On $H_3(p)$ HANKEL DETERMINANT FOR CERTAIN SUBCLASS OF $p-$VALENT FUNCTIONS

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Abstract. The aim of this paper is to obtain an upper bound to the $H_3(p)$ Hankel determinant for certain subclass of p-valent functions. To do so, we obtain best possible bounds for the functionals $|a_{p+3} - a_{p+1}a_{p+2}|$ and $\left|a_{p+2} - a_{p+1}^2\right|$, then using known upper bound for the functional $\left|a_{p+1}a_{p+3} - a_{p+2}^2\right|$ we obtain the required sharp upper bound to the $H_3(p)$ Hankel determinant.

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1. Introduction

Let $\mathcal{A}_p$ denote the class of functions f of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, (p \in \mathbb{N} = \{1, 2, 3, ...\}) \tag{1.1}$$

in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let S be the subclass of $\mathcal{A}_1 := \mathcal{A}$, consisting of univalent functions.

The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke ([17], [16]) as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & a_{n+q} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \tag{1.2}$$

This determinant has been considered by many authors in the literature [14]. For example, Noor [15] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions given by (1.1) with bounded boundary. Ehrenborg [2] studied the Hankel determinant of exponential polynomials. In [7], Janteng et al. studied the Hankel determinant for the classes of starlike and convex functions. Again Janteng et al. discussed the Hankel determinant problem for the classes of starlike functions with respect to symmetric points and convex functions with respect to symmetric points in [5] and for the functions whose derivative has a positive real part in [6]. Also Hankel determinant for various subclasses of $p-$valent functions was investigated by various authors including Krishna and Ramreddy [8] and Hayami and Owa [4].

In this paper, we consider the Hankel determinant in the case of $q = 3$ and $n = p$:

$$H_3(p) = \begin{vmatrix} a_p & a_{p+1} & a_{p+2} \\ a_{p+1} & a_{p+2} & a_{p+3} \\ a_{p+2} & a_{p+3} & a_{p+4} \end{vmatrix}.$$

For $f \in \mathcal{A}_p$, $a_p = 1$, we have

$$H_3(p) = a_{p+2}(a_{p+1}a_{p+3} - a_{p+2}^2) - a_{p+3}(a_{p+3} - a_{p+1}a_{p+2}) + a_{p+4}(a_{p+2} - a_{p+1}^2),$$

and by applying the triangle inequality, we obtain

$$|H_3(p)| \leq |a_{p+2}||a_{p+1}a_{p+3} - a_{p+2}^2| + |a_{p+3}||a_{p+3} - a_{p+1}a_{p+2}| + |a_{p+4}||a_{p+2} - a_{p+1}^2|. \tag{1.3}$$

Incidentally, the sharp upper bound for the functional $\left|a_{p+1}a_{p+3} - a_{p+2}^2\right|$ on the right hand side of the inequality (1.3) for the class of functions which is of our interest in this paper was obtained by Vamshee Krishna and Ramreddy [9]. Thus, in this paper we obtain upper bounds to the functionals $|a_{p+3} - a_{p+1}a_{p+2}|$ and $\left|a_{p+2} - a_{p+1}^2\right|$, then the sharp upper bound on $H_3(p)$ follows as simple corollary.

Definition 1.1 A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{H}_{p,\alpha}$, if it satisfies the condition

$$\operatorname{Re}\left\{(1-\alpha)\frac{f(z)}{z^p}+\alpha\frac{f'(z)}{pz^{p-1}}\right\}>0, \quad (0\le\alpha\le 1), \forall z\in\mathbb{U}. \tag{1.4}$$

For $\alpha = 1$, the class $\mathcal{H}_{p,1}$ reduced to Vamshee Krishna, et al. [10].

In the next section, we state the necessary lemmas, while in Section 3 we present our main results.

