MATCHING EXTENSION IN COMPLEMENTARY PRISM OF REGULAR GRAPHS

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Abstract. Let $\overline{G}$ denote the complement of a simple graph G. The complementary prism of G, denoted by $G\overline{G}$, is obtained by taking a copy of G and a copy of $\overline{G}$ and then adding a perfect matching that joins corresponding vertices. A connected graph G of order at least $2k+2$ is k-extendable if for every matching M of size k in G, there is a perfect matching in G containing all edges of M. In this paper, we establish some sufficient conditions for the complementary prism of regular graphs to be 2-extendable.

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1. Introduction

All graphs considered in this paper are finite, undirected and simple. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The complement of G is denoted by $\overline{G}$. For $S \subseteq V(G)$, $G[S]$ denotes the induced subgraph of G by S.

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A neighbor set of a vertex v in G is denoted by $N_G(v) = \{u \in V(G) | uv \in E(G)\}$. For $v \in V(G)$ and $T \subseteq V(G)$, a neighbor set of a vertex v in T is denoted by $N_T(v) = \{u \in T | uv \in E(G)\}$ and if $X \subseteq V(G)$, $N_G(X)$ denotes $\bigcup_{v \in X} N_G(v)$. The number of components of G, the number of odd components of G and the number of even components of G are denoted by $c(G), c_o(G)$ and $c_e(G)$, respectively. A complete graph of order r is denoted by K_r. For graphs H and G, G is called H-free if G does not contain H as an induced subgraph. A subgraph H is called a clique if $H \cong K_r$, for some r. A set $M \subseteq E(G)$ is called a matching if no two edges of M have a common end vertex. A vertex u is saturated by M if there is an edge in M incident with u. For simplicity, the set of all vertices saturated by M is denoted by $V(M)$. M is called a maximum matching in G if there is no matching N in G of size greater than M. A perfect matching in G is a matching that saturates all vertices of G. A connected graph G of order at least $2k + 2$ is k-extendable if for every matching M of size k in G, there is a perfect matching in G containing all edges of M. A graph G is k-factor-critical if, for every set $S \subseteq V(G)$ with $|S| = k$, the graph $G - S$ contains a perfect matching. For $k = 1$ and $k = 2$, a k-factor-critical graph is also called factor-critical and bicritical, respectively.

The concept of k-extendable graphs was introduced, in 1980, by Plummer [9]. He gave a sufficient condition for a graph to be k-extendable in terms of minimum degree. A fundamental theorem (see Theorem 2.2) that is mainly used in studying matching extension was established. He also proved that 2-extendable non-bipartite graphs are bicritical. Some sufficient conditions for special classes of graphs to be k-extendable have been established (see [8], [10], [14]). For a comprehensive survey of this topic, the reader is referred to Plummer [11]–[13].

The concept of k-factor-critical graphs was introduced, in 1996, by Favaron [6]. She gave a necessary and sufficient condition for a graph to be k-factor-critical and also provided a relationship between n-extendable graphs and k-factor- critical graphs.

A *complementary prism* of G, denoted by $G\overline{G}$, is the graph obtained by taking a copy of G and a copy of its complement $\overline{G}$ and then joining corresponding vertices by an edge. A complementary prism is a specific case of complementary product of graphs introduced by Haynes et al.[3] in 2007. Haynes et al. ([3], [4], [5]) studied some parameters of complementary prism of graphs such as the vertex independence number, the chromatic number and the domination number.

According to the definition of the complementary prism of G, it is easy to see that $G\overline{G}$ contains a perfect matching. A problem that arises is that of investigating properties of G so that $G\overline{G}$ is k-extendable for some k. In [7], Janseana et al. established that if G is a 2-regular, H-free graph where $H \in \{C_3, C_4, C_5\}$, then $G\overline{G}$ is 2-extendable. In this paper, we concentrate on connected r-regular graphs for $r \geq 3$. Let $F = K_{2,3}$ with the addition of an edge shown in Figure 1. We prove that for a connected graph G of order p, if G is either 3-regular, F-free where $p \geq 8$ or r_0-regular where $p \geq 2r_0 + 1 \geq 9$, then $G\overline{G}$ is 2-extendable. We further extend this result to disconnected graphs. We show that if each component G_i of

G is 3-regular, F-free of order at least 8 or r_0-regular of order at least $2r_0+1 \geq 9$, then $G\overline{G}$ is 2-extendable. These results are presented in Section 3. Section 2 contains some preliminary results that we make use of in our work.

Figure 1: The graph F

2. Preliminary results

In this section, we state some results which are used in establishing our results in Section 3. Our first result is a well known theorem for studying the existence of a perfect matching in graphs established by Tutte.

Theorem 2.1. [2] (Tutte's Theorem) *A graph G has a perfect matching if and only if for any $S \subseteq V(G)$, $c_o(G-S) \leq |S|$.*

In 1980, Plummer [9] established the following fundamental theorem on k-extendable graphs.

Theorem 2.2. [9] *Let G be a graph of order $p \geq 2k+2$ and $k \geq 1$. If G is k-extendable, then*
(a) *G is $(k-1)$-extendable, and*
(b) *G is $(k+1)$-connected.*

Ananchuen and Caccetta [1] gave a necessary condition for a neighbor set of a vertex having minimum degree in extendable graphs. They showed that:

Theorem 2.3. [1] *Let G be a k-extendable graph on $p \geq 2k+2$ vertices with $\delta(G) = k+t$, $1 \leq t \leq k \leq p$. If $d_G(u) = \delta(G)$, then the induced subgraph $G[N_G(u)]$ has at most $t-1$ independent edges.*

A neccessary and sufficient condition for a graph to be k-extendable and to be k-factor-critical was provided by Yu [15] and Favaron [6], respectively.

