", they defined another weak closure operation "cl_t" in [1].

In this paper, we introduce the notions of (strong) quasi prime mapping on the set of all ideals, t-type weak closure operation, and tender (resp., naive, sheer, feeble tender) weak closure operation, and investigates their relations and properties. We provide conditions for a weak closure operation to be of t-type.

We consider conditions for "cl_t" to be a t-type weak closure operation.

We discuss conditions for "cl_t" to be a naive (sheer, feeble tender) weak closure operation.

We show that "cl_t" is the smallest tender weak closure operation containing the weak closure operation "*cl*".

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0 \ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra.*

A BCK-algebra X is called a *lower BCK-semilattice* (see [8]) if X is a lower semilattice with respect to the BCK-order.

A subset A of a BCK/BCI-algebra X is called an *ideal* of X (see [8]) if it satisfies

(2.1) $$0 \in A,$$

(2.2) $$(\forall x \in X)\,(\forall y \in A)\,(x * y \in A \ \Rightarrow\ x \in A)\,.$$

Note that every ideal A of a BCK/BCI-algebra X satisfies the following implication (see [8]).

(2.3) $$(\forall x, y \in X)\,(x \leq y,\ y \in A \ \Rightarrow\ x \in A)\,.$$

For any subset A of X, the ideal generated by A is defined to be the intersection of all ideals of X containing A, and it is denoted by $\langle A\rangle$. If A is finite, then we say that $\langle A\rangle$ is *finitely generated ideal* of X (see [8]).

Let $\mathcal{I}(X)$ and $\mathcal{I}_f(X)$ be the set of all ideals of X and the set of all finitely generated ideals of X, respectively.

We refer the reader to the books [7], [8] for further information regarding BCK/BCI-algebras.

3. t-type weak closure operations

In what follows, let X be a lower BCK-semilattice unless otherwise specified.

Definition 3.1. [4] An element x of X is called a *zeromeet element* of X if the condition

$$(\exists\, y \in X \setminus \{0\})\,(x \wedge y = 0)$$

is valid. Otherwise, x is called a *non-zeromeet element* of X.

Denote by $Z(X)$ the set of all zeromeet elements of X, that is,

$$Z(X) = \{x \in X \mid x \wedge y = 0 \text{ for some nonzero element } y \in X\}.$$

Obviously, $0 \in Z(X)$ and if X has the greatest element 1, then $1 \in X \setminus Z(X)$.

Lemma 3.2. [4] *For any $x, y \in X$, if $x, y \notin Z(X)$, then $x \wedge y \notin Z(X)$, that is, the set $X \setminus Z(X)$ is closed under the operation $\wedge$.*

Definition 3.3. [6] For any nonempty subsets A and B of X, we denote

$$A \wedge B := \langle \{a \wedge b \mid a \in A, b \in B\}\rangle$$

which is called the *meet ideal* of X generated by A and B. In this case, we say that the operation "$\wedge$" is a *meet operation.* If $A = \{a\}$, then $\{a\} \wedge B$ is denoted by $a \wedge B$. Also, if $B = \{b\}$, then $A \wedge \{b\}$ is denoted by $A \wedge b$.

Definition 3.4. [5] For any nonempty subsets A and B of X, we define a set

$$(A :_{\wedge} B) := \{x \in X \mid x \wedge B \subseteq A\}$$

which is called the *relative annihilator* of B with respect to A.

For a nonempty subset B of X, consider the following condition:

$$(\forall x, y \in X)(\forall b \in B)\,((x \wedge b) * (y \wedge b) \leq (x * y) \wedge b)\,. \tag{3.1}$$

Lemma 3.5. [5] *If A and B are ideals of X, then the relative annihilator $(A :_{\wedge} B)$ of B with respect to A is an ideal of X.*

Lemma 3.6. [5] *If A is an ideal of X, then $(A :_{\wedge} X) = A$ and $(A :_{\wedge} A) = X$.*

Definition 3.7. [4] A mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ is called a *weak closure operation* on $\mathcal{I}(X)$ if the following conditions are valid.

$$(\forall A \in \mathcal{I}(X))\,(A \subseteq cl(A)), \tag{3.2}$$

$$(\forall A, B \in \mathcal{I}(X))\,(A \subseteq B \;\Rightarrow\; cl(A) \subseteq cl(B)). \tag{3.3}$$

If a weak closure operation $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ satisfies the condition

$$(\forall A \in \mathcal{I}(X))\,(cl(cl(A)) = cl(A)), \tag{3.4}$$

then we say that "cl" is a closure operation on $\mathcal{I}(X)$ (see [2]). In what follows, we use A^{cl} instead of $cl(A)$.