2. Preliminaries results

Let Q denote the class of functions

$$q(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... = 1 + \sum_{k=1}^{\infty} c_k z^k, \tag{2.1}$$

which are analytic in $\mathbb{U}$ and satisfy $\operatorname{Re}\{q(z)\} > 0$ for any $z \in \mathbb{U}$. To prove our main results in the next section, we shall require the following three Lemmas:

Lemma 2.1 ([18], [19]) *Let $q \in Q$. Then $|c_k| \le 2$ for each $k \in \mathbb{N}$. And the inequality is sharp.*

Lemma 2.2 ([3], [11], [12]) *Let $q \in Q$. Then*

$$2c_2 = c_1^2 + (4 - c_1^2)x,$$
$$4c_3 = c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z,$$

for some x and z satisfying $|x| \le 1$, $|z| \le 1$ and $q_1 \in [0, 2]$.

Lemma 2.3 ([1]) *Let $q \in Q$. Then we have the sharp inequalities for any real number σ,*

$$\left|c_2 - \sigma\frac{c_1^2}{2}\right| \le \begin{cases} 2(1-\sigma) & \text{if } \sigma \le 0, \\ 2 & \text{if } 0 \le \sigma \le 2, \\ 2(\sigma - 1) & \text{if } \sigma \ge 0. \end{cases} \tag{2.2}$$

3. Main results

Theorem 3.1 *Let $f(z) \in \mathcal{H}_{p,\alpha}$. Then, for all $j \in \mathbb{N}$, we have the sharp inequalities:*

$$|a_{p+j}| \le \frac{2p}{p + j\alpha}. \tag{3.1}$$

Proof. Since $f \in \mathcal{H}_{p,\alpha}$, then by Definition 1.1 there exists a function $q \in Q$ such that

$$\left\{(1-\alpha)\frac{f(z)}{z^p}+\alpha\frac{f^{'}(z)}{pz^{p-1}}\right\}=p(z)\Rightarrow p(1-\alpha)f(z)+\alpha zf^{'}(z)=pz^p p(z), \tag{3.2}$$

for some $z \in \mathbb{U}$.

Replacing $f(z)$, $f'(z)$ with their equivalent $p-$ valent series expressions, and $p(z)$ with its equivalent series expression in (3.2), we have

$$\begin{aligned} p(1-\alpha)\left[z^p+\sum_{n=p+1}^{\infty}a_nz^n\right]+\alpha z\left(pz^{p-1}+\sum_{n=p+1}^{\infty}na_nz^{n-1}\right) \\ = pz^p\left(1+\sum_{k=1}^{\infty}c_kz^k\right) \end{aligned} \tag{3.3}$$

Simplifying (3.3) implies

$$\begin{aligned} (p+\alpha)a_{p+1}z^{p+1}+(p+2\alpha)a_{p+2}z^{p+2}+(p+3\alpha)a_{p+3}z^{p+3} \\ +(p+4\alpha)a_{p+4}z^{p+4}+...=pc_1z^{p+1}+pc_2z^{p+2}+pc_3z^{p+3}+pc_4z^{p+4}+... \end{aligned} \tag{3.4}$$

Equating coefficients in (3.4) yields

$$a_{p+j}=\frac{pc_j}{p+i\alpha},\quad \text{for all } j\in\mathbb{N} \tag{3.5}$$

and the result follows by using Lemma 2.2. ∎

Theorem 3.2 *Let $f(z) \in \mathcal{H}_{p,\alpha}$. Then we have the sharp inequalities:*

$$|a_{p+1}a_{p+2}-a_{p+3}|\leq\begin{cases} 2 & \text{if} \quad \alpha=0 \\ \dfrac{2p\left(p^2+3\alpha p+6\alpha^2\right)^{\frac{3}{2}}}{3(p+\alpha)(p+2\alpha)(p+3\alpha)\sqrt{6\alpha^2}} & \text{if} \quad 0<\alpha\leq 1. \end{cases} \tag{3.6}$$

Proof. By (3.5) (for $j = 1, 2, \text{and } 3$), we find that

$$|a_{p+1}a_{p+2}-a_{p+3}|=\left|\frac{p^2c_1c_2}{(p+\alpha)(p+2\alpha)}-\frac{pc_3}{(p+3\alpha)}\right|. \tag{3.7}$$