Theorem 2.4. [15] *A graph G is k-extendable $(k \geq 1)$ if and only if for any $S \subseteq V(G)$,*
(a) *$c_o(G-S) \leq |S|$ and*
(b) *$c_o(G-S) = |S| - 2t, (0 \leq t \leq k-1)$ implies that $F(S) \leq t$, where $F(S)$ is the size of a maximum matching in $G[S]$.*

Theorem 2.5. [6] *A graph G is k-factor-critical if and only if $|V(G)| \equiv k (mod\ 2)$ and for any $S \subseteq V(G)$ with $|S| \geq k, c_o(G-S) \leq |S|-k$.*

We now turn our attention to some results concerning complementary prism of graphs.

Theorem 2.6. [7] *For positive integers l and i where $1 \leq i \leq l$, let $G_1, \ldots, G_l$ be components of G. If for each i, $G_i\overline{G}_i$ is a k-extendable graph of order $p_i \geq 2k+2$ for some positive integer k, then $G\overline{G}$ is k-extendable.*

Theorem 2.7. [7] *Let G be a 2-regular, H-free graph where $H \in \{C_3, C_4, C_5\}$, then $G\overline{G}$ is 2-extendable.*

3. Main results

We begin this section by establishing some lemmas concerning complementary prism of graphs and of regular graphs. These results are essential for establishing Theorem 3.10, the main result of our paper. To simplify our discussion of complementary prisms, G and $\overline{G}$ are referred to as subgraph copies of G and $\overline{G}$, respectively, in $G\overline{G}$. For a vertex v of G, there is exactly one vertex of $\overline{G}$ which is adjacent to v in $G\overline{G}$. This vertex is denoted by $\overline{v}$. That is $\{\overline{v}\} = N_{\overline{G}}(v)$. Conversely, v is the only vertex of G which is adjacent to $\overline{v}$. Similarly, for $\phi \neq X = \{x_1, x_2, \ldots, x_k\} \subseteq V(G)$, $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k\} \subseteq V(\overline{G})$ is denoted by $\overline{X}$ and vice versa. Clearly, $|X| = |\overline{X}|$.

Lemma 3.1. *Let G be a graph. Then $G\overline{G}$ is even and connected.*

Proof. Clearly, $G\overline{G}$ is even. Let $u, v \in V(G\overline{G})$. It is easy to see that if $u, v \in V(G)(V(\overline{G}))$, then either $uv \in E(G)$ or $u\bar{u}\bar{v}v$ is a $u-v$ path. We may now assume that $u \in V(G)$ and $v \in V(\overline{G})$. Clearly, $uv \in E(G\overline{G})$ if $v = \bar{u}$. So suppose that $v = \bar{w}$ for some $w \in V(G) - \{u\}$. Then either $u\bar{u}\bar{w}$ or $uw\bar{w}$ is a $u-v$ path. This proves that $G\overline{G}$ is connected and completes the proof of our lemma. ∎

For a graph G, it is easy to see that $G\overline{G}$ has a perfect matching. It then follows by Theorem 2.1 that for a cutset $S \subseteq V(G\overline{G})$, $c_o(G\overline{G}-S) \leq |S|$. The next lemma provides a relationship of a cutset and the number of odd components in a complementary prism.

Lemma 3.2. *Let $G\overline{G}$ be a complementary prism and let $S = A \cup \overline{B}$ be a cutset of $G\overline{G}$, where $A \subseteq V(G)$ and $\overline{B} \subseteq V(\overline{G})$. Then*

(a) *$c_o(G\overline{G}-S) = |S| - 2t = |A| + |B| - 2t$, for some $t \geq 0$.*

(b) *$c_o(G\overline{G}-S) \leq c_o(G[B-A]) + c_o(\overline{G}[\overline{A}-\overline{B}]) \leq |A| + |B| - 2|A \cap B|$. Consequently, $|A \cap B| \leq t$.*

(c) *If $c_o(G[B-A]) + c_o(\overline{G}[\overline{A}-\overline{B}]) = |A|+|B|-2|A \cap B|$, then each component of $G[B-A] \cup \overline{G}[\overline{A}-\overline{B}]$ is singleton. Consequently, $G[A-B]$ is a clique.*

Proof. (a) Since $G\overline{G}$ contains a perfect matching and is of even order, it follows by Theorem 2.1 that there is a non-negative integer t such that $c_o(G\overline{G}-S) = |S|-2t$, for any cutset $S \subseteq V(G\overline{G})$. Clearly, $|S| = |A|+|B|$. Thus $c(G\overline{G}-S) = |S|-2t = |A|+|B|-2t$ as required.

We first observe that $|B-A|+|\overline{A}-\overline{B}| = |B-A|+|A-B| = |A|+|B|-2|A\cap B|$ since $|A| = |A-B|+|A\cap B|$ and $|B| = |B-A|+|A\cap B|$.

