REPRESENTATIONS OF POLYGROUPS

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Abstract. The purpose of this paper is the study of representation of polygroups based on hyperspaces (hypervector spaces). In this regards we introduce and study representation and weak representation of a given polygroup. In particular, we study irreducible and weak irreducible representation of polygroups and obtain some basic properties of them.

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1. Introduction

The theory of algebraic hyperstructures is a well-established branch of classical algebraic theory. Hyperstructure theory was first proposed in 1934 by Marty, who defined hypergroups and began to investigate their properties with applications to groups, rational fractions and algebraic functions [16]. It was later observed

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that the theory of hyperstructures has many applications in both pure and applied sciences; for example, semi-hypergroups are the simplest algebraic hyperstructures that possess the properties of closure and associativity. The theory of hyperstructures has been widely reviewed (for more see [1]-[7], [12]-[15], [21]).

M.S. Tallini introduced the notion of hyperspaces (hypervector spaces) ([18], [19] and [20]) and studied basic properties of them. R. Ameri and O.R. Dehghan [2] introduced and studied dimension of hyperspaces and, in [17], M. Motameni et.al. studied hypermatrix. In this paper, we consider hyperspaces in the sense of Tallini to give a representation for a given polygroup. In this regards, we give representation and weak representation of a polygroups based on various categories of hyperspaces. Also, we discuss on irreducible and weak irreducible representations of a polygroup. Finally, we investigate the relationship between irreducible and decomposable representations and obtain some results.

2. Preliminaries

In this section we give some notions and results of hypergroups and hyperspaces, which we need to developing our paper.

Definition 2.1. Let H be a set. A map $. : H \times H \longrightarrow P_*(H)$ is called a hyperoperation or join operation, where $P_*(H)$ is the set of all non-empty subsets of H. The join operation is extended to subsets of H in natural way, so that $A.B$ is given by

$$A.B = \bigcup \{a.b : a \in A \textit{ and } b \in B\}.$$

the notations $a.A$ and $A.a$ are used for $\{a\}.A$ and $A.\{a\}$ respectively. Generally, the singleton $\{a\}$ is identified by its element a.

Definition 2.2. [12] A hypergroup is a set H equipped with an associative hyperoperation $. : H \times H \to P_*(H)$ which satisfies the property $x.H = H.x = H$, for all $x \in H$. If the hyperoperation . is associative, then H is called a semihypergroup. In the above definition if $A, B \subseteq H$ and $x \in H$, then we define

$$A.B = \bigcup_{a\in A, b\in B} a.b, \quad x.B = \{x\}.B, \quad \textit{and} \quad A.x = A.\{x\}.$$

A quasicanonical hypergroup, is a special kind of a hypergroup, that first time introduced and studied by Bonansinga and Corsini in [8, 9]. After that this kind of hypergroups studied by Comer [10, 11] as the name of polygroups.

Definition 2.3. [12, 14] A polygroup is a system $\mathcal{P} = \langle P, ., e, ^{-1} \rangle$, where $e \in P$, $^{-1}$ is a unary operation on P, . maps $P \times P$ into nonempty subsets of P, and the following axioms hold for all $x, y, z \in P$:
(P_1) $(x.y).z = x.(y.z)$;
(P_2) $x.e = e.x = x$;
(P_3) $x \in y.z$ implies $y \in x.z^{-1}$ and $z \in y^{-1}.x$.

The following elementary facts about polygroups follow easily from the axioms: $e \in x.x^{-1} \cap x^{-1}.x$, $e^{-1} = e$, $(x^{-1})^{-1} = x$, and $(x.y)^{-1} = y^{-1}.x^{-1}$, where $A^{-1} = \{a^{-1} | a \in A\}$. A polygroup in which every element has order 2 (i.e., $x^{-1} = x$ for all x) is called symmetric. As in group theory it can be shown that a symmetric polygroup is commutative.

The concept of hyperspace, which is a generalization of the concept of ordinary vector space.

Definition 2.4. [18] Let K be a field and $(V, +)$ be an abelian group. We define a hyperspace over K to be the quadrupled $(V, +, \circ, K)$, where $\circ$ is a mapping

$$\circ : K \times V \longrightarrow P_*(V),$$

such that the following conditions hold:
(H_1) $\forall\ a \in K,\ \forall\ x, y \in V,\ a \circ (x + y) \subseteq a \circ x + a \circ y$, right distributive law;
(H_2) $\forall\ a, b \in K,\ \forall\ x \in V,\ (a + b) \circ x \subseteq a \circ x + b \circ x$, left distributive law;
(H_3) $\forall\ a, b \in K,\ \forall\ x \in V,\ a \circ (b \circ x) = (ab) \circ x$, associative law;
(H_4) $\forall\ a \in K,\ \forall\ x \in V,\ a \circ (-x) = (-a) \circ x = -(a \circ x)$;
(H_5) $\forall\ x \in V,\ x \in 1 \circ x$.

Remark 2.5.

(i) In the right hand side of (H_1) the sum is meant in the sense of Frobenius, that is we consider the set of all sums of an element of $a \circ x$ with an element of $a \circ y$. Similarly we have in (H_2).

