FILTER THEORY OF PSEUDO HOOP-ALGEBRAS

S.Z. Alavi

Department of Mathematics
Islamic Azad University Central Tehran Branch
Tehran
Iran
e-mail: szalavi@yahoo.com

R.A. Borzooei

M. Aaly Kologani

Department of Mathematics
Shahid Beheshti University
Tehran
Iran
e-mails: borzooei@sbu.ac.ir
mona4011@gmail.com

Abstract. In this paper, by considering the notion of pseudo hoop-algebras, which introduced by G. Georgescu et al. in [11] under the name of residuated integral monoids, and pseudo MV-algebras, pseudo Wajsberg-algebras and pseudo-BL algebras arise as particular cases of them, we introduce the notions of some types of filters ((positive) implicative filters, fantastic filters, associative filters) in pseudo hoop-algebras and to investigate their properties. Several characterizations of (positive) implicative, fantastic and associative filters are derived. Finally, the relations among these filters are investigated.

Keywords: pseudo hoop-algebras, filter, (positive) implicative filter, fantastic filter, associative filter.

1. Introduction

In [13], G. Georgescu, L. Leustean and V. Preoteasa presented pseudo hoops which were originally introduced by Bosbach in ([3], [4]) under the name *residuated integral monoids*. The prefix "pseudo" stands for non-commutative or not necessarily commutative type of algebra. It followed naturally after the introduction of pseudo-MV algebras ([10], [11]), pseudo-Wajsberg algebras ([6], [7]) and pseudo-BL algebras ([12], [9], [8]). All the above are non-commutative generalizations of algebras for many-valued logics. Pseudo-hoops are weaker structures, and pseudo-MV, pseudo-Wajsberg, and pseudo-BL algebras arise as particular cases of them. Pseudo hoops are monoids endowed with orders. Moreover, the orders are canonical (actually inverse canonical) they are given by divisibility relations w.r.t.the monoid operation and the orders have residuals.

Now, in this paper, we study some types of filter (implicative, positive implicative, fantastic, associative filters) and investigate definitions that are equivalent to those and we get the relation between them.

2. Preliminaries

In this section, we recollect some definitions and results which will be used in this paper.

Definition 2.1. [13] A *pseudo hoop* is an algebra $(A, \wedge, \odot, \rightarrow, \rightsquigarrow, 1)$ of type $(2, 2, 2, 2, 0)$ such that, for all $x, y, z \in A$,
(PH-1) $(A, \wedge, 1)$ is a $\wedge$-semilattice,
(PH-2) $(A, \odot, 1)$ is a monoid with unit 1,
(PH-3) $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$ if and only if $y \leq x \rightsquigarrow z$,
(PH-4) $(x \rightarrow y) \odot x = x \odot (x \rightsquigarrow y)$. (Divisibility condition)

A pseudo hoop can be thought of as an algebra $(A, \odot, \rightarrow, \rightsquigarrow, 1)$, where for all $x, y \in A$, $x \wedge y = x \odot (x \rightsquigarrow y)$. Then, it is easy to see that $(A, \wedge, 1)$ is a $\wedge$-semilattice.

On Pseudo hoop A, we define $x \leq y$ if and only if $x \rightarrow y = 1$ (this is equivalent to $x \rightsquigarrow y = 1$). Then $\leq$ is a partial order relation on A. If $\odot$ is commutative (equivalent $\rightarrow = \rightsquigarrow$), A is said to be a *hoop*. We say that a Pseudo hoop A is *bounded* if it has an element $0 \in A$ such that $0 \leq x$, for all $x \in A$. We let $x^0 = 1, x^n = x^{n-1} \odot x$, for any $n \in \mathbb{N}$. We define two unary operations, $x^- = x \rightarrow 0$ and $x^\sim = x \rightsquigarrow 0$, for all $x \in A$. If $(x^-)^\sim = (x^\sim)^- = x$, for all $x \in A$, then the bounded Pseudo hoop A is said to have the *pseudo double negation property*, (PDN) for short. (See [13])

Definition 2.2. [14] A *lattice-ordered group* or *ℓ-group* is a group which is also a lattice that satisfies the identities $x(y \wedge z)t = xyt \wedge xzt$ and $x(y \vee z)t = xyt \vee xzt$. Throughout we write $x \leq y$ instead of $x \vee y = y$ or $x \wedge y = x$, and ℓ-group as an abbreviation for lattice-ordered group.

Example 2.3. [13] Let $(G, +, -, 0, \wedge, \vee)$ be an arbitrary ℓ-group. For an arbitrary element $0 \leq u \in G$ define the following operations, on the set $[0, u]$,

$$x \odot y = (x - u + y) \vee 0, \quad x \rightarrow y = (y - x + u) \wedge u \quad and \quad x \rightsquigarrow y = (u - x + y) \wedge u$$

for any $x, y \in [0, u]$. By routine calculation, we can see that $([0, u], \odot, \rightarrow, \rightsquigarrow, u)$ is a bounded Pseudo hoop.

Example 2.4. [13] Let $G = (G, +, -, 0, \vee, \wedge)$ be an arbitrary ℓ-group and $N(G)$ be the negative cone of G, that is $N(G) = a \in G \mid a \leq 0$. On $N(G)$ we define the following operations:

$$a \odot b = a + b, \quad a \rightarrow b = (b - a) \wedge 0 \quad and \quad a \rightsquigarrow b = (-a + b) \wedge 0$$

Then $N(G) = (N(G), \odot, \rightarrow, \rightsquigarrow, 0)$ is an unbounded pseudo hoop. The following proposition provide some properties of pseudo hoops.

The following proposition provide some properties of pseudo hoops.