(b) Let $C = V(G)-(A\cup B)$. It is easy to see that if $C = \phi$, then $c_o(G\overline{G}-S) = c_o(G[B-A])+c_o(\overline{G}[\overline{A}-\overline{B}]) \leq |B-A|+|\overline{A}-\overline{B}| = |A|+|B|-2|A\cap B|$. We now suppose that $C \neq \phi$. Then, by Lemma 3.1, $G\overline{G}[C\cup\overline{C}]$ is even and connected. Thus $c_o(G\overline{G}-S) \leq c_o(G\overline{G}-(S\cup C\cup\overline{C})) = c_o(G[B-A])+c_o(\overline{G}[\overline{A}-\overline{B}]) \leq |B-A|+|\overline{A}-\overline{B}| = |A|+|B|-2|A\cap B|$ as required.

(c) follows by the fact that $|B-A|+|A-B| = |A|+|B|-2|A\cap B|$. ∎

For an induced subgraph H of G, Com_H denotes the set of all components in H. If $X \subseteq V(G)$, then we use Com_X for $Com_{G[X]}$. For a cutset S of $G\overline{G}$, put $A = S\cap V(G)$, $\overline{B} = S\cap V(\overline{G})$ and $C = V(G)-(A\cup B)$. Thus $S = A\cup\overline{B}$. Further, let $T_{B-A} = \{F \mid F$ is an odd component of $G[B-A]$ and $N_G(u)-V(F) \subseteq A$ for all $u \in V(F)\}$. $T_{\overline{A}-\overline{B}} = \{F \mid F$ is an odd component of $\overline{G}[\overline{A}-\overline{B}]$ and $N_{\overline{G}}(\bar{u})-V(F) \subseteq \overline{B}$ for all $\bar{u} \in V(F)\}$. Finally, let $L = L_G \cup L_{\overline{G}}$, where $L_G = \{F \mid F$ is an odd component in $G[B-A]$ and $N_{G\overline{G}}(V(F))\cap C \neq \phi\}$ and $L_{\overline{G}} = \{F \mid F$ is an odd component in $\overline{G}[\overline{A}-\overline{B}]$ and $N_{G\overline{G}}(V(F))\cap\overline{C} \neq \phi\}$. Note that if $C = \phi$, then $L = \phi$. Clearly, $T_{B-A}\cap L_G = \phi$ and $T_{\overline{A}-\overline{B}}\cap L_{\overline{G}} = \phi$. It is easy to see that, if G is connected and $G[B-A]$ contains only odd components, then $Com_{B-A} = T_{B-A}\cup L_G$. Similarly, if $\overline{G}$ is connected and $\overline{G}[\overline{A}-\overline{B}]$ contains only odd components, then $Com_{\overline{A}-\overline{B}} = T_{\overline{A}-\overline{B}}\cup L_{\overline{G}}$. In what follows, the symbols Com_H, S, A, $\overline{B}$, C, T_{B-A}, $T_{\overline{A}-\overline{B}}$, L, L_G and $L_{\overline{G}}$ are referred to these set up.

The next lemma follows from our set up.

Lemma 3.3. *Let G be an r-regular connected graph of order $p \geq 2r+1$ and $G\overline{G}$ a complementary prism. If $|A| < r$, then T_{B-A} contains no singleton components. Similarly, if $|\overline{B}| < p-r-1$, then $T_{\overline{A}-\overline{B}}$ contains no singleton components.*

Lemma 3.4. *For $r \geq 3$, let G be a connected r-regular graph of order $p \geq 2r+1$. Let $A, B, T_{B-A}, T_{\overline{A}-\overline{B}}$ be defined as above. Then*

(a) *If $G[A] = K_r$, then each component of T_{B-A} is of order at least* 3.

(b) *If $|A\cap B| = 1$ and $G[A-B] \cong K_r$, then the number of singleton components in T_{B-A} is at most* 1.

(c) *If $|A\cap B| = 1$ and $G[A-B] \cong K_{r-1}$, then the number of singleton components in T_{B-A} is at most* 2.

Proof. (a) It follows by the fact that G is connected r-regular of order $p \geq 2r+1$.

(b) Suppose to the contrary that T_{B-A} contains two singleton components, say F_1 and F_2 where $V(F_1) = \{y_1\}$ and $V(F_2) = \{y_2\}$. Because $|A\cap B| = 1$, y_1 and y_2 are adjacent to at least $r-1$ vertices of $A-B$. Since $G[A-B] = K_r$ and $r \geq 3$, it follows that there exists a vertex of $A-B$, say y_3, such that $\{y_1, y_2\}\cup(A-B) \subseteq N_G(y_3)$. Thus $d_G(y_3) \geq r+1$, a contradiction

(c) By applying similar arguments as in the proof of (b), (c) follows. ∎

Let x be a real number, $\lfloor x \rfloor_e$ denotes the greatest even integer less than or equal to x, that is, $\lfloor x \rfloor_e = 2\lfloor x/2 \rfloor$. Note that if x is an integer and $\lfloor x \rfloor_e = k$ then $x = k$ or $x = k + 1$.