(ii) We say that $(V, +, \circ, K)$ is anti-left distributive, if

$$\forall\ a, b \in K,\ \forall\ x \in V,\ (a + b) \circ x \supseteq a \circ x + b \circ x,$$

and strongly left distributive, if

$$\forall\ a, b \in K,\ \forall\ x \in V,\ (a + b) \circ x = a \circ x + b \circ x,$$

In a similar way we define the anti-right distributive and strongly right distributive hyperspaces, respectively. V is called strongly distributive if it is both strongly left and strongly right distributive.

(iii) The left hand side of (H_3) means the set-theoretical union of all the sets $a \circ y$, where y runs over the set $b \circ x$, i.e.,

$$a \circ (b \circ x) = \bigcup_{y \in b \circ x} a \circ y.$$

(iv) Let $\Omega_V = 0 \circ 0_V$, where 0_V is the zero of $(V, +)$, In [18] it is shown if V is either strongly right or left distributive, then Ω_V is a subgroup of $(V, +)$.

Example 2.6. [2] Consider abelian group $(\mathbb{R}^2, +)$. Define hyper-compositions:

$$\begin{cases} \circ : \mathbb{R} \times \mathbb{R}^2 \longrightarrow P_*(\mathbb{R}^2) \\ \quad a \circ (x, y) = ax \times \mathbb{R} \end{cases}$$

and

$$\begin{cases} \diamond : \mathbb{R} \times \mathbb{R}^2 \longrightarrow P_*(\mathbb{R}^2) \\ \quad a \diamond (x, y) = \mathbb{R} \times ay. \end{cases}$$

Then $(\mathbb{R}^2, +, \circ, \mathbb{R})$ and $(\mathbb{R}^2, +, \diamond, \mathbb{R})$ are a strongly distributive hyperspaces.

Example 2.7. [2] Let $(V, +, ., K)$ be a classical vector space and P be a subspace of V. Define the hyper-composition:

$$\begin{cases} \circ : K \times V \longrightarrow P_*(V) \\ \quad a \circ x = a.x + P. \end{cases}$$

Then it is easy to verify that $(V, +, \circ, K)$ is a strongly distributive hyperspace.

Example 2.8. [18] In $(\mathbb{R}^2, +)$ define the hyper-composition $\circ$ as follows:

$$\forall a \in \mathbb{R}, \forall x \in \mathbb{R}^2 \ : \ a \circ x = \begin{cases} \text{line } ox & \text{if } x \neq 0_V, \\ \{0_V\} & \text{if } x = 0_V, \end{cases}$$

where $0_V = (0, 0)$. Then $(\mathbb{R}^2, +, \circ, \mathbb{R})$ is a strongly left, but not right distributive hyperspace.

Proposition 2.9. [18] *Every strongly right distributive hyperspace is strongly left distributive hyperspace. Let $(V, +)$ be an abelian group, Ω a subgroup of V and K a field such that $W = V/\Omega$ is a classical vector space over a field K. If $p : V \longrightarrow W$ is the canonical projection of $(V, +)$ onto $(W, +)$ and set:*

$$\begin{cases} \circ : K \times V \longrightarrow P_*(V) \\ \quad a \circ x = p^{-1}(a.p(x)). \end{cases}$$

Then $(V, +, \circ, K)$ is a strongly distributive hyperspace over a field K. Moreover every strongly distributive hyperspace can be obtained in such a way.

Proposition 2.10. [18] *If $(V, +, \circ, K)$ is a left distributive hyperspace, then for all $a \in K$, and $x \in V$:*

(i) *$0 \circ x$ is a subgroup of $(V, +)$;*

(ii) *Ω_V is a subgroup of $(V, +)$;*

(iii) *$a \circ 0_V = \Omega_V = a \circ \Omega_V$;*

(iv) *$\Omega_V \subseteq 0 \circ x$;*

(v) *$x \in 0 \circ x \Longleftrightarrow 1 \circ x = 0 \circ x \Longleftrightarrow a \circ x = 0 \circ x$.*

Remark 2.11. Let $(V, +, \circ, K)$ be a hyperspace and W be a subhyperspace of V. Consider the quotient abelian group $(V/W, +)$. Define the rule:

$$\begin{cases} * : K \times V/W \longrightarrow P_*(V/W) \\ \quad (a, x + W) \longmapsto a \circ x + W. \end{cases}$$

Then it is easy to verify that $(V/W, +, *, K)$ is a hyperspace over a field K and it is called the quotient hyperspace of V over W.

Definition 2.12. [2] Let V be a hyperspace over a field K. A nonempty subset W of V is called a subhyperspace if W is itself a hyperspace with the hyperoperation on V, i.e.,

$$W \neq \emptyset, \;\; W - W \subseteq W, \;\; \forall a \in K, \;\; a \circ W \subseteq W.$$

In this case we write $W \leq V$.