Proposition 2.5. [3],[4] *Let $(A, \odot, \rightarrow, \rightsquigarrow, 1)$ be a Pseudo hoop. Then the following conditions hold, for all $x, y, z, a \in A$,*

(i) $x \leq y$ *if and only if* $x \rightarrow y = 1$ *if and only if* $x \rightsquigarrow y = 1$,

(ii) $x \rightsquigarrow (y \rightarrow z) = y \rightarrow (x \rightsquigarrow z)$, $x \rightarrow (y \rightsquigarrow z) = y \rightsquigarrow (x \rightarrow z)$,

(iii) $((x \rightarrow y) \rightsquigarrow y) \rightarrow y = x \rightarrow y$, $((x \rightsquigarrow y) \rightarrow y) \rightsquigarrow y = x \rightsquigarrow y$,

(iv) $x \leq y \rightarrow x$, $x \leq y \rightsquigarrow x$,

(v) $1 \rightarrow x = x$, $1 \rightsquigarrow x = x$,

(vi) $x \rightarrow 1 = 1$, $x \rightsquigarrow 1 = 1$,

(vii) $x \odot (x \rightsquigarrow y) \leq y$, $(x \rightarrow y) \odot x \leq y$,

(viii) $x \leq (x \rightarrow y) \rightsquigarrow y$, $x \leq (x \rightsquigarrow y) \rightarrow y$,

(ix) $x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z)$, $x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$,

(x) $(y \rightarrow z) \odot (x \rightarrow y) \leq x \rightarrow z$, $(x \rightsquigarrow y) \odot (y \rightsquigarrow z) \leq x \rightsquigarrow z$,

(xi) $x \leq y$ *implies* $x \odot z \leq y \odot z$, $z \odot x \leq z \odot y$,

(xii) $x \leq y$ *implies* $y \rightarrow z \leq x \rightarrow z$, $y \rightsquigarrow z \leq x \rightsquigarrow z$,

(xiii) $x \leq y$ *implies* $z \rightarrow x \leq z \rightarrow y$, $z \rightsquigarrow x \leq z \rightsquigarrow y$,

(xiv) $z \rightarrow x \leq (y \rightarrow z) \rightarrow (y \rightarrow x)$, $z \rightsquigarrow x \leq (y \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow x)$.

Theorem 2.6. [13] *Let $A = (A, \odot, \rightarrow, \rightsquigarrow, 1)$ be a Pseudo hoop. Then the following identities hold, for all $x, y, z \in A$,*

(i) $x \rightarrow x = x \rightsquigarrow x = 1$,

(ii) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$,

(iii) $(x \odot y) \rightsquigarrow z = y \rightsquigarrow (x \rightsquigarrow z)$.

Proposition 2.7. [3],[4] *Let A be a bounded Pseudo hoop. Then $x^- \leq x \rightarrow y$ and $x^\sim \leq x \rightsquigarrow y$, for all $x, y \in A$.*

Proposition 2.8. [13] *Let A be a Pseudo hoop and for any $x, y \in A$, we define, $x \sqcup_1 y = ((x \rightarrow y) \rightsquigarrow y) \wedge ((y \rightarrow x) \rightsquigarrow x)$ and $x \sqcup_2 y = ((x \rightsquigarrow y) \rightarrow y) \wedge ((y \rightsquigarrow x) \rightarrow x)$. Then the following conditions are equivalent:*

(i) $\sqcup_1$ *and* $\sqcup_2$ *are associative,*

(ii) $x \leq y$ *implies* $x \sqcup z \leq y \sqcup z$, *for all* $x, y, z \in A$ *and* $\sqcup \in \{\sqcup_1, \sqcup_2\}$,

(iii) $x \sqcup (y \wedge z) \leq (x \sqcup y) \wedge (x \sqcup y)$, *for all* $x, y, z \in A$ *and* $\sqcup \in \{\sqcup_1, \sqcup_2\}$,

(iv) $\sqcup_1$ *and* $\sqcup_2$ *are the pseudo join operation on* A.

Definition 2.9. [13] A Pseudo hoop A is called a $\sqcup$-*pseudo hoop*, if $\sqcup$ is a pseudo join operation on A, where $\sqcup \in \{\sqcup_1, \sqcup_2\}$

Proposition 2.10. [13] *Let A be a $\sqcup$-pseudo hoop, for $\sqcup \in \{\sqcup_1, \sqcup_2\}$. Then $(x \sqcup y) \rightsquigarrow z = (x \rightsquigarrow z) \wedge (y \rightsquigarrow z)$, for all $x, y, z \in A$.*

Definition 2.11. [13] Let A be a Pseudo hoop. A non-empty subset F of A is a *filter* if it satisfies,
(F1) $x, y \in F$ implies $x \odot y \in F$,
(F2) $x \in F$ and $x \leq y$ imply $y \in F$, for any $x, y \in A$.

A filter F of A is *proper* if and only if $F \neq A$.

Proposition 2.12. [13] *Let A be a Pseudo hoop and F be a non-empty subset of A such that $1 \in F$. Then the following statements are equivalent, for any $x, y, z \in A$,*

(i) *F is a filter,*
(ii) *if $x, x \to y \in F$, then $y \in F$,*
(iii) *if $x, x \rightsquigarrow y \in F$, then $y \in F$.*

Definition 2.13. [13] Let A be a Pseudo hoop and F be a filter F. Then F is called *normal* if $x \to y \in F$ if and only if $x \rightsquigarrow y \in F$.

Note. From now one in this paper, we let $A = (A, \odot, \to, \rightsquigarrow, 1)$ be a pseudo hoop, unless otherwise state.

3. (Positive) Implicative filters

In this section, we introduce the notions of implicative and positive implicative filters in pseudo hoops and investigate some properties of them and the relation between them.

Definition 3.1. Let F be a non-empty subset of A. Then F is called an *implicative filter* of A if,
(IF1) $1 \in F$,
(IF2) $x \to ((y \to z) \rightsquigarrow y) \in F$ and $x \in F$ imply $y \in F$, for any $x, y, z \in A$,
(IF3) $x \rightsquigarrow ((y \rightsquigarrow z) \to y) \in F$ and $x \in F$ imply $y \in F$, for any $x, y, z \in A$.

Example 3.2. Let $A = \{0, a, b, c, d, 1\}$. We define $\odot$, $\to$ and $\rightsquigarrow$ on A as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	d	0	d	a
b	0	d	c	c	0	b
c	0	0	c	c	0	c
d	0	d	0	0	0	d
1	0	a	b	c	d	1

$\to = \rightsquigarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Routine calculations show that A is a pseudo hoop and $F = \{b, c, 1\}$ is an implicative filter.