Lemma 3.5. *Let $G\overline{G}$ be a complementary prism and $L = L_G \cup L_{\overline{G}}$ be defined as above. Then $c_o(G\overline{G} - S) = c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) - \lfloor |L| \rfloor_e$. Consequently, $c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) - c_o(G\overline{G} - S) \leq |L| \leq c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) - c_o(G\overline{G} - S) + 1$.*

Proof. If $C = \phi$, then $|L| = 0$ and thus $c_o(G\overline{G} - S) = c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}])$ as required. We now suppose that $C \neq \phi$. By Lemmas 3.2(a) and (b), $c_o(G\overline{G} - S) \leq c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}])$. By Lemma 3.1, $G\overline{G}[C \cup \overline{C}]$ is even and connected. So it must be contained in some component of $G\overline{G} - S$, say F. If $x \in V(F) - (C \cup \overline{C})$, then x is in some component of $G[B - A] \cup \overline{G}[\overline{A} - \overline{B}]$, say M. So $V(M) \subseteq V(F)$. If M is odd, then $M \in L$. Note that each odd component of L is a subgraph of F. Hence, $|V(F)|$ has the same parity with $|L|$ and $c_o(G\overline{G} - S) = c_o(G[B - A] \cup \overline{G}[\overline{A} - \overline{B}]) - |L| + \epsilon$, where $\epsilon = 1$ if $|L|$ is odd and $\epsilon = 0$ if $|L|$ is even. So $c_o(G\overline{G} - S) = c_o(G[B - A] \cup \overline{G}[\overline{A} - \overline{B}]) - \lfloor |L| \rfloor_e$. Thus $\lfloor |L| \rfloor_e = c_o(G[B - A] \cup \overline{G}[\overline{A} - \overline{B}]) - c_o(G\overline{G} - S)$. By properties of $\lfloor x \rfloor_e$, our result follows. This proves our lemma. ∎

Lemma 3.6. *If G is an r-regular graph of order $p \geq 2r + 1$, then $\overline{G}$ is connected.*

Proof. Note that $\overline{G}$ is $(p - r - 1)$-regular graph of order p. Suppose $\overline{G}$ is disconnected. Then each component must have order at least $p - r$. So $p \geq 2(p - r)$ and thus $p \leq 2r$, a contradiction. This proves our lemma. ∎

Lemma 3.7. *Let G be a connected r-regular graph of order $p \geq 2r + 1$. Let S be a cutset of $G\overline{G}$. Then $S \cap V(G) \neq \phi$ and $S \cap V(\overline{G}) \neq \phi$.*

Proof. By Lemma 3.6, $\overline{G}$ is connected. Hence, G and $\overline{G}$ are connected. Suppose without loss of generality that $S \cap V(G) = \phi$. So $S \subseteq V(\overline{G})$. Since $G = G\overline{G} - V(\overline{G})$ is connected and each vertex $\overline{u}$ of $V(\overline{G}) - S$ is adjacent to a vertex u in G, it follows that $G\overline{G} - S$ is connected, a contradiction. Hence, $S \cap V(G) \neq \phi$. By a similar argument, $S \cap V(\overline{G}) \neq \phi$. This proves our lemma. ∎

Theorem 3.8. *Let G be a connected r-regular graph of order $p \geq 2r + 1$, for some $r \geq 2$. Then $G\overline{G}$ is bicritical. Consequently, $G\overline{G}$ is 1-extendable.*

Proof. Suppose $G\overline{G}$ is not bicritical. By Theorem 2.5, there is a cutset $S \subseteq V(G\overline{G})$, where $|S| \geq 2$ such that $c_o(G\overline{G} - S) > |S| - 2$. It follows by Lemmas 3.2(a) that $c_o(G\overline{G} - S) = |S|$ for $|S| \geq 2$. Note that, by Lemma 3.7, $A = S \cap V(G)$ and $\overline{B} = S \cap V(\overline{G})$ are not empty. Thus $\overline{A}$ and B are not empty. By Lemma 3.2 (b), $A \cap B = \phi$ and thus $c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) = c_o(G[B]) + c_o(\overline{G}[\overline{A}])) = c_o(G\overline{G} - S) = |S| = |B| + |\overline{A}|$. By Lemma 3.2(c), each component of $G[B]$ and $\overline{G}[\overline{A}]$ is singleton. Hence, $G[A] \cong K_{|A|}$. Since G is r-regular of order $p \geq 2r + 1$, $|A| \leq r + 1$. If $|A| = r + 1$, then $G[A] \cong K_{r+1}$ is a disconnected component in G, a contradiction. So $1 \leq |A| \leq r$. By Lemmas 3.3 and 3.4(a), no singleton component in $G[B]$ belongs to T_{B-A}. Since each component of $G[B]$ is singleton, $T_{B-A} = \phi$. Because $c_o(G\overline{G} - S) = c_o(\overline{G}[\overline{A}]) + c_o(G[B])$, it follows by Lemma 3.5 that $0 \leq |L| \leq 1$. Since $B \neq \phi$ and $G[B]$ contains only singleton components, it follows that $1 \leq |B| = |T_{B-A}| + |L_G| \leq 1$. Hence, $|B| = |L_G| = 1$. Therefore, $|\overline{B}| = 1 < r \leq p - r - 1$. By Lemma 3.3, $T_{\overline{A}-\overline{B}}$ contains no singleton components. Hence, $T_{\overline{A}-\overline{B}} = \phi$. Since each component of $\overline{G}[\overline{A}]$ is singleton, it is contained in $L_{\overline{G}}$. So $|L_{\overline{G}}| = |\overline{A}| = |A| \geq 1$. Therefore, $|L| = |L_G| + |L_{\overline{G}}| \geq 2$, a contradiction. Hence, $G\overline{G}$ is bicritical. It then follows that $G\overline{G}$ is 1-extendable. This proves our theorem. ∎

The next lemma follows by Theorem 2.3.