For any mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ and every ideal A of X, let

$$K := \cup\{((a \wedge A)^{cl} :_{\wedge} \langle b\rangle) \mid a, b \in X \setminus Z(X)\}. \tag{3.5}$$

Then the mapping

$$cl^{*} : \mathcal{I}(X) \to \mathcal{I}(X),\;\; A \mapsto \langle K\rangle \tag{3.6}$$

is not a weak closure operation on $\mathcal{I}(X)$ as seen in the following example.

Example 3.8. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	4	0

There are six ideals: $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 1, 2\}$, $A_4 = \{0, 1, 2, 3\}$ and $A_5 = X$.

Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_1$, $A_1^{cl} = A_0$, $A_2^{cl} = A_1$, $A_3^{cl} = A_1$, $A_4^{cl} = A_2$ and $A_5^{cl} = A_3$. Then "cl" is not a weak closure operation on $\mathcal{I}(X)$ because $A_3 \nsubseteq A_1 = A_3^{cl}$.

Note that $Z(X) = \{0, 1, 2\}$. For non-zeromeet elements $3, 4$ of X, we have
$((3 \wedge A_2)^{cl} :_{\wedge} \langle 3\rangle) = (A_2^{cl} :_{\wedge} A_4) = (A_1 :_{\wedge} A_4) = A_1$,
$((3 \wedge A_2)^{cl} :_{\wedge} \langle 4\rangle) = (A_2^{cl} :_{\wedge} A_5) = (A_1 :_{\wedge} A_5) = A_1$,
$((4 \wedge A_2)^{cl} :_{\wedge} \langle 3\rangle) = (A_2^{cl} :_{\wedge} A_4) = (A_1 :_{\wedge} A_4) = A_1$,
$((4 \wedge A_2)^{cl} :_{\wedge} \langle 4\rangle) = (A_2^{cl} :_{\wedge} A_5) = (A_1 :_{\wedge} A_5) = A_1$.
It follows that

$$A_2^{cl^{*}} = \langle \cup\{((a \wedge A_2)^{cl} :_{\wedge} \langle b\rangle) \mid a, b \in X \setminus Z(X)\}\rangle = \langle A_1\rangle = A_1 \nsupseteq A_2$$

which shows that "cl^{*}" is not a weak closure operation on $\mathcal{I}(X)$.

If "cl" is a weak closure operation on $\mathcal{I}(X)$, then K in (3.5) is an ideal of X containing A^{cl} (see [1, Theorem 3.28]).

Assume that X has the greatest element 1. For a weak closure operation "cl" on $\mathcal{I}(X)$, we define a new function

$$cl_t : \mathcal{I}(X) \to \mathcal{I}(X),\ A \mapsto \cup\{((a \wedge A)^{cl} :_\wedge \langle b\rangle) \mid a, b \in X \setminus Z(X)\}. \tag{3.7}$$

Then "cl_t" is also a weak closure operation on $\mathcal{I}(X)$ (see [1, Theorem 3.29]).

We investigate relations between "cl" and "cl_t". The following example shows that they are not equal, that is, there exists $A \in \mathcal{I}(X)$ such that $A^{cl} \neq A^{cl_t}$.

Example 3.9. Consider the lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ which is given in Example 3.8. Note that the element 4 is the greatest element of X and we have 6 ideals, $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 1, 2\}$, $A_4 = \{0, 1, 2, 3\}$ and $A_5 = X$.

Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_1$, $A_1^{cl} = A_3$, $A_2^{cl} = A_3$, $A_3^{cl} = A_4$, $A_4^{cl} = A_4$ and $A_5^{cl} = A_5$. Then "cl" is a weak closure operation on $\mathcal{I}(X)$.