Proposition 3.3. *Every implicative filter of A is a filter.*

Proof. Let F be an implicative filter and $x, x \to y \in F$, for $x, y \in A$. By Proposition 2.5(vi) and (v), $x \to ((y \to 1) \rightsquigarrow y) = x \to y \in F$. Since $x \in F$ and F is an implicative filter, by (IF2), $y \in F$. Therefore, F is a filter. ∎

Remark 3.4. The converse of Proposition 3.3 may not be true. In Example 3.2, $F = \{a, 1\}$ is a filter, but is not an implicative filter of A, because, $1 \to ((b \to c) \rightsquigarrow b) = 1 \in F$, but $b \notin F$.

Theorem 3.5. *Let A be bounded and F be a filter of A. Then, for all $x, y \in A$, the following conditions are equivalent,*

(i) *F is an implicative filter,*

(ii) *if $(x \to y) \rightsquigarrow x \in F$, then $x \in F$ and if $(x \rightsquigarrow y) \to x \in F$, then $x \in F$,*

(iii) *$((x \rightsquigarrow y) \to x) \to x \in F$ and $((x \to y) \rightsquigarrow x) \rightsquigarrow x \in F$,*

(iv) *$(x^- \rightsquigarrow x) \rightsquigarrow x \in F$ and $(x^\sim \to x) \to x \in F$,*

(v) *if $(z^- \odot x) \rightsquigarrow y \in F$ and $y \rightsquigarrow z \in F$, then $x \rightsquigarrow z \in F$ and if $(x \odot z^\sim) \to y \in F$ and $y \to z \in F$, then $z \to x \in F$,*

(vi) *if F is a normal filter and $(y^- \odot x) \rightsquigarrow y \in F$, then $x \rightsquigarrow y \in F$ and if $(x \odot y^\sim) \to y \in F$, then $y \to x \in F$.*

Proof. (i)⇒(ii) Let F be an implicative filter and $(x \to y) \rightsquigarrow x \in F$. By Proposition 2.5(v), $1 \to ((x \to y) \rightsquigarrow x) = (x \to y) \rightsquigarrow x \in F$. Since $1 \in F$ and F is an implicative filter, by (IF2), $x \in F$. The proof of other case is similar.

(ii)⇒(i) Suppose that (ii) holds. Let $x \to ((y \to z) \rightsquigarrow y) \in F$ and $x \in F$. Since F is a filter, by Proposition 2.12(ii), $(y \to z) \rightsquigarrow y \in F$. Now, by (ii), $y \in F$. Therefore, F is an implicative filter.

(iii)⇒(i) Let $x \to ((y \to z) \rightsquigarrow y) \in F$ and $x \in F$. Since F is a filter, by Proposition 2.12(ii), $(y \to z) \rightsquigarrow y \in F$ and so by (iii), $((y \to z) \rightsquigarrow y) \rightsquigarrow y \in F$. Thus, by Proposition 2.12(iii), $y \in F$. Therefore, F is an implicative filter.

(iii)⇒(iv) Since A is bounded, it is enough to take $y = 0$ in (iii). Then $((x \rightsquigarrow 0) \to x) \to x \in F$. Hence, $(x^\sim \to x) \to x \in F$, for all $x \in A$. The proof of other case is similar.

(iv)⇒(iii) Since A is bounded, for any $y \in A$, $0 \leq y$. By Proposition 2.5(xiii), for any $x \in A$, $x \to 0 \leq x \to y$, and so $x^- \leq x \to y$. Now, by Proposition 2.5(xii), $(x \to y) \rightsquigarrow x \leq x^- \rightsquigarrow x$, then $(x^- \rightsquigarrow x) \rightsquigarrow x \leq ((x \to y) \rightsquigarrow x) \rightsquigarrow x$. Since F is a filter and by assumption $(x^- \rightsquigarrow x) \rightsquigarrow x \in F$, by (F2), $((x \to y) \rightsquigarrow x) \rightsquigarrow x \in F$, for all $x, y \in A$. The proof of other case is similar.

(v)⇒(vi) Suppose that F is a normal filter and $(y^- \odot x) \rightsquigarrow y \in F$, for $x, y \in A$. Then by Theorem 2.6(iii) and (i), $x \rightsquigarrow (y^- \rightsquigarrow y) \in F$ and $y \rightsquigarrow y = 1 \in F$. Then by (v), $x \rightsquigarrow y \in F$. The proof of other case is similar.

(vi)⇒(v) Suppose that $(x \odot z^\sim) \to y \in F$ and $y \to z \in F$. Since F is a filter, by (F1), $(y \to z) \odot ((x \odot z^\sim) \to y) \in F$. By Proposition 2.5(x),

$(y \to z) \odot ((x \odot z^{\sim}) \to y) \leq (x \odot z^{\sim}) \to z$. Since $(y \to z) \odot ((x \odot z^{\sim}) \to y) \in F$, by (F2), $(x \odot z^{\sim}) \to z \in F$. Hence, by (vi), $z \to x \in F$. The proof of other case is similar.

(vi)⇒(iv) By Theorem 2.6(iii),(i), $(x^- \odot (x^- \rightsquigarrow x)) \rightsquigarrow x = 1 \in F$. Then by (vi), $(x^- \rightsquigarrow x) \rightsquigarrow x \in F$. Therefore, (iv) holds.