Lemma 3.9. *Let G be a connected r-regular graph of order $p \geq 2r + 1$, for some $r \geq 2$. If G contains a triangle, then $G\overline{G}$ is not r-extendable.*

By Lemma 3.9, if G is a 3-regular graph of order $p \geq 8$ containing a triangle, then $G\overline{G}$ is not 3-extendable. The next theorem provides a sufficient condition for a connected r-regular graph so that $G\overline{G}$ is 2-extendable, for $r \geq 4$. In case $r = 3$, if G contains the graph F in Figure 1 as an induced subgraph, then $\{yz, \bar{w}\bar{x}\}$ cannot be extended to a perfect matching in $G\overline{G}$. Hence, $G\overline{G}$ is not 2-extendable. We next show that the complementary prism of connected 3-regular, F-free graphs and connected r-regular graphs for $r \geq 4$ are 2-extendable.

Theorem 3.10. *Suppose G is a connected graph of order p. If G is either 3-regular, F-free where $p \geq 8$ and F is the graph in Figure 1 or r_0-regular where $p \geq 2r_0 + 1 \geq 9$, then $G\overline{G}$ is 2-extendable.*

Proof. Observe that $\overline{G}$ is $(p - r - 1)$-regular where $r \in \{3, r_0\}$ and $p - r - 1 \geq 4$. By Theorem 3.8, $G\overline{G}$ is bicritical. Suppose to the contrary that $G\overline{G}$ is not 2-extendable. Then there is a matching $M \subseteq E(G\overline{G})$ of size two such that $G\overline{G} - V(M)$ contains no perfect matching. By Theorem 2.1, there is a cutset $T \subseteq V(G\overline{G}) - V(M)$ such that $c_o(G\overline{G} - (V(M) \cup T)) > |T|$. Let $S = T \cup V(M)$. Clearly, $|S| \geq 4$. Thus $c_o(G\overline{G} - S) > |S| - 4$. Because $G\overline{G}$ is bicritical, by Theorem 2.5, $c_o(G\overline{G} - S) \leq |S| - 2$. It follows by parity that $c_o(G\overline{G} - S) = |S| - 2$ and $G\overline{G}[S]$ contains a matching of size at least two. Let $A = S \cap V(G)$ and $\overline{B} = S \cap V(\overline{G})$. By Lemma 3.2 (b), $|A \cap B| \leq 1$. Further, by Lemma 3.7, $A \neq \phi$ and $\overline{B} \neq \phi$. So $\overline{A} \neq \phi$ and $B \neq \phi$. We distinguish 2 cases according to $|A \cap B|$.

Case 1. $|A \cap B| = 1$. Put $\{u\} = A \cap B$. By Lemma 3.2(b) $c_o(G\overline{G} - S) = c_o(G[B - A]) + c_o(\overline{G}[\overline{A} - \overline{B}]) = |S| - 2$. By Lemma 3.5, $|L| \leq 1$. Further, by Lemma 3.2(c), each component of $\overline{G}[\overline{A} - \overline{B}] \cup G[B - A]$ is singleton. Thus, $G[A - B]$ is a clique, $|Com_{\overline{A}-\overline{B}}| = |\overline{A} - \overline{B}|$ and $|Com_{B-A}| = |B - A|$. Since G is connected, it is easy to see that if $|A - B| \geq r + 1$, then $G[A - B] \cong K_{|A-B|}$ contains a vertex of degree greater than r or $G \cong K_{r+1}$ is a graph of order less than p, a contradiction. Hence, $|A - B| \leq r$.

We first show that $|T_{B-A}| \geq 2$. Suppose to the contrary that $|T_{B-A}| \leq 1$. Since $G[B - A]$ contains only singleton components and $|L_G| \leq |L| \leq 1$, it follows that $|B-A| = |Com_{B-A}| = |T_{B-A}| + |L_G| \leq 2$. Thus $|\overline{B}| = |B| = |B-A| + |B \cap A| \leq 3 < 4 \leq p-r-1$. By Lemma 3.3, $T_{\overline{A}-\overline{B}}$ contains no singleton components. Thus $T_{\overline{A}-\overline{B}} = \phi$. Consequently, $Com_{\overline{A}-\overline{B}} = T_{\overline{A}-\overline{B}} \cup L_{\overline{G}} = L_{\overline{G}}$. Therefore, $|\overline{A}-\overline{B}| = |L_{\overline{G}}| \leq 1$ since $\overline{G}[\overline{A}-\overline{B}]$ contains only singleton components. So $|A| = |\overline{A}| = |\overline{A}-\overline{B}| + |\overline{A} \cap \overline{B}| \leq 2 < r$. By Lemma 3.3, T_{B-A} contains no singleton components. So $T_{B-A} = \phi$. Since $T_{\overline{A}-\overline{B}} = \phi$ and $T_{B-A} = \phi$, it follows that every odd component of $\overline{G}[\overline{A}-\overline{B}] \cup G[B-A]$ is in L. Because $|L| \leq 1$ and $\overline{G}[\overline{A}-\overline{B}] \cup G[B-A]$ contains only singleton components, it follows that $|\overline{A}-\overline{B}| + |B-A| \leq 1$. Hence, $|S| = |A-B| + |A \cap B| + |\overline{A} \cap \overline{B}| + |\overline{B}-\overline{A}| = |A-B| + 2|A \cap B| + |\overline{B}-\overline{A}| \leq 3 < 4$, contradicting the fact that $|S| \geq 4$. Therefore, $|T_{B-A}| \geq 2$.