Substituting the values of c_2 and c_3 from Lemma 2.2 in (3.7) and letting $c_1 = c$, we have

$$\begin{aligned} |a_{p+1}a_{p+2}-a_{p+3}| = \frac{p}{4(p+\alpha)(p+2\alpha)(p+3\alpha)}\big|(p^2+3\alpha p-2\alpha^2)c^3-4\alpha^2c(4-c^2)x \\ +c(p^2+3\alpha p+2\alpha^2)(4-c^2)x^2-2(p^2+3\alpha p+2\alpha^2)(4-c^2)(1-|x|^2)z\big|. \end{aligned}$$

Without loss of generality, assume that $c = c_1 \in [0, 2]$ (see Lemma 2.1). Then, by applying the triangle inequality with $\delta = |x|$ and noticing that $p^2+3\alpha p-2\alpha^2 \geq 0$, since $0 \leq \alpha \leq 1$ by definition, we get

$$\begin{aligned}
|a_{p+1}a_{p+2}-a_{p+3}| \leq &\frac{p}{4(p+\alpha)(p+2\alpha)(p+3\alpha)}\big[(p^2+3\alpha p-2\alpha^2)c^3+4\alpha^2 c(4-c^2)\delta \\
&+c(p^2+3\alpha p+2\alpha^2)(4-c^2)\delta^2+2(p^2+3\alpha p+2\alpha^2)(4-c^2)(1-\delta^2)\big] \\
&= \frac{p}{4(p+\alpha)(p+2\alpha)(p+3\alpha)}\big[2(p^2+3\alpha p+2\alpha^2)(4-c^2)+(p^2+3\alpha p-2\alpha^2)c^3 \\
&+4\alpha^2 c(4-c^2)\delta+(p^2+3\alpha p+2\alpha^2)(c-2)(4-c^2)\delta^2\big] \\
&:= F(\delta), \text{ where } 0 \leq \delta \leq 1.
\end{aligned} \tag{3.8}$$

We then maximize the function $F(\delta)$ on the closed interval $[0, 1]$:

$$F'(\delta) = \frac{p\alpha^2 c(4-c^2)}{(p+\alpha)(p+2\alpha)(p+3\alpha)} + \frac{p(c-2)(4-c^2)}{2(p+3\alpha)}\delta.$$

Note that $F'(\delta) \geq F'(1^-) > 0$ for all $c \in (0, 2)$ and $F'(\delta) \leq 0$ otherwise. Hence, there exists $c^* \in [0, 2]$ such that $F'(\delta) > 0$ for $c \in (c^*, 2]$ and $F'(\delta) \leq 0$ otherwise. Thus, for $c \in (c^*, 2]$ we observe that $F(\delta) \leq F(1)$, that is:

$$|a_{p+1}a_{p+2}-a_{p+3}| \leq \frac{p(p^2+3\alpha p+6\alpha^2)}{(p+\alpha)(p+2\alpha)(p+3\alpha)}c - \frac{2\alpha^2 p}{(p+\alpha)(p+2\alpha)(p+3\alpha)}c^3 := G_1(c). \tag{3.9}$$

If $\alpha = 0$, we have $G_1(c) = c \leq 2$. Otherwise, simplifying the relations (3.8) and (3.9), we get

$$G_1(c) = \frac{p(p^2+3\alpha p+6\alpha^2)}{(p+\alpha)(p+2\alpha)(p+3\alpha)}c - \frac{2\alpha^2 p}{(p+\alpha)(p+2\alpha)(p+3\alpha)}c^3 \tag{3.10}$$

$$G_1^{'}(c) = \frac{p(p^2+3\alpha p+6\alpha^2)}{(p+\alpha)(p+2\alpha)(p+3\alpha)} - \frac{6\alpha^2 p}{(p+\alpha)(p+2\alpha)(p+3\alpha)}c^2 \tag{3.11}$$

$$G_1^{''}(c) = -\frac{12\alpha^2 p}{(p+\alpha)(p+2\alpha)(p+3\alpha)}c \tag{3.12}$$