Note that $Z(X) = \{0, 1, 2\}$. For non-zeromeet element 3 of X, we have

$$((3 \wedge A_3)^{cl} :_\wedge \langle 3\rangle) = (A_3^{cl} :_\wedge \{0, 1, 2, 3\}) = (A_4 :_\wedge A_4) = X.$$

Thus $A_3^{cl_t} = \cup\{((a \wedge A_3)^{cl} :_\wedge \langle b\rangle) \mid a, b \in X \setminus Z(X)\} = X$. Therefore

$$A_3^{cl} = A_4 \neq X = A_3^{cl_t}.$$

Proposition 3.10. *Assume that X has the greatest element 1. If "cl" is a weak closure operation on $\mathcal{I}(X)$, then "cl" is contained in "cl_t", that is, $A^{cl} \subseteq A^{cl_t}$ for all $A \in \mathcal{I}(X)$.*

Proof. Suppose that $x \in A^{cl}$. Since $1 \wedge A = A$ and $\langle 1\rangle = X$, we have

$$A^{cl} = ((1 \wedge A)^{cl} :_\wedge \langle 1\rangle) \subseteq A^{cl_t}.$$

by Lemma 3.6. Therefore, $x \in A^{cl_t}$ and $A^{cl} \subseteq A^{cl_t}$ for all $A \in \mathcal{I}(X)$. ■

Definition 3.11. Assume that X has the greatest element 1. A weak closure operation "cl" on $\mathcal{I}(X)$ is said to be of *t-type* if the following assertion is valid.

$$(\forall A \in \mathcal{I}(X)) \left(A^{cl} = A^{cl_t}\right). \tag{3.8}$$

Example 3.12. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	2	1	0	3
4	4	4	4	4	0

The element 4 is the greatest element of X and we have 5 ideals: $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 1, 2, 3\}$ and $A_4 = X$.

Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_0$, $A_1^{cl} = A_1$, $A_2^{cl} = A_2$, $A_3^{cl} = A_4$ and $A_4^{cl} = A_4$. Then "cl" is a weak closure operation on $\mathcal{I}(X)$.

Note that $Z(X) = \{0, 1, 2\}$. For non-zeromeet elements 3 and 4 of X, we have $\langle 3\rangle = A_3$ and $\langle 4\rangle = A_4$. Also,

$((3 \wedge A_0)^{cl} :_\wedge \langle 3\rangle) = (A_0^{cl} :_\wedge A_3) = (A_0 :_\wedge A_3) = A_0$.
$((3 \wedge A_0)^{cl} :_\wedge \langle 4\rangle) = (A_0^{cl} :_\wedge A_4) = (A_0 :_\wedge A_4) = A_0$.
$((4 \wedge A_0)^{cl} :_\wedge \langle 3\rangle) = (A_0^{cl} :_\wedge A_3) = (A_0 :_\wedge A_3) = A_0$.
$((4 \wedge A_0)^{cl} :_\wedge \langle 4\rangle) = (A_0^{cl} :_\wedge A_4) = (A_0 :_\wedge A_4) = A_0$.

Hence $A_0^{cl_t} = A_0^{cl}$. Similarly

$((3 \wedge A_1)^{cl} :_\wedge \langle 3\rangle) = (A_1^{cl} :_\wedge A_3) = (A_1 :_\wedge A_3) = A_1$.
$((3 \wedge A_1)^{cl} :_\wedge \langle 4\rangle) = (A_1^{cl} :_\wedge A_4) = (A_1 :_\wedge A_4) = A_1$.
$((4 \wedge A_1)^{cl} :_\wedge \langle 3\rangle) = (A_1^{cl} :_\wedge A_3) = (A_1 :_\wedge A_3) = A_1$.
$((4 \wedge A_1)^{cl} :_\wedge \langle 4\rangle) = (A_1^{cl} :_\wedge A_4) = (A_1 :_\wedge A_4) = A_1$.

Thus $A_1^{cl_t} = A_1^{cl}$. By the similar way, we have

$$A_i^{cl_t} = A_i^{cl}, \ i = \{2, 3, 4\}.$$

Therefore "cl" is a t-type weak closure operation on $\mathcal{I}(X)$.

Given a weak closure operation "cl" on $\mathcal{I}(X)$, we discuss conditions for "cl" to be of t-type.