(i)⇒(vi) Let F be an implicative filter. By Proposition 2.5(iv), $y \leq x \to y$. Then by Proposition 2.5(xii), $(x \to y) \to 0 \leq y \to 0$, thus by Proposition 2.5(xii), $(y^- \rightsquigarrow y) \leq (x \to y)^- \rightsquigarrow y$. Now, by Proposition 2.5(xiii), $x \rightsquigarrow (y^- \rightsquigarrow y) \leq x \rightsquigarrow ((x \to y)^- \rightsquigarrow y)$. Hence, by Theorem 2.6(iii), $(y^- \odot x) \rightsquigarrow y \leq ((x \to y)^- \odot x) \rightsquigarrow y$. Since $(y^- \odot x) \rightsquigarrow y \in F$, by (F2), $((x \to y)^- \odot x) \rightsquigarrow y \in F$. Then $x \rightsquigarrow ((x \to y)^- \rightsquigarrow y) \in F$. Since F is a normal filter, we get $x \to ((x \to y)^- \rightsquigarrow y) \in F$. By Proposition 2.5(ii), $(x \to y)^- \rightsquigarrow (x \to y) \in F$, thus by Proposition 2.5(v), $1 \to ((x \to y)^- \rightsquigarrow (x \to y)) \in F$. Since F is an implicative filter and $1 \in F$, by (IF2), $x \to y \in F$, and so $x \rightsquigarrow y \in F$. ■

Corollary 3.6. *Let A be bounded and F be an implicative filter of A. Then $x^{\sim -} \to x \in F$ and $x^{-\sim} \rightsquigarrow x \in F$, for all $x \in A$.*

Proof. Let F be an implicative filter of A. Since A is a bounded Pseudo hoop, we have $0 \leq x$, for all $x \in A$. Then by Proposition 2.5(xiii), $x^- \rightsquigarrow 0 \leq x^- \rightsquigarrow x$, and so by Proposition 2.5(i), $x^{-\sim} \rightsquigarrow (x^- \rightsquigarrow x) = 1$. Thus, by (IF1), $x^{-\sim} \rightsquigarrow (x^- \rightsquigarrow x) = 1 \in F$ and by (PH-3) $(x^- \odot x^{-\sim}) \rightsquigarrow x \in F$. Hence, by Theorem 3.5(vi), $x^{-\sim} \rightsquigarrow x \in F$. By the similar way, we get $x^{\sim -} \to x \in F$. ■

Proposition 3.7. *Let A be bounded and F and G be two filters of A such that $F \subseteq G$. If F is a normal and implicative filter, then G is an implicative filter, too.*

Proof. Let F and G be two filters of bounded Pseudo hoop A such that $F \subseteq G$ and F be an implicative filter. By Theorem 3.5, it suffices to prove that (vi) holds. Suppose that $(y^- \odot x) \rightsquigarrow y \in G$, for $x, y \in A$. Let $u = (y^- \odot x) \rightsquigarrow y$. By Theorem 2.6(i), $u \to u = 1$, and so $u \to ((y^- \odot x) \rightsquigarrow y) = 1$, then by Proposition 2.5(ii), $(y^- \odot x) \rightsquigarrow (u \to y) = 1$. Since F is a filter and $1 \in F$, then $(y^- \odot x) \rightsquigarrow (u \to y) \in F$. By Proposition 2.5(iv), $y \leq u \to y$, thus, by Proposition 2.5(xii), $(u \to y) \to 0 \leq y \to 0$, then by Proposition 2.5(xi), $(u \to y)^- \odot x \leq y^- \odot x$. Again, by Proposition 2.5(xii), $(y^- \odot x) \rightsquigarrow (u \to y) \leq ((u \to y)^- \odot x) \rightsquigarrow (u \to y)$. Since $(y^- \odot x) \rightsquigarrow (u \to y) \in F$, by (F2), $((u \to y)^- \odot x) \rightsquigarrow (u \to y) \in F$. Since F is an implicative filter, by Theorem 3.5(vi), $x \rightsquigarrow (u \to y) \in F$. Then by Proposition 2.5(ii), $u \to (x \rightsquigarrow y) \in F$, and so $u \to (x \rightsquigarrow y) \in G$. Since $u \in G$, by Proposition 2.12(ii), $x \rightsquigarrow y \in G$. Therefore, G is an implicative filter. ■

Theorem 3.8. *Let A be a bounded $\sqcup$-pseudo hoop and F be a filter of A. If $x \sqcup x^- \in F$, for any $x \in A$ and $\sqcup \in \{\sqcup_1, \sqcup_2\}$, then F is an implicative filter.*

Proof. Let $x \sqcup x^- \in F$. Then by Proposition 2.10, $(x \sqcup x^-) \rightsquigarrow x = (x \rightsquigarrow x) \wedge (x^- \rightsquigarrow x)$. Hence by Theorem 2.6(i), $(x \sqcup x^-) \rightsquigarrow x = x^- \rightsquigarrow x$. Also, by Proposition 2.7, $x^- \leq x \to y$. Then $(x \to y) \rightsquigarrow x \leq x^- \rightsquigarrow x$. Since $(x \to y)$

$\rightsquigarrow x \in F$, by (F2), $x^- \rightsquigarrow x \in F$. Since $(x \sqcup x^-) \rightsquigarrow x = x^- \rightsquigarrow x$, we have $(x \sqcup x^-) \rightsquigarrow x \in F$. Moreover, $x \sqcup x^- \in F$, thus, by Proposition 2.12(iii), $x \in F$. Then by Theorem 3.5(ii), F is an implicative filter. The proof of other case is similar. Therefore, F is an implicative filter. ∎

Definition 3.9. A non-empty subset F of A is called a *positive implicative filter* of A if,
(PIF1) $1 \in F$,
(PIF2) $x \to (y \to z) \in F$ and $x \rightsquigarrow y \in F$ imply $x \to z \in F$, for any $x, y, z \in A$,
(PIF3) $x \rightsquigarrow (y \rightsquigarrow z) \in F$ and $x \to y \in F$ imply $x \rightsquigarrow z \in F$, for any $x, y, z \in A$.

Example 3.10. Let $A = \{0, a, b, c, 1\}$. We define $\odot$, $\to$ and $\rightsquigarrow$ on A as follows:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

$\to = \rightsquigarrow$	0	a	b	c	1
0	0	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	1	a	b	c	1

Routine calculations show that A is a pseudo hoop and $F = \{b, 1\}$ is a positive implicative filter.