Definition 2.13. [2] Let V be a hyperspace over a field K. If W is a nonempty subset of V, then the linear span of W is defined by

$$\begin{aligned} L(W) &= \{t \in V : t \in \sum_{i=1}^{n} a_i \circ w_i, a_i \in K, w_i \in W, n \in \mathbb{N}\} \\ &= \{t_1 + t_2 + ... + t_n : t_i \in a_i \circ w_i, a_i \in K, w_i \in W, n \in \mathbb{N}\}. \end{aligned}$$

Lemma 2.14. [2] *$L(W)$ is the smallest subhyperspace of V containing W.*

Definition 2.15. [2] Let V be a hyperspace over a field K. A subset W of V is called linearly independent if for every vectors $v_1, v_2, ..., v_n$ in W, $c_1, c_2, ..., c_n \in K$, and $0_V \in c_1 \circ v_1 + ... + c_n \circ v_n$, implies that $c_1 = c_2 = ... = c_n = 0$. A subset W of V is called linearly dependent if it is not linearly independent.

Definition 2.16. [2] Let V be a hyperspace over a field K. A basis for V is a linearly independent subset of V such that span V. We say that V has finite dimensional if it has a finite basis.

Definition 2.17. [2] Let V and W be two hyperspaces over a field K. A mapping $T : V \longrightarrow W$ is called

(i) weak linear transformation iff
$T(x+y) = T(x)+T(y)$ and $T(a \circ x) \cap a \circ T(x) \neq \emptyset$, for all $x, y \in V$, $a \in K$.

(ii) linear transformation iff
$T(x + y) = T(x) + T(y)$ and $T(a \circ x) \subseteq a \circ T(x)$, for all $x, y \in V$, $a \in K$.

(iii) strong linear transformation iff
$T(x + y) = T(x) + T(y)$ and $T(a \circ x) = a \circ T(x)$, for all $x, y \in V$, $a \in K$.

A (resp. weak, strong) linear isomorphism is defined as usual. If $T : V \longrightarrow W$ is a (resp. weak, strong) linear isomorphism, then it is denoted by (resp. $V \cong_w W$, $V \cong_s W$) $V \cong W$.

Definition 2.18. [2] Let V and W be two hyperspaces over a field K and $T : V \longrightarrow W$ be a linear transformation. The kernel and image of T are denoted by $kerT$ and ImT, respectively, are defined by

$$kerT = \{x \in V \mid T(x) \in \Omega_W\},$$

and

$$ImT = \{y \in W \mid y = T(x) \text{ for some } x \in V\}.$$

Proposition 2.19. [2] *Let $T : V \to W$ be a strong linear transformation.*

(i) *If Z is a subhyperspace of V, then the image of Z, $T(Z)$ is a subhyperspace of W. In particular ImT is a subhyperspace of W.*

(ii) *If L be a subhyperspace of W, then the preimage of L, $T^{-1}(L)$ is a subhyperspace of V containing $kerT$.*

Proposition 2.20. [2] *Let V and W be strongly left distributive hyperspaces over a field K, and $T : V \to W$ be a linear transformation. Then $kerT$ is a subhyperspace of V. Moreover, $\Omega_V \subseteq kerT$.*

Proposition 2.21. [2] *Let V and W be strongly left distributive hyperspaces over a field K, and $T : V \to W$ be a linear transformation. Then*

$$V/kerT \cong T(V)/\Omega_W,$$

Moreover if T is onto, then

$$V/kerT \cong W/\Omega_W.$$

Corollary 2.22. [2] *Let V be a strongly left distributive hyperspace over a field K, and let $B = \{x_1, ..., x_n\}$ be a basis for V. Then $V/0 \circ \omega \cong K^n$, where $\omega = \sum_{i=1}^{n} x_i$.*

Proposition 2.23. *Let $(V, +, \circ, K)$ be a strongly left distributive hyperspace over a field K. If $T : V \longrightarrow V$ be a linear transformation such that $T(0) = 0$, and $(-T)(\alpha) = T(-\alpha)$. We define $l(V) = \{T \mid T \text{ is invertible}\}$ and the operation $+$ as follows:*

$$(T + U)(\alpha) = T(\alpha) + U(\alpha),$$

Also, we define the external composition as

$$(a \circ T)(\alpha) = a \circ T(\alpha).$$

Then $(l(V), +, \circ, K)$ as defined above is a vector space over a field K.

Proof. The external composition $\circ$ is defined as follows:

$$\begin{array}{rcl} \circ : K \times l(V) & \longrightarrow & l(V) \\ (a, T) & \longmapsto & a \circ T. \end{array}$$

First, we show that $(l(V),+)$ is an abelian group. Let T,U, and Z be three linear transformations that belong to $l(V)$. Then

$$\begin{aligned}(T+(U+Z))(\alpha) &= T(\alpha)+(U+Z)(\alpha)\\ &= T(\alpha)+(U(\alpha)+Z(\alpha))\\ &= (T(\alpha)+U(\alpha))+Z(\alpha),\ \text{since V is hyperspace}\\ &= ((T+U)+Z)(\alpha).\end{aligned}$$

Also,

$$(T+U)(\alpha)=T(\alpha)+U(\alpha)=U(\alpha)+T(\alpha)=(U+T)(\alpha).$$

Thus associativity and commutativity is hold. We consider the transformation $0 : V \longrightarrow 0$ as a 0 for the group and $1_V : V \longrightarrow V$ as the identity. Then there exists a unique inverse $(-T)$ such that

$$\begin{aligned}(T+(-T))(\alpha) &= T(\alpha)+(-T)(\alpha)\\ &= T(\alpha)+T(-\alpha)\\ &= T(\alpha+(-\alpha))\\ &= T(0)\\ &= 0.\end{aligned}$$

Therefore, $(l(V),+)$ is an abelian group.