Theorem 3.13. *Assume that X has the greatest element 1. If the greatest element 1 is the only non-zeromeet element of X, then every weak closure operation on $\mathcal{I}(X)$ is of t-type.*

Proof. Let "cl" be a weak closure operation on $\mathcal{I}(X)$. For any $A \in \mathcal{I}(X)$, we have $1 \wedge A = A$ and $\langle 1\rangle = X$. It follows from Lemma 3.6 that

$$\begin{aligned} A^{cl_t} &= \cup\{((a \wedge A)^{cl} :_\wedge \langle b\rangle) \mid a, b \in X \setminus Z(X)\} \\ &= ((1 \wedge A)^{cl} :_\wedge \langle 1\rangle) = (A^{cl} :_\wedge X) = A^{cl}. \end{aligned}$$

Therefore "cl" is a t-type weak closure operation on $\mathcal{I}(X)$. ∎

Definition 3.14. A mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ is said to be

- *quasi-prime* if it satisfies:

(3.9) $$(\forall a \in X \setminus Z(X))\, (\forall A \in \mathcal{I}(X))\, \left(a \wedge A^{cl} \subseteq (a \wedge A)^{cl}\right).$$

- *strong quasi-prime* if it satisfies:

(3.10) $$(\forall a \in X \setminus Z(X))\, (\forall A \in \mathcal{I}(X))\, \left(a \wedge A^{cl} = (a \wedge A)^{cl}\right).$$

Example 3.15. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}(B_{5-1-2})$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	0	0
3	3	3	3	0	3
4	4	4	4	4	0

There are five ideals: $A_0 = \{0\}$, $A_1 = \{0, 1, 2\}$, $A_2 = \{0, 1, 2, 3\}$, $A_3 = \{0, 1, 2, 4\}$ and $A_4 = X$.

Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_0$, $A_1^{cl} = A_2$, $A_2^{cl} = A_3$, $A_3^{cl} = A_4$ and $A_4^{cl} = A_4$. It is routine to verify that "cl" is a quasi-prime mapping. But it is not a weak closure operation on $\mathcal{I}(X)$ since $A_2 \nsubseteq A_3 = A_2^{cl}$.

Lemma 3.16. *Every ideal A of X satisfies the following assertion.*

$$(\forall a, b, z \in X)\,(a \wedge b \in A \;\Rightarrow\; a \wedge \langle b \wedge z\rangle \subseteq A)\,. \tag{3.11}$$

Proof. Let $p \in \langle b \wedge z\rangle$. Then $p * (b \wedge z)^n = 0$ for some $n \in \mathbb{N}$. Since $b \wedge z \leq b$, we have $(b \wedge z)^n \leq b$, which implies that

$$p * b \leq p * (b \wedge z)^n = 0.$$

Hence $p * b = 0$, that is, $p \leq b$. It follows that

$$a \wedge p \leq a \wedge b \in A$$

and so that $a \wedge p \in A$. Therefore $a \wedge \langle b \wedge z\rangle \subseteq A$. ■

Theorem 3.17. *Assume that X has the greatest element 1. If "cl" is a quasi-prime weak closure operation on $\mathcal{I}(X)$, then "cl_t" is a t-type weak closure operation on $\mathcal{I}(X)$.*

Proof. Note that "cl_t" is a weak closure operation on $\mathcal{I}(X)$. Let $x \in A^{cl_t}$. Then $x \in ((a \wedge A)^{cl} :_\wedge \langle b\rangle)$, and so $x \wedge \langle b\rangle \subseteq (a \wedge A)^{cl}$ for some $a, b \in X \setminus Z(X)$ by (3.7). It follows from the quasi-primeness of "cl" that

$$x \wedge c \wedge z \in (a \wedge A)^{cl} \wedge z = z \wedge (a \wedge A)^{cl} \subseteq (z \wedge a \wedge A)^{cl} = (z \wedge (a \wedge A))^{cl}$$

for all $c, z \in X \setminus Z(X)$. Thus $x \wedge z \in ((z \wedge (a \wedge A))^{cl} :_\wedge c)$, and so

$$x \wedge z \in ((z \wedge (a \wedge A))^{cl} :_\wedge \langle b\rangle) \subseteq (a \wedge A)^{cl_t}. \tag{3.12}$$

Now suppose that $w \in X \setminus Z(X)$. Then $z \wedge w \in X \setminus Z(X)$ by Lemma 3.2. Using Lemma 3.16 and (3.12) induces $x \wedge \langle z \wedge w\rangle \subseteq (a \wedge A)^{cl_t}$, and thus

$$x \in ((a \wedge A)^{cl_t} :_\wedge \langle z \wedge w\rangle) \subseteq \cup\{((a \wedge A)^{cl_t} :_\wedge \langle b\rangle) \mid a, b \in X \setminus Z(X)\}.$$