Let $D_1, D_2 \in T_{B-A}$. Since $G[B-A]$ contains only singleton components, $D_i \cong K_1$, for $1 \leq i \leq 2$. Put $\{v_i\} = V(D_i)$. By Lemma 3.3, $|A| \geq r$. Consequently, $|A-B| \geq r-1$. Because $|A-B| \leq r$, $r-1 \leq |A-B| \leq r$. Since $G[A-B]$ is clique, $|A \cap B| = 1$ and $|T_{B-A}| \geq 2$, it follows by Lemmas 3.4 (b) and (c) that $|A-B| = r-1$ and $|T_{B-A}| = 2$. Thus $|A| = |A-B| + |A \cap B| = r$. Because $r-1 = |A-B| = |\overline{A}-\overline{B}| = |Com_{\overline{A}-\overline{B}}| = |T_{\overline{A}-\overline{B}}| + |L_{\overline{G}}| \leq |T_{\overline{A}-\overline{B}}| + 1$, it follows that $|T_{\overline{A}-\overline{B}}| \geq r-2 \geq 1$. Thus $T_{\overline{A}-\overline{B}}$ contains a singleton component. By Lemma 3.3, $|\overline{B}| \geq p-r-1 \geq 4$. Therefore, $|\overline{B}-\overline{A}| = |\overline{B}| - |\overline{B} \cap \overline{A}| \geq p-r-2 \geq 3$. On the other hand, $|B-A| = |Com_{B-A}| = |T_{B-A}| + |L_G| \leq 3$. Then $|B-A| = |\overline{B}-\overline{A}| = 3$. Thus $3 = |T_{B-A}| + |L_G| = 2 + |L_G|$. It follows that $L = L_G = \{K_1\}$ and consequently $L_{\overline{G}} = \phi$. Since $|A| = r$, $deg_G v_1 = deg_G v_2 = r$ and $N_G(v_1) = N_G(v_2) \subseteq A$, it follows that $N_G(v_1) = N_G(v_2) = A$.

We now put $\{\bar{w}\} = V(K_1)$ where $K_1 \in T_{\overline{A}-\overline{B}}$. Clearly, $N_{\overline{G}}(\bar{w}) \subseteq \overline{B} - \{\bar{v}_1, \bar{v}_2\}$ since v_1 and v_2 are adjacent to every vertex in A. Because $|\overline{B}| = |\overline{B}-\overline{A}| + |\overline{A} \cap \overline{B}| = 3 + 1 = 4$, $|N_{\overline{G}}(\bar{w})| \leq |\overline{B}| - |\{\bar{v}_1, \bar{v}_2\}| = 2$ thus $\overline{G}$ is t-regular where $t \leq 2$. This contradicts the fact that $\overline{G}$ is $(p-r-1)$-regular where $p-r-1 \geq 4$. Therefore, Case 1 cannot occur.

Case 2. $|A \cap B| = 0$. By Lemmas 3.2(a) and (b), $|S| - 2 = c_o(G\overline{G} - S) \leq c_o(\overline{G}[\overline{A}]) + c_o(G[B]) \leq |\overline{A}| + |B| = |S|$. By parity, $c_o(\overline{G}[\overline{A}]) + c_o(G[B]) = |S|$ or $c_o(\overline{G}[\overline{A}]) + c_o(G[B]) = |S| - 2$. We distinguish 2 cases.

Case 2.1. $c_o(\overline{G}[\overline{A}]) + c_o(G[B]) = |S| = |\overline{A}| + |B|$. Clearly, each component of $\overline{G}[\overline{A}] \cup G[B]$ is a singleton. So $G[A] \cong K_{|A|}$. It is easy to see that if $|A| \geq r+1$, then $G[A]$ contains a vertex of degree greater than r or $G[A]$ is a disconnected component in G, a contradiction. Hence, $|A| \leq r$. By Lemmas 3.3 and 3.4(a), T_{B-A} contains no singleton components. Therefore, $T_{B-A} = \phi$. Thus $|L_G| = |B|$. Because $c_o(G[B]) + c_o(\overline{G}[\overline{A}]) - c_o(G\overline{G} - S) = |S| - (|S| - 2) = 2$, by Lemma 3.5, $2 \leq |L| \leq 3$. Since $B \neq \phi$ and $|B| = |L_G| \leq |L|$, it follows that $1 \leq |B| \leq 3$. Because $|\overline{B}| = |B| \leq 3 < 4 \leq p-r-1$, by Lemma 3.3, $T_{\overline{A}-\overline{B}}$ contains no singleton components. Thus $T_{\overline{A}-\overline{B}} = \phi$. Hence, $|L_{\overline{G}}| = |\overline{A}| = |A|$. Therefore, $|L| = |L_G| + |L_{\overline{G}}| = |B| + |\overline{A}| = |S|$ and thus $2 \leq |S| \leq 3$ since $2 \leq |L| \leq 3$, contradicting the fact that $|S| \geq 4$. Hence, Case 2.1 cannot occur.

Case 2.2. $c_o(\overline{G}[\overline{A}]) + c_o(G[B]) = |S| - 2 = |\overline{A}| + |B| - 2$. Put $s = |S|$. It is easy to see that $\overline{G}[\overline{A}] \cup G[B]$ contains all singleton components except exactly one non-singleton component which is of order 2 or 3. Hence, $\overline{G}[\overline{A}] \cup G[B]$ is isomorphic to a graph in $\{(s-2)K_1 \cup K_2, (s-3)K_1 \cup P_3, (s-3)K_1 \cup K_3\}$. If $|\overline{A}| \geq r+2 \geq 5$, then $\overline{G}[\overline{A}]$ must contain a singleton component, say F, where $V(F) = \{\overline{u}\}$. It follows that $deg_G u \geq r+1$, a contradiction. Hence, $|A| = |\overline{A}| \leq r+1$. Since $c_o(\overline{G}[\overline{A}]) + c_o(G[B]) - c_o(G\overline{G} - S) = (|S|-2) - (|S|-2) = 0$, by Lemma 3.5, $|L| \leq 1$. We distinguish 2 subcases according to the non-singleton component.