Now, we check that $I(V)$ is a vector space. Let $a,b \in K$ and $T,U \in I(V)$. Then we have

(1) $(a \circ (T+U))(\alpha) = a \circ (T+U)(\alpha) = (a \circ T(\alpha)) + (a \circ U(\alpha))$,

(2) $((a+b) \circ T)(\alpha) = (a \circ T(\alpha)) + (b \circ T(\alpha))$.

The other conditions will be obtained immediately. Therefore, $(l(V),+,\circ,K)$ is a vector space. ■

The composition $S \odot T : V \longrightarrow V$ of linear transformations $T : V \longrightarrow V$ and $S : V \longrightarrow V$ is defined as follows:

$$(S \odot T)(x) = ST(x) = S(T(x)).$$

Proposition 2.24. *Let $(V,+,\circ,K)$ be a hyperspace over a field K. Then $(l(V),\odot)$ is a group.*

Proof. First, we show that $l(V)$ is an associative. Let T,U, and Z be three linear transformations that belong to $l(V)$. Then

$$T \odot (U \odot Z))(\alpha) = T(U \odot Z)(\alpha) = T(U(Z(\alpha))),$$

Also,

$$(T \odot U) \odot Z)(\alpha) = (T \odot U)(Z(\alpha)) = T(U(Z(\alpha))).$$

Therefore $T \odot (U \odot Z) = (T \odot U) \odot Z$. Also,

$$\exists 1_V \in l(V);\ \forall T \in l(V),\ T \odot 1_V = 1_V \odot T = T,$$

and

$$\forall T \in l(V),\ \exists T^{-1} \in l(V);\ T \odot T^{-1} = T^{-1} \odot T = 1_V.$$

Thus $(l(V), \odot)$ is a group. ■

Suppose $\dim V = n$, $K = \mathbb{R}$, $M_n(\mathbb{R}) = \{n \times n$ matrices with entries in $\mathbb{R}\}$, and $GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid A$ is invertible$\}$. Then $l(V) \cong GL_n(\mathbb{R})$ (for more details see [17]).

Definition 2.25. Let V and W be two hyperspaces over a field K. A multivalued linear transformation (MLT), $T : V \longrightarrow P_*(W)$ is a mapping such that, for all $x, y \in V$ and $a \in K$,

(1) $T(x + y) \subseteq T(x) + T(y)$;

(2) $T(a \circ x) \subseteq a \circ T(x)$;

(3) $T(-a) = -T(a)$.

Remark 2.26.

(i) In Definition 2.25 (1) and (2), if the equality holds, then T is called a strong multivalued linear transformation $(SMLT)$.

(ii) In Definition 2.25, if we use the all conditions of "=" and "$\subseteq$" we can find three types of $SMLT$. Here we consider only inclusion and equality cases.

(iii) If T is a MLT, then $0 \in T(x)$. Since $T(x) \neq \emptyset$, so there exists $y \in T(x)$; $0 = y - y \in T(x) - T(x) = T(x) + T(-x) = T(x + (-x)) = T(x - x) = T(0)$.

Definition 2.27. [1] Let V and W be two hyperspaces over a field K and $T : V \longrightarrow P_*(W)$ be a $SMLT$. Then multivalued kernel and multivalued image of T, denoted by $\overline{Ker}T$ and $\overline{Im}T$, respectively, are defined as follows:

$$\overline{Ker}T = \{x \in V \mid 0_W \in T(x)\},$$

and

$$\overline{Im}T = \{y \in W \mid y \in T(x) \text{ for some } x \in V\}.$$

Remark 2.28.

(i) Note that $\overline{Ker}T \neq \emptyset$, by Remark 2.26(iii).

(ii) For hyperspaces V and W over a field K, by $Hom_K(V, W)$ and $Hom^s_K(V, W)$, we mean the set of all MLT and $SMLT$, respectively and sometimes we use morphism instead multivalued linear transformation, respectively.

In the following, we briefly introduced the categories of hyperspaces and study the relationship between monomorphism, epimorphism, isomorphism and monic, epic and iso objects in this category.

Definition 2.29. The category of hyperspaces over a field K denoted by $\mathcal{HV}_K$ is defined as follows:

(1) The objects of $\mathcal{HV}_K$ are all hyperspaces over K;

(2) For the objects V and W of $\mathcal{HV}_K$, the set of all morphisms from V to W denoted by $Hom_K(V,W)$, is the set of all MLT from V to W;

(3) The composition $S \odot T : V \longrightarrow P_*(W)$ of morphisms $T : V \longrightarrow P_*(L)$ and $S : L \longrightarrow P_*(W)$ is defined as follows:

$$S \odot T(x) = ST(x) = \bigcup_{t \in T(x)} S(t).$$

(4) For any object V, the morphism $1_V : V \longrightarrow P_*(V)$, $x \to \{x\}$ is the identity.