Conversely, suppose that $x \in A^{(cl_t)_t}$. Then $x \in ((a \wedge A)^{cl_t} :_\wedge \langle b \rangle)$ for some $a, b \in X \setminus Z(X)$. Then $x \wedge z \in (a \wedge A)^{cl_t}$ for all $z \in \langle b \rangle$. It follows from (3.7) that there exist $p, q \in X \setminus Z(X)$ such that

$$x \wedge \langle b \rangle \subseteq ((p \wedge (a \wedge A))^{cl} :_\wedge \langle q \rangle).$$

Thus $x \wedge \langle b \wedge q \rangle \subseteq x \wedge \langle b \rangle \wedge \langle q \rangle \subseteq (p \wedge (a \wedge A))^{cl}$, which implies that

$$x \in ((p \wedge a \wedge A)^{cl} :_\wedge \langle b \wedge q \rangle)$$

Since $p \wedge a$ and $b \wedge q$ are elements of $X \setminus Z(X)$ by Lemma 3.2, we conclude that $x \in A^{cl_t}$. Consequently, "cl_t" is a t-type weak closure operation on $\mathcal{I}(X)$. ∎

Corollary 3.18. *Assume that X has the greatest element 1. If "cl" is a strong quasi-prime weak closure operation on $\mathcal{I}(X)$, then "cl_t" is a t-type weak closure operation on $\mathcal{I}(X)$.*

Definition 3.19. A weak closure operation "*cl*" on $\mathcal{I}(X)$ is said to be

• *tender* if for any $A \in \mathcal{I}(X)$ and any non-zeromeet elements a and b of X, the equality

$$((a \wedge A)^{cl} :_\wedge \langle b \rangle) = A^{cl} \tag{3.13}$$

is valid,

• *feeble tender* if for any $A \in \mathcal{I}(X)$ and any non-zeromeet element a of X, the equality

$$((a \wedge A)^{cl} :_\wedge \langle a \rangle) = A^{cl} \tag{3.14}$$

is valid,

• *naive* if for any $A \in \mathcal{I}(X)$ there exist non-zeromeet elements a and b of X such that the equality (3.13) is valid.

• *sheer* if for any $A \in \mathcal{I}(X)$ there exists non-zeromeet element a of X such that the equality (3.14) is valid.

Example 3.20. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	4	0

X has 6 ideals: $A_0 = \{0\}$, $A_1 = \{0, 1\}$, $A_2 = \{0, 2\}$, $A_3 = \{0, 1, 2\}$, $A_4 = \{0, 1, 2, 3\}$ and $A_5 = X$.

Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_0$, $A_1^{cl} = A_3$, $A_2^{cl} = A_3$, $A_3^{cl} = A_3$, $A_4^{cl} = X$ and $A_5^{cl} = X$. Then "cl" is a weak closure operation on $\mathcal{I}(X)$.

Note that $Z(X) = \{0, 1, 2\}$. For non-zeromeet elements 3 and 4 of X, we have $\langle 3\rangle = A_4$ and $\langle 4\rangle = X$. Also,

$((3 \wedge A_0)^{cl} :_\wedge \langle 3\rangle) = (A_0^{cl} :_\wedge A_4) = (A_0 :_\wedge A_4) = A_0 = A_0^{cl}$.
$((4 \wedge A_0)^{cl} :_\wedge \langle 4\rangle) = (A_0^{cl} :_\wedge A_5) = (A_0 :_\wedge A_5) = A_0 = A_0^{cl}$.
$((3 \wedge A_1)^{cl} :_\wedge \langle 3\rangle) = (A_1^{cl} :_\wedge A_4) = (A_3 :_\wedge A_4) = A_3 = A_1^{cl}$.
$((4 \wedge A_1)^{cl} :_\wedge \langle 4\rangle) = (A_1^{cl} :_\wedge A_5) = (A_3 :_\wedge A_5) = A_3 = A_1^{cl}$.
$((3 \wedge A_2)^{cl} :_\wedge \langle 3\rangle) = (A_2^{cl} :_\wedge A_4) = (A_3 :_\wedge A_4) = A_3 = A_2^{cl}$.
$((4 \wedge A_2)^{cl} :_\wedge \langle 4\rangle) = (A_2^{cl} :_\wedge A_5) = (A_3 :_\wedge A_5) = A_3 = A_2^{cl}$.
$((3 \wedge A_3)^{cl} :_\wedge \langle 3\rangle) = (A_3^{cl} :_\wedge A_4) = (A_3 :_\wedge A_4) = A_3 = A_3^{cl}$.
$((4 \wedge A_3)^{cl} :_\wedge \langle 4\rangle) = (A_3^{cl} :_\wedge A_5) = (A_3 :_\wedge A_5) = A_3 = A_3^{cl}$.
$((3 \wedge A_4)^{cl} :_\wedge \langle 3\rangle) = (A_4^{cl} :_\wedge A_4) = (A_5 :_\wedge A_4) = A_5 = A_4^{cl}$.
$((4 \wedge A_4)^{cl} :_\wedge \langle 4\rangle) = (A_4^{cl} :_\wedge A_5) = (A_5 :_\wedge A_5) = A_5 = A_4^{cl}$.
$((3 \wedge A_5)^{cl} :_\wedge \langle 3\rangle) = (A_4^{cl} :_\wedge A_4) = (A_5 :_\wedge A_4) = A_5 = A_5^{cl}$.
$((4 \wedge A_5)^{cl} :_\wedge \langle 4\rangle) = (A_5^{cl} :_\wedge A_5) = (A_5 :_\wedge A_5) = A_5 = A_5^{cl}$.