Subcase 2.2.1. The only non-singleton component in $\overline{G}[\overline{A}] \cup G[B]$ is contained in $G[B]$. So $\overline{G}[\overline{A}] \cong |\overline{A}|K_1$ and $G[A] \cong K_{|\overline{A}|} \cong K_{|A|}$. Clearly, $|A| \leq r$ otherwise $G[A]$ is a disconnected component in G. By Lemmas 3.3 and 3.4(a), T_{B-A} contains no singleton components. So every singleton component in $G[B]$ is contained in L_G. Since $|L_G| \leq |L| \leq 1$, $G[B]$ contains at most 1 singleton component. We first show that $T_{\overline{A}-\overline{B}} = \phi$. Suppose this is not the case. Then there is $K_1 \in T_{\overline{A}-\overline{B}}$ since $\overline{G}[\overline{A}]$ contains only singleton components. By Lemma 3.3, $|B| = |\overline{B}| \geq p-r-1 \geq 4$. Because $G[B]$ contains a non-singleton component of order either 2 or 3 and at most 1 singleton component, it follows that $G[B]$ is isomorphic to a graph in $\{K_1 \cup P_3, K_1 \cup K_3\}$. Thus $|B| = 4$ and either $T_{B-A} = \{P_3\}$ or $T_{B-A} = \{K_3\}$, and $L_G = \{K_1\}$. Thus $L_{\overline{G}} = \phi$. So $Com_{\overline{A}} = T_{\overline{A}-\overline{B}} \cup L_{\overline{G}} = T_{\overline{A}-\overline{B}}$. Therefore, each vertex of $\overline{A}$ is adjacent to every vertex of $\overline{B}$ since $\overline{G}$ is $(p-r-1)$-regular and $p-r-1 \geq 4$. It follows that there is no edge joining vertices of A and B. But this contradicts the fact that $T_{B-A} \neq \phi$. Hence, $T_{\overline{A}-\overline{B}} = \phi$ as required.

Therefore, $Com_{\overline{A}} = L_{\overline{G}}$. Since $|L_{\overline{G}}| \leq |L| \leq 1$ and $|\overline{A}| = |A| \neq 0$, it follows that $|Com_{\overline{A}}| = |L_{\overline{G}}| = 1$. Further, $L_G = \phi$ and $\overline{G}[\overline{A}] = K_1$. Thus $Com_B = T_{B-A}$. Because $|A| = |\overline{A}| = 1 < r \leq 3$, by Lemma 3.3, T_{B-A} contains no singleton components. So $G[B]$ contains no singleton components and $G[B]$ is isomorphic to a graph in $\{P_3, K_3\}$ since $|B| = |S| - |A| \geq 3$. Then $G\overline{G}[S] = G[A] \cup \overline{G}[\overline{B}]$ contains a matching of size less than two, contradicting the fact that $G\overline{G}[S]$ contains a matching of size at least two. Hence, Subcase 2.2.1 cannot occur.

Subcase 2.2.2. The only non-singleton component in $\overline{G}[\overline{A}] \cup G[B]$ is contained in $\overline{G}[\overline{A}]$. So $G[B] \cong |B|K_1$. We first show that $T_{B-A} \neq \phi$. Suppose this is not the case. Then $T_{B-A} = \phi$ and thus $Com_B = T_{B-A} \cup L_G = L_G$. Since $B \neq \phi$ and $|L_G| + |L_{\overline{G}}| = |L| \leq 1$, it follows that $|L_G| = 1$ and $|L_{\overline{G}}| = 0$. Consequently, $|B| = 1$ since $G[B] \cong |B|K_1$. Because $|\overline{B}| = |B| = 1 < r$, $T_{\overline{A}-\overline{B}}$ contains no singleton components by Lemma 3.3. Hence, $G[\overline{A}]$ contains exactly one non-singleton component of order 2 or 3. Thus $|A| = |\overline{A}| \leq 3$. It is easy to see that $G\overline{G}[S] = G[A] \cup \overline{G}[\overline{B}]$ contains a matching of size at most one since $|\overline{B}| = 1$. This contradicts the fact that $G\overline{G}[S]$ contains a matching of size at least two. Hence, $T_{B-A} \neq \phi$. Further, $|T_{B-A}| \geq |B| - 1$ since $|L_G| \leq |L| \leq 1$ and $|T_{B-A}| + |L_G| = |B|$.