Remark 2.30. If in Definition 2.29(2), we replace $Hom_K(V,W)$ by $Hom_K^s(V,W)$, the set of all $SMLT$, then we will obtain a new category, which it denotes by $\mathcal{HV}_K^s$. In fact, $\mathcal{HV}_K^s \preceq \mathcal{HV}_K$ (by $A \preceq B$ we mean A is a subcategory of B). Also, denote the category of all vector spaces over a field K by $\mathcal{V}_K$. Clearly, $\mathcal{V}_K \preceq \mathcal{HV}_K^s$ (for more details see [1]).

Definition 2.31. Let $T : V \longrightarrow P_*(W)$ be a $SMLT$ of hyperspaces. We say that T is weakly injective if

$$\forall\ x, y \in V,\ T(x) \cap T(y) \neq \emptyset \Rightarrow x = y.$$

We say that T is strongly injective if

$$\forall\ x, y \in V,\ T(x) = T(y) \Rightarrow x = y.$$

Remark 2.32. Clearly, every weakly injective morphism is also strongly injective. Note that T is strongly injective, means that T is injective as a function with values in $P_*(W)$.

Proposition 2.33. *Let V and W be strongly left distributive hyperspaces such that $|1 \circ x| = 1$, for all $x \in V$. If $T : V \longrightarrow P_*(W)$ is monic in $\mathcal{HV}_K^s$, then T is strongly injective.*

Proof. Suppose that $T : V \longrightarrow P_*(W)$ is a monic. Fix $x_1, x_2 \in V$ and let $T(x_1) = T(x_2)$. Define mappings $\hat{x_1}, \hat{x_2} : K \longrightarrow P_*(V)$ by $\hat{x_1}(a) = a \circ x_1$ and $\hat{x_2}(a) = a \circ x_2$ (here K is viewed as a hyperspace). $\hat{x_1}$ and $\hat{x_2}$ are well-defined morphisms of hyperspaces since for all $a, b \in K$, and $i \in \{1, 2\}$, we have

$$\hat{x_i}(a + b) = (a + b) \circ x_i = a \circ x_i + b \circ x_i = \hat{x_i}(a) + \hat{x_i}(b),$$

$$\hat{x_i}(ab) = (ab) \circ x_i = a \circ (b \circ x) = a \circ \hat{x_i}(b),$$

$$\hat{x_i}(-a) = (-a) \circ x_i = -(a \circ x_i) = -\hat{x_i}(a).$$

Moreover, $T\hat{x_1}(a) = T(a \circ x_1) = a \circ T(x_1) = a \circ T(x_2) = T(a \circ x_2) = T\hat{x_2}(a)$, and hence $\hat{x_1} = \hat{x_2}$, since T is monic. In particular $\hat{x_1}(1) = \hat{x_2}(1)$, that $1 \circ x_1 = 1 \circ x_2$, then $x_1 = x_2$. ∎

Proposition 2.34. *In $\mathcal{HV}_K^s$, if $S : V \longrightarrow P_*(W)$ is weakly injective, then it is a monic.*

Proof. Say $S : V \longrightarrow P_*(W)$ is weakly injective and let $T, U : Z \longrightarrow P_*(V)$ be two morphisms such that $ST = SU$. It suffices to show that $T(x) = U(x)$, for all $x \in Z$. Indeed fix $x \in Z$ and $y \in U(x)$. Then $S(y) \subseteq SU(x) = ST(x)$, and therefore for some $z \in T(x), S(y) \cap S(z) \neq \emptyset$. Thus $y = z$ and hence $y \in T(x)$. The other inclusion is proved analogously. ∎

Similarly, we introduce the notions of weakly and strongly surjective. A morphism $T : V \longrightarrow P_*(W)$ of hyperspaces is said to be weakly surjective if for every $y \in W$ there exists $x \in V$ such that $y \in T(x)$ and is strongly surjective, if for every non-empty subset Z of W, there exists $x \in V$ such that $Z = T(x)$.

Remark 2.35. Clearly, every strongly surjective morphism is weakly surjective. But the converse is not true. For example the identity function on every hyperspace is weakly surjective, but is not strongly surjective.

Proposition 2.36. *Let $T : V \longrightarrow P_*(W)$ be a $SMLT$:*

(i) *If Z is a subhyperspace of V, then $T(Z)$ is also a subhyperspace of W. In particular, $\overline{Im}T$ is a subhyperspace of W.*

(ii) *If L is a subhyperspace of W, then $T^{-1}(L)$ is also a subhyperspace of V containing $\overline{ker}T$, where $T^{-1}(L) = \{x \in V \mid T(x) \subseteq L\}$.*

Proof. Straightforward. ∎

Proposition 2.37. *In $\mathcal{HV}_K^s$, every epic is weakly surjective.*

Proof. Suppose T is an epic. Define morphisms $0, \pi : W \longrightarrow P_*(W/\overline{Im}T)$ by $0(a) = \{\overline{0}\}$ and $\pi(x) = \{\overline{x}\}$. Then:

$$\pi T = \{\overline{0}\} = 0T,$$

so that $\pi = 0$, thus $\pi(x) = 0(x)$ and $x + \overline{Im}T = 0 + \overline{Im}T$, so $x \in \overline{Im}T$ that is T is a weakly surjective. ∎