Thus "cl" is a (feeble) tender weak closure operation on $\mathcal{I}(X)$.

Obviously, every tender weak closure operation is a naive weak closure operation, but the converse is not true in general as seen in the following example.

Example 3.21. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3\}$ with the following Cayley table.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	1
3	3	3	3	0

We have 3 ideals of X, and they are $A_0 = \{0\}$, $A_1 = \{0, 1, 2\}$ and $A_2 = X$.

Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_1^{cl} = A_1$ and $A_2^{cl} = A_2$.

We can easily check that "cl" is a naive weak closure operation on $\mathcal{I}(X)$. But, it is not a tender weak closure operation on $\mathcal{I}(X)$. In fact, we know that there are two non-zeromeet elements 2 and 3. Thus

$$((3 \wedge A_1)^{cl} :_\wedge \langle 2\rangle) = (A_1^{cl} :_\wedge \langle 2\rangle) = (A_1 :_\wedge A_1) = X \neq A_1 = A_1^{cl}.$$

Obviously, every feeble tender weak closure operation is a sheer weak closure operation, but the converse is not true in general as seen in the following example.

Example 3.22. Consider a lower BCK-semilattice $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	0
3	3	3	3	0	3
4	4	4	4	4	0

There are six ideals: $A_0 = \{0\}$, $A_1 = \{0,1\}$, $A_2 = \{0,1,2\}$, $A_3 = \{0,1,2,3\}$, $A_4 = \{0,1,2,4\}$ and $A_5 = X$. Define a mapping $cl : \mathcal{I}(X) \to \mathcal{I}(X)$ by $A_0^{cl} = A_0$, $A_1^{cl} = A_4$, $A_2^{cl} = A_4$, $A_3^{cl} = X$, $A_4^{cl} = X$ and $A_5^{cl} = X$. Then "*cl*" is a weak closure operation on $\mathcal{I}(X)$.

Note that $Z(X) = \{0\}$. For non-zeromeet elements 3 and 4 of X, we have $\langle 3\rangle = A_3$ and $\langle 4\rangle = A_4$. Also,

$((3 \wedge A_0)^{cl} :_\wedge \langle 3\rangle) = (A_0^{cl} :_\wedge A_3) = (A_0 :_\wedge A_3) = A_0 = A_0^{cl}$.
$((3 \wedge A_1)^{cl} :_\wedge \langle 3\rangle) = (A_1^{cl} :_\wedge A_3) = (A_4 :_\wedge A_3) = A_4 = A_1^{cl}$.
$((3 \wedge A_2)^{cl} :_\wedge \langle 3\rangle) = (A_2^{cl} :_\wedge A_3) = (A_4 :_\wedge A_3) = A_4 = A_2^{cl}$.
$((4 \wedge A_3)^{cl} :_\wedge \langle 4\rangle) = (A_2^{cl} :_\wedge A_4) = (A_4 :_\wedge A_4) = X = A_3^{cl}$.
$((4 \wedge A_4)^{cl} :_\wedge \langle 4\rangle) = (A_4^{cl} :_\wedge A_4) = (X :_\wedge A_4) = X = A_4^{cl}$.
$((4 \wedge A_5)^{cl} :_\wedge \langle 4\rangle) = (A_4^{cl} :_\wedge A_4) = (X :_\wedge A_4) = X = A_5^{cl}$.