For an optimum value of $G_1(c)$, consider $G_1^{'}(c) = 0$. From (3.11), we have

$$c^2 = \frac{p^2+3\alpha p+6\alpha^2}{6\alpha^2} \tag{3.13}$$

Using the obtained value of c^2 from (3.13) in (3.12), after simplifying, we get

$$G_1^{''}(c) = -\frac{12\alpha^2 p}{(p+\alpha)(p+2\alpha)(p+3\alpha)}\sqrt{\frac{p^2+3\alpha p+6\alpha^2}{6\alpha^2}} < 0 \text{ for } p \in \mathbb{N}.$$

Therefore, by the second derivative test, $G_1(c)$ has maximum value at c, where c^2 is given by (3.13). Substituting c^2 value in the expression (3.10), upon simplification, the maximum value of $G_1(c)$ at c^2 is obtained as

$$G_1(c)_{\max} = \frac{p}{(p+\alpha)(p+2\alpha)(p+3\alpha)} \frac{2\left(p^2+3\alpha p+6\alpha^2\right)^{\frac{3}{2}}}{3\sqrt{6\alpha^2}}.$$

Upon simplification, we obtain

$$|a_{p+1}a_{p+2} - a_{p+3}| \leq \frac{2p\left(p^2+3\alpha p+6\alpha^2\right)^{\frac{3}{2}}}{3(p+\alpha)(p+2\alpha)(p+3\alpha)\sqrt{6\alpha^2}}.$$

This completes the proof. ∎

Theorem 3.3 *Let $f(z) \in \mathcal{H}_{p,\alpha}$. Then*

$$\left|a_{p+2} - a_{p+1}^2\right| \leq \frac{2p}{p+2\alpha}. \tag{3.14}$$

Proof. Since $f(z) \in \mathcal{H}_{p,\alpha}$, then using equation (3.5), (for $j = 1$ and $j = 2$), we find that

$$\left|a_{p+2} - a_{p+1}^2\right| = \left|\frac{pc_2}{(p+2\alpha)} - \frac{p^2c_1^2}{(p+\alpha)^2}\right| = \frac{p}{p+2\alpha}\left|c_2 - \frac{2p(p+2\alpha)}{(p+\alpha)^2}\left(\frac{c_1^2}{2}\right)\right|.$$

Now, since $0 \leq \sigma = \frac{2p(p+2\alpha)}{(p+\alpha)^2} \leq 2$, then Lemma 2.3 yeilds

$$\left|a_{p+2} - a_{p+1}^2\right| \leq \frac{2p}{p+2\alpha}. \tag{3.15}$$

This completes the proof. ∎

Substituting the above results in (1.3) together with the known inequality $\left|a_{p+1}a_{p+3} - a_{p+2}^2\right| \leq \frac{4p^2}{(p+2\alpha)^2}$ ([9]) and $|a_k| \leq \frac{2}{k}$, where $k \in \{p+1,\ p+2,\ p+3, ...\}$ ([13]), after simplifying, we obtain the sharp inequalities:

Corollary 3.4 *Let $f(z) \in \mathcal{H}_{p,\alpha}$. Then*

$$|H_3(p)| \leq \begin{cases} 4\left[\frac{2p^2}{(p+2)(p+2\alpha)^2} + \frac{p}{(p+4)(p+2\alpha)} + \frac{1}{p+3}\right] & \text{if } \alpha = 0 \\ \frac{2}{p+2\alpha}\left[\frac{4p^2}{(p+2)(p+2\alpha)} + \frac{\sqrt{2}p\left(p^2+3\alpha p+6\alpha^2\right)^{\frac{3}{2}}}{3\sqrt{3\alpha^2}(p+3)(p+\alpha)(p+3\alpha)} + \frac{2p}{p+4}\right] & \text{if } 0 < \alpha \leq 1. \end{cases} \tag{3.16}$$