Proposition 3.11. *Every positive implicative filter is a filter.*

Proof. Suppose that F is a positive implicative filter. Then by Proposition 2.12(iii), it is enough to prove that if $x, x \rightsquigarrow y \in F$, then $y \in F$, for any $x, y \in A$. By Proposition 2.5(v), $1 \rightsquigarrow (x \rightsquigarrow y) \in F$ and $1 \to x = x \in F$. Since F is a positive implicative filter, we have $1 \rightsquigarrow y = y \in F$. Therefore, F is a filter. ∎

Proposition 3.12. *Let F be a positive implicative filter of A. Then, for all $x, y \in A$ the following statements are hold:*

(i) *if $x \to (x \to y) \in F$, then $x \to y \in F$,*

(ii) *if $x \rightsquigarrow (x \rightsquigarrow y) \in F$, then $x \rightsquigarrow y \in F$,*

(iii) $x \to x^2 \in F$,

(iv) $x \rightsquigarrow x^2 \in F$.

Proof. We prove (ii) and (iii), the proofs of (i) and (iv) are similar.

(ii) Since F is a positive implicative filter, by Proposition 3.11, F is a filter. Now, let $x \rightsquigarrow (x \rightsquigarrow y) \in F$, for $x, y \in A$. Then, by Theorem 2.6(i), $x \to x = 1 \in F$, thus, by (PIF3), $x \rightsquigarrow y \in F$.

(iii) Since F is a filter, by Theorem 2.6(i), for any $x \in A$, $(x \odot x) \to (x \odot x) = 1 \in F$. Then by Theorem 2.6(iii), $x \to (x \to (x \odot x)) \in F$ and $x \rightsquigarrow x \in F$. Hence, by (PIF2), $x \to (x \odot x) \in F$. ∎

Theorem 3.13. *Let A be a chain and F be a positive implicative filter of A. Then F is an implicative filter if and only if $(x \to y) \rightsquigarrow y \in F$ implies $(y \to x) \rightsquigarrow x \in F$, for any $x, y \in A$.*

Proof. ($\Leftarrow$) Let $(x \to y) \rightsquigarrow x \in F$, for $x, y \in A$. Since A is a chain, $x \leq y$ or $y \leq x$. If $x \leq y$, then $x \to y = 1$. Thus, by Proposition 2.5(v), $x = 1 \rightsquigarrow x = (x \to y) \rightsquigarrow x \in F$. Now, let $y \leq x$. Since $x \leq (x \to y) \rightsquigarrow y$, by Proposition 2.5(xiii), $(x \to y) \rightsquigarrow x \leq (x \to y) \rightsquigarrow ((x \to y) \rightsquigarrow y)$. By Proposition 3.11, F is a filter and $(x \to y) \rightsquigarrow x \in F$. Then $(x \to y) \rightsquigarrow ((x \to y) \rightsquigarrow y) \in F$. Since $(x \to y) \rightsquigarrow (x \to y) = 1 \in F$ and F is a positive implicative filter, by (PIF3), $(x \to y) \rightsquigarrow y \in F$. By assumption, $(y \to x) \rightsquigarrow x \in F$. Since $y \leq x$, we have $y \to x = 1$. Then $x = 1 \rightsquigarrow x = (y \to x) \rightsquigarrow x \in F$. By the similar way, if $(x \rightsquigarrow y) \to x \in F$, then $x \in F$. Hence, by Theorem 3.5(ii), F is an implicative filter of A.

($\Rightarrow$) Let F be an implicative filter and $(x \to y) \rightsquigarrow y \in F$, for $x, y \in A$. By Proposition 2.5(viii), $y \leq (y \to x) \rightsquigarrow x$. Then by Proposition 2.5(xiii), $(x \to y) \rightsquigarrow y \leq (x \to y) \rightsquigarrow ((y \to x) \rightsquigarrow x)$. Since $(x \to y) \rightsquigarrow y \in F$, by (F2), $(x \to y) \rightsquigarrow ((y \to x) \rightsquigarrow x) \in F$. Also, since $x \leq (y \to x) \rightsquigarrow x$, we have $((y \to x) \rightsquigarrow x) \to y \leq x \to y$. Hence, $(x \to y) \rightsquigarrow ((y \to x) \rightsquigarrow x) \leq (((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow ((y \to x) \rightsquigarrow x)$, by Proposition 3.11 and (F2), $(((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow ((y \to x) \rightsquigarrow x) \in F$. Since F is an implicative filter, by Theorem 3.5(ii), $(y \to x) \rightsquigarrow x \in F$. ∎

Theorem 3.14. *Let F be a positive implicative normal filter of A such that $(x \to y) \rightsquigarrow y \in F$ implies $(y \to x) \rightsquigarrow x \in F$, for any $x, y \in A$. Then F is an implicative filter.*

Proof. Let $(x \to y) \rightsquigarrow x \in F$, for any $x, y \in A$. By Proposition 2.5(viii), $x \leq (x \to y) \rightsquigarrow y$. Then by Proposition 2.5(xiii), $(x \to y) \rightsquigarrow x \leq (x \to y) \rightsquigarrow ((x \to y) \rightsquigarrow y)$. Since $(x \to y) \rightsquigarrow x \in F$, by Proposition 3.11, F is a filter and so we have, $(x \to y) \rightsquigarrow ((x \to y) \rightsquigarrow y) \in F$. Since F is a positive implicative filter, by Theorem 3.12(ii), $(x \to y) \rightsquigarrow y \in F$, and so by assumption, $(y \to x) \rightsquigarrow x \in F$. Also, by Proposition 2.5(iv), $y \leq x \to y$. Then, by Proposition 2.5(xii), $(x \to y) \rightsquigarrow x \leq y \rightsquigarrow x$, thus by (F2), $y \rightsquigarrow x \in F$. Since F is a normal filter, $y \to x \in F$. Then, by Proposition 2.12(ii), $x \in F$. By the similar way, if $(x \rightsquigarrow y) \to x \in F$, we get $x \in F$. Therefore, F is an implicative filter. ∎

Proposition 3.15. *Let F be a normal filter of A and*

(i) *if $x \rightsquigarrow (x \rightsquigarrow y) \in F$, then $x \rightsquigarrow y \in F$, for any $x, y \in A$.*

(ii) *if $x \to (x \to y) \in F$, then $x \to y \in F$, for any $x, y \in A$.*

Then F is a positive implicative filter of A.