Thus "*cl*" is a sheer weak closure operation. But it is not feeble tender since

$$((3 \wedge A_4)^{cl} :_\wedge \langle 3\rangle) = (A_2^{cl} :_\wedge A_3) = (A_4 :_\wedge A_3) = A_4 \neq X = A_4^{cl}.$$

Theorem 3.23. *Assume that X has the greatest element 1. If "cl" is a quasi-prime weak closure operation on $\mathcal{I}(X)$, then "cl_t" is a naive weak closure operation on $\mathcal{I}(X)$.*

Proof. Note that "cl_t" is a weak closure operation on $\mathcal{I}(X)$. Suppose that A is an ideal of X and $x \in A^{cl_t}$. Then there exist $p, q \in X \setminus Z(X)$ such that $x \in ((p \wedge A)^{cl} :_\wedge \langle q\rangle)$. So $x \wedge \langle q\rangle \subseteq (p \wedge A)^{cl}$. Let $a \in X \setminus Z(X)$. Then

$$a \wedge x \wedge \langle q\rangle \subseteq a \wedge (p \wedge A)^{cl} \subseteq (a \wedge p \wedge A)^{cl}$$

by the quasi-primeness of "*cl*", and thus

$$x \wedge a \in ((p \wedge (a \wedge A))^{cl} :_\wedge \langle q\rangle) \subseteq (a \wedge A)^{cl_t}.$$

It follows from Lemma 3.16 that

$$x \wedge \langle a \wedge b\rangle \subseteq (a \wedge A)^{cl_t}$$

for $b \in X \setminus Z(X)$. Therefore $x \in ((a \wedge A)^{cl_t} :_\wedge \langle a \wedge b\rangle)$ which means that there exist non-zeromeet elements s, t such that $x \in ((s \wedge A)^{cl_t} :_\wedge \langle t\rangle)$.

Conversely, let $x \in ((a \wedge A)^{cl_t} :_\wedge \langle b\rangle)$ for some $a, b \in X \setminus Z(X)$. Then $x \wedge \langle b\rangle \subseteq (a \wedge A)^{cl_t}$, and so there exist $p, q \in X \setminus Z(X)$ such that

$$x \wedge \langle b\rangle \subseteq ((p \wedge (a \wedge A))^{cl} :_\wedge \langle q\rangle).$$

Thus $x \wedge \langle q \wedge b\rangle \subseteq x \wedge \langle q\rangle \wedge \langle b\rangle \subseteq (p \wedge (a \wedge A))^{cl}$, which means that

$$x \in (((p \wedge a) \wedge A)^{cl} :_\wedge \langle q \wedge b\rangle) \subseteq A^{cl_t}.$$

Consequently, "cl_t" is a naive weak closure operation on $\mathcal{I}(X)$. ∎

Corollary 3.24. *Assume that X has the greatest element 1. If "cl" is a strong quasi-prime weak closure operation on $\mathcal{I}(X)$, then "cl_t" is a naive weak closure operation on $\mathcal{I}(X)$.*

Lemma 3.25. [4] *Assume that X has the greatest element 1. If "cl" is a tender weak closure operation on $\mathcal{I}(X)$, then so is the function "cl_t" in* (3.7).

Theorem 3.26. *Assume that X has the greatest element 1. If "cl" is a tender weak closure operation on $\mathcal{I}(X)$, then "cl_t" is the smallest tender weak closure operation on $\mathcal{I}(X)$ such that "cl" is contained in "cl_t", that is, $A^{cl} \subseteq A^{cl_t}$ for all $A \in \mathcal{I}(X)$.*

Proof. By using Proposition 3.10 and Lemma 3.25, "cl_t" is a tender weak closure operation which contains "cl". Now suppose that "cl_1" is a tender weak closure operation which contains "cl". For any $A \in \mathcal{I}(X)$, if $x \in A^{cl_t}$, then $x \in ((p \wedge A)^{cl} :_\wedge \langle q \rangle)$ for some $p, q \in X \setminus Z(X)$. Since $A^{cl} \subseteq A^{cl_1}$ and "cl_1" is a tender weak closure operation, we have

$$x \in ((p \wedge A)^{cl_1} :_\wedge \langle q \rangle) = A^{cl_1},$$

which shows that $A^{cl_t} \subseteq A^{cl_1}$. ∎

Theorem 3.27. *Assume that X has the greatest element 1. If "cl" is a quasi-prime weak closure operation on $\mathcal{I}(X)$, then "cl_t" is a feeble tender weak closure operation on $\mathcal{I}(X)$.*