Remark 3.5 As a final remark, for the choice of $\alpha = 1$, from expressions (3.6) and (3.16), we obtain

$$|a_{p+1}a_{p+2} - a_{p+3}| \leq \frac{\sqrt{2}p\left(p^2+3p+6\right)^{\frac{3}{2}}}{3\sqrt{3}(p+1)(p+2)(p+3)}$$

and

$$|H_3(p)| \leq \frac{2}{p+2}\left[\frac{4p^2}{(p+2)^2} + \frac{\sqrt{2}p\,(p^2+3p+6)^{\frac{3}{2}}}{3\sqrt{3}(p+3)^2(p+1)} + \frac{2p}{p+4}\right],$$

respectively. These inequalities are sharp and coincide with the results obtained by ([10]).

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References

[1] Babalola, K.O., Opoola, T.O., *On the coefficients of a certain class of analytic functions*, Advances in Inequalities for Series, S.S. Dragomir and A. Sofo (Eds.), Nova Science Publishers, 2006, 5-17. (2008), 5-17.

[2] Ehrenborg, R., *The Hankel determinant of exponential polynomials*, American Mathematical Monthly, 2000, 557-560.

[3] Grenander, U., Szegö,G., *Toeplitz Forms and Their Applications*, 2nd edn (New York: Chelsea), 1984.

[4] Hayami, T., Owa, S., *Hankel determinant for $p-$valently starlike and convex functions of orderα*, General Math., 17 (4) (2009), 29-44.

[5] Janteng, A., Halim, S.A., Darus, M., *Hankel determinant for functions starlike and convex with respect to symmetric points*, Journal of Quality Measurement and Analysis, 2 (1) (2006), 37-43.

[6] Janteng, A., Halim, S.A., Darus, M., *Coefficient inequality for a function whose derivative has a positive real part*, Journal of Inequalities in Pure and Applied Mathematics, 7 (2) (2006), 1–5.

[7] Janteng, A., Halim, S.A., Darus, M., *Hankel determinant for starlike and convex functions*, Int. J. Math. Anal., 13 (1) (2007), 619-625.

[8] Krishna, D.V., Ramreddy, T., *Hankel determinant for $p-$valent starlike and convex functions of order α*, Novi Sad J. Math., 42 (2) (2012), 89-96.

[9] Krishna, D.V., Ramreddy, T., *Coefficient inequality for certain subclass of p-valent functions*, Pales. Poly. Univ., 2(2015), 223-228.

[10] Vamshee Krishna, D., Venkateswarlu, B., Ramreddy, T., *Third Hankel determinant for certain subclass of $p-$valent functions*, Complex Variables and Elliptic Equations 60. 9(2015), 1301-1307.

[11] Libera, R.J., Złotkiewicz, E.J., *Early coefficients of the inverse of a regular convex function*, Proceedings of the American Mathematical Society 85. 2(1982), 225-230.

[12] Libera, R.J., Złotkiewicz, E.J., *Coefficient bounds for the inverse of a function with derivative in P*, Proceedings of the American Mathematical Society 87. 2(1983), 251-257.

[13] MacGregor, T.H., *Functions whose derivative has a positive real part*, Transactions of the American Mathematical Society, 104 (3) (1962), 532–537.

[14] Noonan, J.W., Thomas, D.K., *On the second Hankel determinant of areally mean $p-$valent functions*, Transactions of the American Mathematical Society, 223 (1976), 337-346.

[15] Noor, K.I., *Hankel determinant problem for the class of functions with bounded boundary rotation*, Revue Roumaine de Mathématiques Pures et Appliquées, 28 (8) (1983), 731-739.

[16] Pommerenke, C., *On the coefficients and Hankel determinants of univalent functions*, Journal of the London Mathematical Society, 1 (1) (1966), 111-122.

[17] Pommerenke, C., *On the Hankel determinants of univalent functions*, Mathematika, 14 (1) (1967), 108-112.

[18] Pommerenke, C., Jensen, G., *Univalent functions*, Vol. 25, Göttingen: Vandenhoeck und Ruprecht, 1975.

[19] Simon, B., *Orthogonal polynomials on the unit circle*, Part 1, vol. 54 of American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2005.

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