Proposition 2.38. *In $\mathcal{HV}_K^s$, if $T : V \longrightarrow P_*(W)$ is a strongly surjective, then it is epic.*

Proof. Let $T : V \longrightarrow P_*(W)$ be a strongly surjective and $S, U : W \longrightarrow P_*(Z)$ be two morphisms such that $ST = UT$. For a fixed $y \in W$, suppose $x \in V$ be such that $T(x) = \{y\}$. Then

$$S(y) = S(T(x)) = ST(x) = UT(x) = U(T(x)) = U(y).$$ ∎

Proposition 2.39. *Let V be a hyperspace such that $|1 \circ x| = 1$ for all $x \in V$. Then a morphism T in $\mathcal{HV}_K^s$ is iso if and only if it is a single valued bijective morphism.*

Proof. First, suppose that $T : V \longrightarrow P_*(W)$ is an iso in $\mathcal{HV}^s_K$ and suppose that T is not single valued, that is for some $x \in V$ there exist $y_1, y_2 \in T(a)$ with $y_1 \neq y_2$. Since T is an iso, there exists a morphism $S : W \longrightarrow P_*(V)$ such that $TS = 1_W$ and $ST = 1_V$. In particular, S is an iso and thus it is strongly injective, by Proposition 2.33 Moreover, $S(y_1) = \{x\}$ and $S(y_2) = \{x\}$, so that $y_1 = y_2$ which is a contradiction. Hence T is single valued, strongly injective and weakly surjective by Propositions 2.33 and 2.37, and thus it is a bijective function.

Conversely, suppose that $T : V \longrightarrow P_*(W)$ is a single valued bijective morphism. Define the mapping $S : W \longrightarrow P_*(V)$ by $S(y) = \{x\}$ if and only if $T(x) = \{y\}$. Clearly, $TS = 1_W$ and $ST = 1_V$, and it is easy to check that S is a morphism in $\mathcal{HV}^s_K$. Now, for $y_1, y_2 \in W$, consider $x_1, x_2 \in V$ such that $T(x_1) = y_1$ and $T(x_2) = y_2$. Therefore

$$\begin{aligned} x \in S(y_1 + y_2) &\iff x = S(y_1 + y_2) \\ &\iff y_1 + y_2 = T(x) \\ &\iff T(x) = T(x_1) + T(x_2) = T(x_1 + x_2) \\ &\iff x = x_1 + x_2 \\ &\iff x = S(y_1) + S(y_2). \end{aligned}$$

Thus,

$$S(y_1 + y_2) = S(y_1) + S(y_2).$$

Also,

$$\begin{aligned} x_1 \in S(a \circ y) &\iff x_1 = S(z) \text{ and } z \in a \circ y \\ &\iff z = T(x_1) \text{ and } z \in a \circ y \\ &\iff T(x_1) \in a \circ y = a \circ T(x_2) = T(a \circ x_2) \\ &\iff x_1 \in a \circ x_2 \\ &\iff x_1 \in a \circ S(y). \end{aligned}$$

Therefore S is a morphism in $\mathcal{HV}^s_K$. ∎

Proposition 2.40. *In $\mathcal{HV}^s_K$ a morphism $T : V \longrightarrow P_*(W)$ is weakly injective if and only if $\overline{Ker}T = \{0\}$.*

Proof. First, let $x \in \overline{Ker}T$. Then $0 \in T(x)$, also $0 \in T(0)$ by Remark 2.26(iii). Thus $T(0) \cap T(x) \neq \emptyset$ implies $0 = x$. Conversly, let $x, y \in V$ such that $T(x) \cap T(y) \neq \emptyset$. So, there exists $z \in T(x) \cap T(y)$ and

$$0 = z - z \in T(x) - T(y) = T(x - y) \Rightarrow x - y \in \overline{Ker}T = \{0\} \Rightarrow x = y. \quad ∎$$

Proposition 2.41. *Let V and W be two left distributive hyperspaces over a field K and $T : V \longrightarrow P_*(W)$ be a $SMLT$. Then $\overline{Ker}T$ is a subhyperspace of V.*

Proof. Let $x, y \in \overline{Ker}T$. Then $0 \in T(x)$ and $0 \in T(y)$, hence $0 = 0 - 0 \in T(x) - T(y) = T(x - y)$ and so $x - y \in \overline{Ker}T$. Also for all $a \in \overline{Ker}T$, $a \circ 0 \subseteq a \circ T(x)$ and by Proposition 2.10, $a \circ 0 = \Omega_W$. Also

$$0 \in \Omega_W \subseteq a \circ T(x) = T(a \circ x).$$

Therefore, $a \circ x \in \overline{Ker}T$. Thus $\overline{Ker}T \leq V$. ∎

3. Representation of polygroups

In the sequel, by V we mean a hyperspace over a field K.

Definition 3.1. A representation of a polygroup P is a homomorphism $\varphi : P \longrightarrow l(V)$ for some (finite-dimensional) non-zero hyperspace V. The dimension of V is called the degree of φ.

We usually write φ_g for $\varphi(g)$ and $\varphi_g(x)$, or simply $\varphi_g x$, for the action of φ_g on $x \in V$.