Proof. Suppose that (i) holds. Let $x \rightsquigarrow (y \rightsquigarrow z) \in F$ and $x \to y \in F$, for $x, y \in A$. Since F is a normal filter and $x \rightsquigarrow (y \rightsquigarrow z) \in F$, we have $x \to (y \rightsquigarrow z) \in F$. By Proposition 2.5(ii), $y \rightsquigarrow (x \to z) \in F$. Since F is a normal filter, we have $y \to (x \to z) \in F$ and $x \to y \in F$. By (F1), $(y \to (x \to z)) \odot (x \to y) \in F$. Then by Proposition 2.5(x), $(y \to (x \to z)) \odot (x \to y) \leq x \to (x \to z)$, then by (F2), $x \to (x \to z) \in F$. Also, by (ii), $x \to z \in F$. Since F is a normal filter, we get $x \rightsquigarrow z \in F$. Then we have (PIF3). The proof of (PIF2) is similar. Therefore, F is a positive implicative filter. ∎

Definition 3.16. Let A be bounded and $x \in A$. If there exists a smallest positive integer number $n \in \mathbb{N}$ such that $x^n = 0$, then we say that the order of x is n and we denote by $ord(x) = n$. We say $ord(x) = \infty$, if no such n exists.

Example 3.17. In Example 3.2, $ord(a) = ord(b) = ord(c) = ord(1) = \infty$ and $ord(d) = 2$.

Proposition 3.18. *Let A be bounded such that for any $x \in A$, $ord(x) = 2$. If F is a positive implicative filter, then $x^-, x^\sim \in F$.*

Proof. By Theorem 3.12, The proof is clear. ∎

Theorem 3.19. *Let F and G be two filters of A such that $F \subseteq G$. If F is a positive implicative filter of A, then G is a positive implicative filter of A, too.*

Proof. Let F and G be two filters of Pseudo hoop A such that $F \subseteq G$ and for any $x, y \in A$, $x \rightsquigarrow (x \rightsquigarrow y) \in G$. Let $u = x \rightsquigarrow (x \rightsquigarrow y) \in G$. Then by Theorem 2.6(iii), $u = (x \odot x) \rightsquigarrow y \in G$ and by Theorem 2.6(i), $u \to ((x \odot x) \rightsquigarrow y) = 1 \in F$. Thus, by Proposition 2.5(ii), $(x \odot x) \rightsquigarrow (u \to y) \in F$. Since F is a positive implicative filter and $x \to x = 1 \in F$, by (PIF3), $x \rightsquigarrow (u \to y) \in F$. By Proposition 2.5(ii), $u \to (x \rightsquigarrow y) \in F$. Since $F \subseteq G$, $u \in G$, and by Proposition 2.12(ii), $x \rightsquigarrow y \in G$. ∎

Theorem 3.20. *Let A be bounded and F be an implicative filter of A.*

(i) *If $x^{-\sim} \in F$ or $x^{\sim -} \in F$, for any $x \in A$, then F is a positive implicative filter,*

(ii) *If for any $x \in A$, $x^- \rightsquigarrow x \in F$ or $x^\sim \to x \in F$, then F is a positive implicative filter.*

Proof. (i) Let F be an implicative filter and $x^{-\sim} \in F$, for any $x \in A$. Suppose that $x \to (y \to z) \in F$ and $x \rightsquigarrow y \in F$, for $x, y \in A$. Since $z^{-\sim} \in F$, we have $(z \to 0) \rightsquigarrow 0 \in F$. So by Theorem 3.13, $(0 \to z) \rightsquigarrow z \in F$. Then by Proposition 2.5(i), (v), $1 \to z = z \in F$. And by similar way $x \in F$. Thus, $x \to z \in F$. Therefore, F is a positive implicative filter. The proof of other case is similar.

(ii) Let F be an implicative filter of A and $x^- \rightsquigarrow x \in F$, for any $x \in A$. Suppose that, $x \to (y \to z) \in F$ and $x \rightsquigarrow y \in F$, for $x, y \in A$. Since F is an implicative filter, by Theorem 3.5(iv), $(x^- \rightsquigarrow x) \rightsquigarrow x \in F$, for any $x \in A$. Since $x^- \rightsquigarrow x \in F$, by Proposition 2.12(ii), $x \in F$. Since $x \rightsquigarrow y \in F$ and $x \in F$, we get $y \in F$. Now, y, $y \to z \in F$, then $z \in F$. Since, $z \leq x \to z$, we get $x \to z \in F$. Therefore, F is a positive implicative filter. ∎

4. Fantastic and associative filters

In this section, we introduce the notions of fantastic and associative filters in pseudo hoops and investigate their properties of them and we study the relation between them.

Definition 4.1. Let F be a non-empty subset of A. Then F is called a *fantastic filter* A if it satisfies the following properties,
(FF1) $1 \in F$,
(FF2) $z \to (x \to y) \in F$ and $z \in F$ imply $((y \to x) \rightsquigarrow x) \to y \in F$, for any $x, y, z \in A$,
(FF3) $z \rightsquigarrow (x \rightsquigarrow y) \in F$ and $z \in F$ imply $((y \rightsquigarrow x) \to x) \rightsquigarrow y \in F$, for any $x, y, z \in A$.

Example 4.2. According to Example 2.3, $([0, u], \odot, \to, \rightsquigarrow, u)$ is a bounded Pseudo hoop. By [13], Let K be a normal convex ℓ-group of G and $F=\{a \in [o, u] \mid u-a \in k\}$ is a normal filter of $[0, u]$. suppose that $z \to (x \to y) \in F$ and $z \in F$ then $((y \to x) \rightsquigarrow x) \to y = (((x-y+u) \wedge u) \rightsquigarrow x) \to y = ((u-((x-y+u) \wedge u)+x) \wedge u) \to y = (((u-u+y-x+x) \vee (u-u+x)) \wedge u) \to y = (y \vee x) \to y = (y-(y \vee x)+u) \wedge u = u \in F$. thus we have (FF2). The similar way (FF3), holds. Therefore, F is a fantastic filter.