Because $G[B] \cong |B|K_1$, there exists $K_1 \in T_{B-A}$. By Lemma 3.3, $|A| \geq r$. So $r \leq |A| \leq r+1$. We first suppose that $|A| = r+1$. Let F_t be the non-singleton component of order t in $\overline{G}[\overline{A}]$ and let $\overline{A}_1 = V(F_t)$. Then $2 \leq t \leq 3$ and $\overline{G}[\overline{A}] \cong (r+1-t)K_1 \cup F_t$. It is easy to see that $G[A]$ contains $r+1-t$ vertices of degree r and each vertex of $A_1 = \overline{\overline{A}}_1$ has degree, in $G[A]$, at least $r+1-t$

and at most $r-1$. Let $\{w\} = V(K_1)$ where $K_1 \in T_{B-A}$, then $N_G(w) \subseteq A_1$ and thus $3 \leq r = deg_G(w) \leq t \leq 3$. It then follows that $N_G(w) = A_1$ and $t = r = 3$. Thus $\bar{w}$ is not adjacent to any vertex of $\overline{A}_1$ and $\overline{G}[\overline{A}] \cong K_1 \cup F_3$. Further, each vertex of A_1 has degree at least $|T_{B-A}|+1 = |B| - |L_G| + 1 \geq |B|$ since $|L_G| \leq 1$. Thus $|B| \leq 3$ since G is now 3-regular. Because $\overline{G}$ is $(p-r-1)$-regular where $p-r-1 \geq 4$ and each vertex of $V(F_3) = \overline{A}_1$ has degree at most 3 in $\overline{G}[\overline{A} \cup \overline{B}]$ since it must be adjacent to at most one vertex in $\overline{B}$, it follows that $F_3 \in L_{\overline{G}}$. Since $|L_{\overline{G}}| \leq |L| \leq 1$, the only singleton component, K_1, of $\overline{G}[\overline{A}]$ must be in $T_{\overline{A}-\overline{B}}$. By Lemma 3.3, $|\overline{B}| \geq p-r-1 \geq 4$. But this contradicts the fact that $|\overline{B}| = |B| \leq 3$. Therefore, $|A| = r$.

Consequently, for each $w \in V(K_1)$ where $K_1 \in T_{B-A}$, $N_G(w) = A$. Now let $\bar{v} \in \overline{A}$. Then $deg_{\overline{B}}(\bar{v}) \leq |\overline{B}| - |T_{B-A}| = |B| - |T_{B-A}| = |L_G| \leq 1$. Further, $deg_{\overline{A}}(\bar{v}) \leq 2$ since each component of $\overline{G}[\overline{A}]$ has order at most 3. Because $\overline{G}$ is $(p-r-1)$-regular where $p-r-1 \geq 4$, $\bar{v}$ is adjacent to some vertex of $\overline{C}$. Consequently, each odd component of $\overline{G}[\overline{A}]$ is contained in $L_{\overline{G}}$. Because $|\overline{A}| = |A| = r \geq 3$, $\overline{G}[\overline{A}]$ contains a non-singleton component of order either 2 or 3 and $|L_{\overline{G}}| \leq |L| \leq 1$, it follows that $c_o(\overline{G}[\overline{A}]) = 1$. Therefore, $\overline{G}[\overline{A}]$ is isomorphic to a graph in $\{K_1 \cup K_2, P_3, K_3\}$. Hence, $r = |A| = 3$, $|L| = |L_{\overline{G}}| = 1$, $Com_B = T_{B-A} = \{|B|K_1\}$. Further, for $x \in B, y \in A, N_G(x) = A$ and $deg_G(y) = r = 3 \geq |B| = |\overline{B}|$.

We first suppose that $\overline{G}[\overline{A}] \cong K_3$. Then $G[A]$ is independent and thus $\overline{G}[\overline{B}]$ must contain a matching of size at least two since $G\overline{G}[S]$ contains a matching of size at least two. So $|B| = |\overline{B}| \geq 4$. But this contradicts the fact that $|B| = |\overline{B}| \leq 3$. Hence, $\overline{G}[\overline{A}] \neq K_3$. Therefore, $\overline{G}[\overline{A}]$ is isomorphic to a graph in $\{P_3, K_1 \cup K_2\}$. In either case, $G[A]$ contains a maximum matching of size one. Then $2 \leq |\overline{B}| \leq 3$ since $G\overline{G}[A \cup \overline{B}]$ contains a matching of size at least two.

We now suppose that $\overline{G}[\overline{A}] \cong K_1 \cup K_2$. Then $G[A] \cong P_3$ and then the vertex of degree two in P_3 has degree, in G, greater than $r = 3$, again a contradiction. Hence, $\overline{G}[\overline{A}] \neq K_1 \cup K_2$. Consequently, $\overline{G}[\overline{A}] \cong P_3$ and then $G[A] \cong K_1 \cup K_2$. Clearly, $|B| \neq 3$ otherwise $G[A]$ contains a vertex of degree greater than $r = 3$. So $|B| = 2$ and thus $G[A \cup B]$ contains the graph F in Figure 1 as an induced subgraph. But this contradicts our hypothesis that G is 3-regular, F-free graph. This completes the proof of our theorem. ∎

It is clear that a connected 3-regular graph containing F, in Figure 1, as an induced subgraph contains v as a cut vertex. So 2-connected 3-regular graphs are F-free. The next corollary follows by this fact and Theorem 3.10.

Corollary 3.11. *If G is a 2-connected r-regular graph of order $p \geq 2r+1$, for $r \geq 3$, then $G\overline{G}$ is 2-extendable.*

According to Theorems 2.6 and 3.10, we have the following theorem.

Theorem 3.12. *If each component G_i of G is 3-regular, F-free of order at least 8 where F is the graph in Figure 1 or r_0-regular of order at least $2r_0+1 \geq 9$, then $G\overline{G}$ is 2-extendable.*

We conclude our paper by posing following problem.

Problem. Establish sufficient condition for a complementary prism of r-regular graphs to be k-extendable for $r \geq k \geq 3$.

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