Proof. Note that "cl_t" is a weak closure operation on $\mathcal{I}(X)$. Suppose that A is an ideal of X and $x \in A^{cl_t}$. Then there exist $p, q \in X \setminus Z(X)$ such that $x \in ((p \wedge A)^{cl} :_\wedge \langle q \rangle)$. So $x \wedge \langle q \rangle \subseteq (p \wedge A)^{cl}$. Let $a \in X \setminus Z(X)$ be an arbitrary element. Using the quasi-primeness of "cl" implies

$$a \wedge x \wedge \langle q \rangle \subseteq a \wedge (p \wedge A)^{cl} \subseteq (a \wedge p \wedge A)^{cl}.$$

Thus $x \wedge a \in ((p \wedge (a \wedge A))^{cl} :_\wedge \langle q \rangle) \subseteq (a \wedge A)^{cl_t}$. It follows from Lemma 3.16 that

$$x \wedge \langle a \rangle = x \wedge \langle a \wedge a \rangle \subseteq (a \wedge A)^{cl_t}$$

and so that $x \in ((a \wedge A)^{cl_t} :_\wedge \langle a \rangle)$ for all $a \in X \setminus Z(X)$

Conversely, let $x \in ((a \wedge A)^{cl_t} :_\wedge \langle a \rangle)$ for $a \in X \setminus Z(X)$. Then $x \wedge z \in (a \wedge A)^{cl_t}$ for every element $z \in \langle a \rangle$. So there exist $p, q \in X \setminus Z(X)$ such that

$$x \wedge z \in ((p \wedge (a \wedge A))^{cl} :_\wedge \langle q \rangle).$$

Hence $x \wedge \langle a \rangle \subseteq ((p \wedge (a \wedge A))^{cl} :_\wedge \langle q \rangle)$, and so $x \wedge \langle q \wedge a \rangle \subseteq x \wedge \langle q \rangle \wedge \langle a \rangle \subseteq (p \wedge (a \wedge A))^{cl}$. Therefore

$$x \in (((p \wedge a) \wedge A)^{cl} :_\wedge \langle q \wedge a \rangle) \subseteq A^{cl_t}.$$

Consequently, "cl_t" is a feeble tender weak closure operation on $\mathcal{I}(X)$. ∎

Corollary 3.28. *Assume that X has the greatest element 1. If "cl" is a quasi-prime weak closure operation on $\mathcal{I}(X)$, then "cl_t" is a sheer weak closure operation on $\mathcal{I}(X)$.*

Corollary 3.29. *Assume that X has the greatest element 1. If "cl" is a strong quasi-prime weak closure operation on $\mathcal{I}(X)$, then "cl_t" is a feeble tender weak closure operation on $\mathcal{I}(X)$ and so a sheer weak closure operation on $\mathcal{I}(X)$.*

References

[1] Bordbar, H., Jun, Y.B., Novak, M., *Tender and naive weak closure operations on lower BCK-semilattices,* Math. Slovaca (submitted).

[2] Bordbar, H., Zahedi, M.M., *A finite type of closure operations on BCK-algebra,* Appl. Math. Inf. Sci. Lett., 4 (2) (2016), 1–9.

[3] Bordbar, H., Zahedi, M.M., *Semi-prime closure operations on BCK-algebra,* Commun. Korean Math. Soc., 30 (5) (2015), 385–402.

[4] Bordbar, H., Zahedi, M.M., Ahn, S.S., Jun, Y.B., *Weak closure operations on ideals of BCK-algebras,* J. Comput. Anal. Appl. (in press).

[5] Bordbar, H., Zahedi, M.M., Jun, Y.B., *Relative annihilators in lower BCK-semilattices,* Demonstratio Mathematica (submitted).

[6] Bordbar, H., Ahn, S.S., Zahedi, M.M., Jun, Y.B., *Semiring structures based on meet and plus ideals in lower BCK-semilattices,* J. Comput. Anal. Appl. (submitted).

[7] Huang, Y., *BCI-algebra,* Science Press, Beijing 2006.

[8] Meng, J., Jun, Y.B., *BCK*-algebras, Kyung Moon Sa Co., Seoul 1994.

Accepted: 08.11.2016