Proposition 4.3. *Every fantastic filter of A is a filter.*

Proof. Let F be a fantastic filter and $x, x \to y \in F$. By Proposition 2.5(v), $x \to y = x \to (1 \to y) \in F$. Since $x \in F$ and F is a fantastic filter, by (FF2), $((y \to 1) \rightsquigarrow 1) \to y \in F$. Then, by Proposition 2.5(v) and (vi), $y \in F$. Therefore, F is a filter. ∎

Proposition 4.4. *Let F be a filter of A. Then F is a fantastic filter if and only if $x \to y \in F$ implies $((y \to x) \rightsquigarrow x) \to y \in F$, for any $x, y \in A$.*

Proof. ($\Leftarrow$) Let F be a fantastic filter and $y \to x \in F$, for any $x, y \in A$. Then by Proposition 2.5(v), $y \to x = 1 \to (y \to x) \in F$. Since F is a filter, $1 \in F$, and by (FF2), $((y \to x) \rightsquigarrow x) \to y \in F$.

($\Rightarrow$) Since F is a filter, (FF1) holds. Now, let $z \to (y \to x) \in F$ and $z \in F$, for any $x, y \in A$. Since F is a filter, by Proposition 2.12(ii), $y \to x \in F$. Then, by assumption, $((x \to y) \rightsquigarrow y) \to x \in F$. Therefore, F is a fantastic filter. ∎

Proposition 4.5. *Let F and G be two filters of A such that $F \subseteq G$. If F is a fantastic filter, then G is a fantastic filter, too.*

Proof. Let F and G be two filters of A such that $F \subseteq G$. Suppose that $y \to x \in G$, for any $x, y \in A$. By Theorem 2.6(i), $(y \to x) \rightsquigarrow (y \to x) = 1$. Then by Proposition 2.5(ii), $y \to ((y \to x) \rightsquigarrow x) = 1$ and so $y \to ((y \to x) \rightsquigarrow x) = 1 \in F$. Since F is a fantastic filter and $y \to ((y \to x) \rightsquigarrow x) \in F$, we have $((((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow y) \to ((y \to x) \rightsquigarrow x) = (y \to x) \rightsquigarrow (((((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow y) \to x) \in F$. Since $F \subseteq G$, $(y \to x) \rightsquigarrow (((((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow y) \to x) \in G$.

Also, since $y \to x \in G$, by Proposition 2.12(ii), $(((((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow y) \to x) \in G$. Let $\alpha = (((((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow y) \to x)$. Now, by Proposition 2.5(xii),(vi),(ii), and Theorem 2.6(i), we get

$$\begin{aligned} \alpha \rightsquigarrow (((x \to y) \rightsquigarrow y) \to x) &\geq ((x \to y) \rightsquigarrow y) \to ((((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow y) \\ &\geq (((y \to x) \rightsquigarrow x) \to y) \rightsquigarrow (x \to y) \\ &\geq x \to ((y \to x) \rightsquigarrow x) \\ &= (y \to x) \rightsquigarrow (x \to x) \\ &= 1 \end{aligned}$$

By assumption, G is a filter, then $1 \in G$ and by (F2), $\alpha \rightsquigarrow (((x \to y) \rightsquigarrow y) \to x) \in G$. Since $\alpha \in G$, by Proposition 2.12(ii), $((x \to y) \rightsquigarrow y) \to x \in G$. Therefore, G is a fantastic filter. ∎

Theorem 4.6. *Let A be bounded. If F is an implicative filter, then F is a fantastic filter.*

Proof. Let F be an implicative filter and $y \rightsquigarrow x \in F$. By Proposition 2.7, $x^- \leq x \to y$. Then $(x \to y) \rightsquigarrow y \leq x^- \rightsquigarrow y$ and so $((x \to y) \rightsquigarrow y) \rightsquigarrow (x^- \rightsquigarrow y) = 1$. By Proposition 3.3, F is a filter, and so $((x \to y) \rightsquigarrow y) \rightsquigarrow (x^- \rightsquigarrow y) \in F$, then by Theorem 2.6(iii), $(x^- \odot ((x \to y) \rightsquigarrow y)) \rightsquigarrow y \in F$. Also, by assumption, $y \rightsquigarrow x \in F$. Since F is an implicative filter, by Theorem 3.5(v), $((x \rightsquigarrow y) \rightsquigarrow y) \rightsquigarrow x \in F$. Therefore, F is a fantastic filter. ∎

Theorem 4.7. *Let F be a fantastic and positive implicative filter of A. Then F is an implicative filter.*

Proof. Let F be a fantastic and positive implicative filter and $(x \to y) \rightsquigarrow x \in F$, for $x, y \in A$. Since F is a fantastic filter, by (FF2), $((x \to (x \to y)) \rightsquigarrow (x \to y)) \to x \in F$. Then by Theorem 2.6(ii), $((x^2 \to y) \rightsquigarrow (x \to y)) \to x \in F$. By Proposition 2.5(ix), $x \to x^2 \leq (x^2 \to y) \rightsquigarrow (x \to y)$. Since F is a positive implicative filter, by Theorem 3.12(iii), $x \to x^2 \in F$, and so by (F2), $(x^2 \to y) \rightsquigarrow (x \to y) \in F$. Since $((x^2 \to y) \rightsquigarrow (x \to y)) \to x \in F$, then by Proposition 2.12(ii), $x \in F$. Hence, by Theorem 3.5(ii), F is an implicative filter. ∎

Definition 4.8. Let F be a non-empty subset of A. Then F is called an *associative filter* of A if, for any $x, y, z \in A$;
(AF1) $1 \in F$,
(AF2) $x \to (y \to z) \in F, x \rightsquigarrow y \in F$ imply $z \in F$,
(AF3) $x \rightsquigarrow (y \rightsquigarrow z) \in F, x \rightsquigarrow y \in F$ imply $z \in F$.

Example 4.9. Let $A = \{0, a, b, c, d, 1\}$. We define $\odot, \to$ and $\rightsquigarrow$ on A as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	b	a	d	b	a
b	0	a	b	0	0	b
c	0	d	0	c	d	c
d	0	b	0	d	0	d
1	0	a	b	c	c	1

$\to = \rightsquigarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

Then, A with these operations is a pseudo hoop. Let $F = \{1, a, b\}$. Then routine calculations show that F is an associative filter of A.