Definition 3.2. Two representations $\varphi : P \to l(V)$ and $\psi : P \to l(W)$ are equivalent if there exists an isomorphism $T : V \to W$ such that $\psi_g = T\varphi_g T^{-1}$ for all $g \in P$, i.e., $\psi_g T = T\varphi_g$ for all $g \in P$. In this case, we write $\varphi \sim \psi$. In picture, we have the diagram

$$\begin{array}{ccc} V & \xrightarrow{\varphi_g} & V \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ W & \xrightarrow[\psi_g]{} & W \end{array}$$

commutes.

Definition 3.3. Let $\varphi : P \to l(V)$ be a representation. A subhyperspace $W \leq V$ is P-invariant if, for all $g \in P$ and $w \in W$, one has $\varphi_g w \in W$.

Definition 3.4. Suppose that representations $\varphi^{(1)} : P \to l(V_1)$ and $\varphi^{(2)} : P \to l(V_2)$ are given. Then their direct sum

$$\varphi^{(1)} \oplus \varphi^{(2)} : P \to l(V_1 \oplus V_2)$$

is given by

$$(\varphi^{(1)} \oplus \varphi^{(2)})_g(v_1, v_2) = (\varphi_g^{(1)}(v_1), \varphi_g^{(2)}(v_2)).$$

Definition 3.5. A representation $\varphi : P \to l(V)$ is said to be irreducible if the only P-invariant subhyperspaces of V are $\{0\}$ and V.

Definition 3.6. Let P be a polygroup. A representation $\varphi : P \to l(V)$ is said to be completely reducible if $V = V_1 \oplus V_2 \oplus ... \oplus V_n$ where the V_i are non-zero P-invariant subhyperspaces and $\varphi\mid_{V_i}$ are irreducible for all $i = 1, ..., n$.

Equivalently, φ is completely reducible if $\varphi \sim \varphi^{(1)} \oplus \varphi^{(2)} \oplus ... \oplus \varphi^{(n)}$ where the $\varphi^{(i)}$ are irreducible representations.

Definition 3.7. We say that φ is decomposable if $V = V_1 \oplus V_2$ with V_1 and V_2 non-zero P-invariant subhyperspaces. Otherwise, V is called indecomposable.

Lemma 3.8. *Let $\varphi : P \to l(V)$ be equivalent to decomposable representation. Then φ is decomposable.*

Proof. Let $\psi : P \to l(W)$ be a decomposable representation with $\psi \sim \varphi$ and $T : V \to W$ a hyperspace isomorphism with $\varphi_g = T^{-1}\psi_g T$. Suppose that W_1 and W_2 are non-zero invariant subhyperspaces of W with $W = W_1 \oplus W_2$. Since T is an equivalence we have

$$\begin{array}{ccc} V & \xrightarrow{\varphi_g} & V \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ W & \xrightarrow[\psi_g]{} & W \end{array}$$

commutes, i.e., $T\varphi_g = \psi_g T$, for all $g \in P$. Let $V_1 = T^{-1}(W_1)$ and $V_2 = T^{-1}(W_2)$. First we claim $V = V_1 \oplus V_2$, Indeed, if $v \in V_1 \cap V_2$, then $Tv \in W_1 \cap W_2 = \{0\}$ and so $Tv = 0$. But T is injective so this implies $v = 0$. Next, if $v \in V$, then $Tv = w_1 + w_2$ for some $w_1 \in W_1$ and $w_2 \in W_2$. Then $v = T^{-1}w_1 + T^{-1}w_2 \in V_1 + V_2$. Thus $V = V_1 \oplus V_2$.

Next, we show that V_1 and V_2 are P-invariant. If $v \in V_i$ such that $i \in \{1, 2\}$, then $\varphi_g v = T^{-1}\psi_g T v$. But $Tv \in W_i$ implies $\psi_g T v \in W_i$, since W_i is P-invariant. Therefore, we conclude that $\varphi_g v = T^{-1}\psi_g T v \in T^{-1}(W_i) = V_i$, as required. ∎

Similarly, we have the following results, whose proofs we omit.

Lemma 3.9. *Let $\varphi : P \to l(V)$ be equivalent to an irreducible representation. Then φ is irreducible.*

Lemma 3.10. *Let $\varphi : P \to l(V)$ be equivalent to a completely reducible representation. Then φ is completely reducible.*

4. Weak representation of polygroups

Definition 4.1. A weak representation of a polygroup P is a homomorphism $\varphi : P \longrightarrow L(V)$ for some (finite-dimensional) non-zero hyperspace V such that $L(V) = \{T : V \longrightarrow P_*(V) \mid T \textit{ is } MLT\}$. The dimension of V is called the degree of φ.

If $T : V \longrightarrow P_*(W)$ is a MLT, then T induced a map $\overline{T} : P_*(V) \longrightarrow P_*(W)$ by $\overline{T}(A) = \bigcup_{a \in A} T(a)$.

Definition 4.2. Two weak representations $\varphi : G \to L(V)$ and $\psi : G \to L(W)$ are equivalent if there exists an iso $T : V \to P_*(W)$ such that $\overline{\psi_g} T = \overline{T}\varphi_g$ for all $g \in P$. In this case, we write $\varphi \sim \psi$. In picture, we have the diagram

$$\begin{array}{ccc} V & \xrightarrow{\varphi_g} & P_*(V) \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle \overline{T}} \\ P_*(W) & \xrightarrow[\overline{\psi_g}]{} & P_*(W) \end{array}$$

commutes.