Theorem 4.10. *Let F be an associative filter of A. Then*

(i) *F is a filter,*

(ii) *F is an implicative filter,*

(iii) *F is a positive implicative filter,*

(iv) *F is a fantastic filter, if A is bounded.*

Proof. (i) Suppose that F is an associative filter and $x, x \to y \in F$, for $x, y \in A$. Then by Proposition 2.5(v), $1 \to (x \to y) = x \to y \in F$ and $1 \rightsquigarrow x = x \in F$. Since F is an associative filter, we have $y \in F$. Therefore, F is a filter.

(ii) Let F be an associative filter and $(x \to y) \rightsquigarrow x \in F$. Then by Proposition 2.5(v),(iii), $(x \to y) \rightsquigarrow (1 \rightsquigarrow x) \in F$ and $(x \to y) \to 1 = 1 \in F$. Since F is associative filter, by (AF3), $x \in F$. Therefore, F is an implicative filter.

(iii) Let F be an associative filter such that $x \to (y \to z) \in F$ and $x \rightsquigarrow y \in F$. Then by (AF2), $z \in F$. Since F is a filter and by Proposition 2.5(iv), $z \leq x \to z$, and so $x \to z \in F$. Therefore, F is a Positive implicative filter.

(iv) By (ii) and 4.6, the proof is clear. ∎

Example 4.11. (i) Let G be a Pseudo hoop similar as Example 2.3. We show that $\{u\}$ is a filter of $G[u]$. $u \in \{u\}$ so (F1), holds. If $x, y \in \{u\}$, then $x \odot y = u \odot u = (u - u + u) \vee 0 = u \in \{u\}$. This is clear that (F2), holds, and so $\{u\}$ is a filter. Let $x = z \neq u$ and $y = u$. We obtained, $z \to (u \to z) = u \in \{u\}, z \rightsquigarrow u = u \in \{u\}$. Since $\{u\}$ is an associative filter, by (AF2), $z \in \{u\}$, which is a contradiction. Therefore, F is not an associative filter.

(ii) Let A be a Pseudo hoop similar as Example 3.2. Then $F = \{b, c, 1\}$ is an implicative filter but it is not an associative filter, because, $a \to (b \to a)$ $= a \to a = 1 \in F$ and $a \to b = b \in F$, but $a \notin F$.

(iii) In Example 3.2, $F = \{b, c, 1\}$, is a positive implicative filter, but it is not an associative filter.

(iv) In Example 3.2, $F = \{b, c, 1\}$, is a fantastic filter, but it is not an associative filter.

Proposition 4.12. *Let A be bounded and F be an associative filter of A. Then $x^-, x^\sim \in F$, for any $x \in A$.*

Proof. Let F be an associative filter of A. By Proposition 2.5(iv),(v), $(x \to 0) \to (1 \to (x \to 0)) = 1 \in F$ and $(x \to 0) \rightsquigarrow 1 = 1 \in F$. Since F is an associative filter, by (AF2), $x^- \in F$. By similar way, we can see that $x^\sim \in F$. ∎

Theorem 4.13. *Let A be bounded and F be an implicative filter of A. If one of following conditions holds, then F is an associative filter.*

(i) *if $x^- \rightsquigarrow x \in F$, for any $x \in A$.*

(ii) *if $x^\sim \to x \in F$, for any $x \in A$.*

Proof. (i) Let F be an implicative filter and $x \to (y \to z) \in F$ and $x \rightsquigarrow y \in F$. Since F is an implicative filter, by Theorem 3.5(iv), $(x^- \rightsquigarrow x) \rightsquigarrow x \in F$, for any $x \in A$. Since $x^- \rightsquigarrow x \in F$, then by Proposition 2.12(iii), $x \in F$. Since $x \to (y \to z) \in F$ and $x \rightsquigarrow y \in F$, we get $y \to z \in F$, and $y \in F$, then $z \in F$. Therefore, F is an associative filter.

(ii) The proof of (ii) is similar. ■

Corollary 4.14. *Let A is bounded. If F is a fantastic and positive implicative filter of A and $x^- \rightsquigarrow x \in F$ or $x^\sim \to x \in F$, for $x \in A$, then F is an associative filter.*

Proof. By Theorems 4.7 and 4.13, the proof is clear. ■

Theorem 4.15. *Let A be bounded and F be an implicative filter of A. If one of following conditions holds, then F is an associative filter.*

(i) *if $x^{-\sim} \in F$, for any $x \in A$,*

(ii) *if $x^{\sim -} \in F$, for any $x \in A$.*

Proof. (i) Let F be an implicative filter and $x^{-\sim} \in F$, for any $x \in A$. Suppose that $x \to (y \to z) \in F$ and $x \rightsquigarrow y \in F$. Since $z^{-\sim} \in F$, we have $(z \to 0) \rightsquigarrow 0 \in F$. So by Theorem 3.13, $(0 \to z) \rightsquigarrow z \in F$. Then by Proposition 2.5(i),(v), $1 \rightsquigarrow z = z \in F$. Therefore, (AF2) holds. The proof of (AF3), is similar.

(ii) Let F be an implicative filter and $x \to (y \to z) \in F$ and $x \rightsquigarrow y \in F$. Since F is an implicative filter, by Corollary 3.6, $x^{\sim -} \to x \in F$. Since $x^{\sim -} \in F$, then by Proposition 2.12(ii), $x \in F$. Now, since $x \to (y \to z) \in F$ and $x \in F$, we get $y \to z$. Since $y \to z \in F$ and $y \in F$, by Proposition 2.12(ii), $z \in F$. Therefore, (AF2) holds. By similar way (AF3), holds. ■

Proposition 4.16. *Let A be bounded and F and G be two filters A such that $F \subseteq G$ and $x^- \rightsquigarrow x \in F$, for any $x \in A$. If F is an associative filter, then G is an associative filter, too.*

Proof. By Proposition 3.7 and Theorems 4.10 and 4.13 the proof is clear. ■

5. Conclusions and future works

The aim of this paper is to introduce the notions of (positive)implicative, fantastic and associative filters in Pseudo hoops and to investigate their properties. Several characterizations of these filters are derived but in this paper we try to investigate the relation between them. I want to examine quotient algebra associated with the filters in the future.

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