Definition 4.3. Let $\varphi : P \to L(V)$ be a weak representation. A subhyperspace $W \leq V$ is weak P-invariant if, for all $g \in P$ and $w \in W$, one has $\varphi_g w \cap W \neq \emptyset$.

Definition 4.4. Let $\varphi : P \to L(V)$ be a weak representation. A a subhyperspace $W \leq V$ is P-invariant if, for all $g \in P$ and $w \in W$, one has $\varphi_g w \subseteq W$.

Remark 4.5. *Clearly P-invariant subhyperspace is weak P-invariant subhyperspace.*

Definition 4.6. Suppose that weak representations $\varphi^{(1)} : P \to L(V_1)$ and $\varphi^{(2)} : P \to L(V_2)$ are given. Then their direct sum

$$\varphi^{(1)} \oplus \varphi^{(2)} : P \to L(V_1 \oplus V_2)$$

is given by

$$(\varphi_g^{(1)} \oplus \varphi^{(2)})_g(v_1, v_2) = (\varphi_g^{(1)}(v_1), \varphi_g^{(2)}(v_2)).$$

Definition 4.7. A weak representation $\varphi : P \to L(V)$ is said to be weak irreducible if the only weak P-invariant subhyperspaces of V are $\{0\}$ and V.

Definition 4.8. A weak representation $\varphi : P \to L(V)$ is said to be irreducible if the only P-invariant subhyperspaces of V are 0 and V.

Definition 4.9. Let P be a polygroup. A weak representation $\varphi : P \to L(V)$ is said to be completely reducible if $V = V_1 \oplus V_2 \oplus ... \oplus V_n$ where the V_i are non-zero P-invariant subhyperspaces and $\varphi\,|_{V_i}$ are irreducible for all $i = 1, ..., n$.

Equivalently, φ is completely reducible if $\varphi \sim \varphi^{(1)} \oplus \varphi^{(2)} \oplus ... \oplus \varphi^{(n)}$ where the $\varphi^{(i)}$ are irreducible representations.

Definition 4.10. We say that φ is decomposable if $V = V_1 \oplus V_2$ with V_1 and V_2 non-zero P-invariant subhypervector spaces. Otherwise, V is called indecomposable.

Proposition 4.11. *Let V be a strongly left distributive hyperspace such that $|1 \circ x| = 1$, for all $x \in V$, and $\varphi : P \to L(V)$ be equivalent to decomposable representation. Then φ is decomposable.*

Proof. Let $\psi : P \to L(W)$ be a decomposable representation with $\psi \sim \varphi$ and $T : V \to P_*(W)$ a hyperspace iso with $\overline{T}\varphi_g = \overline{\psi_g}T$. Suppose that W_1 and W_2 are non-zero P-invariant subhyperspaces of W with $W = W_1 \oplus W_2$. Since T is an equivalence we have

$$\begin{array}{ccc} V & \xrightarrow{\varphi_g} & P_*(V) \\ {\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle \overline{T}} \\ P_*(W) & \xrightarrow[\overline{\psi_g}]{} & P_*(W) \end{array}$$

commutes, i.e., $\overline{T}\varphi_g = \overline{\psi_g}T$, for all $g \in G$. Let $V_1 = T^{-1}(W_1)$ and $V_2 = T^{-1}(W_2)$. First we claim $V = V_1 \oplus V_2$, Indeed, if $v \in V_1 \cap V_2$, then $T(v) \in W_1 \cap W_2 = \{0\}$ and so $Tv = \{0\}$. But T is monic by Proposition 2.33 implies $v = 0$. Next, if

$v \in V$, then for all $w \in Tv$, $w = w_1 + w_2$ for some $w_1 \in W_1$ and $w_2 \in W_2$. Then $v \in T^{-1}(w) = T^{-1}(w_1) + T^{-1}(w_2) \subseteq V_1 + V_2$. Thus $V = V_1 \oplus V_2$.

Next, we show that V_1 and V_2 are P-invariant. If $v \in V_i$ such that $i \in \{1, 2\}$, then $\overline{T}\varphi_g v = \overline{\psi_g} Tv$. But $Tv \subseteq W_i$ implies $\overline{\psi_g} Tv \subseteq W_i$ since W_i is P-invariant. Therefore, we conclude that $\overline{T}\varphi_g v = \overline{\psi_g} Tv \subseteq W_i$, thus $\bigcup_{a \in \varphi_g v} T(a) \subseteq W_i$. Therefore $Ta \subseteq W_i$ and $a \in T^{-1}(W_i) = V_i$, for all $a \in \varphi_g v$. Then $\varphi_g v \subseteq V_i$, as required. ∎

Similarly, we have the following results, whose proofs was omitted.

Proposition 4.12. *Let $\varphi : P \to L(V)$ be equivalent to an irreducible weak representation. Then φ is irreducible.*

Proposition 4.13. *Let $\varphi : P \to L(V)$ be equivalent to a completely reducible weak representation. Then φ is completely reducible.*

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