NEW RESULTS ON FIXED POINTS FOR AN INFINITE SEQUENCE OF MAPPINGS IN G-METRIC SPACE

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Abstract. In this paper, we prove new results on coincidence and common fixed points for a sequence of mappings satisfying generalized $(\Psi - \Phi)$ contractive conditions in G-metric space. Also we investigate the existence of common fixed point for a sequence of mappings satisfying the almost generalized cyclic weak contractive condition in G-metric space. An example supporting our results is included.

1. Introduction

Fixed point theory is one of the most useful tools in analysis. The first result of fixed point theorem is given by Banach S. [4] by the general setting of complete metric space using which is known as Banach Contraction Principle. This principle has been generalized by many researchers in many ways like by [2], [5], [6], [11], [24]-[26], and so on.

In 2006, Mustafa and Sims [16] introduced a new structure to generalize the usual notion of metric space (X, d). This new structure leads to generalized metric spaces or to what are called G-metric spaces that allow to develop and introduce a new fixed point theory for various mappings. Later, several fixed point theorems were obtained in these new metric spaces for mappings satisfying certain contractive conditions. For example, in [9], [10], [13], [14] some fixed point results and theorems for self mappings satisfying some kind of contractive type conditions on complete G-metric spaces were proved. Abbas et. al. [1] studied common fixed point theorems for three maps in G-metric spaces. For other work on common fixed points using different conditions and considering more than three maps one can see [12, 23] and references therein. Fixed point results for cyclic ϕ-contraction mappings on metric spaces were proved by Pacurar and Rus [20]. Also, in [8], Karapinar obtained a unique fixed point of cyclic weak ϕ- contraction mappings. Whereas, Aydi [3] proved some fixed point theorems in G-metric spaces involving generalized cyclic contractions. Many researchers have also studied what are called coincidence points in G-metric spaces and obtained results for existence and uniqueness for such points, see for example [23], and for a recent work in metric spaces see [7].

In this paper, we prove some results on coincidence and common fixed points for a sequence of mappings in G-metric spaces.

Now, we give first in what follows preliminaries and basic definitions which will be used throughout the paper.

Definition 1.1 [16] Let X be a non-empty set, and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties.

(G1) $G(x, y, z) = 0$ if $x = y = z$.

(G2) $0 < G(x, x, y)$ whenever $x \neq y$, for all $x, y \in X$.

(G3) $G(x, x, y) \leq G(x, y, z)$ whenever $y \neq z$, for all $x, y, z \in X$.

(G4) $G(x, y, z) = G(x, z, y) = G(y, x, z) = \ldots$ (Symmetry in all three variables).

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$. (Rectangle Inequality).

Then the function G is called a generalized metric, or more specifically, a G-metric on X, and the pair (X, G) is called a generalized metric space, G-metric space.

Example 1.2 [13] Let (X, d) be any metric space. Define the mappings G_s and G_m on $X \times X \times X \to \mathbb{R}^+$ by

$$\begin{aligned} G_s(x, y, z) &= d(x, y) + d(y, z) + d(x, z), \\ G_m(x, y, z) &= \max\{d(x, y), d(y, z), d(x, z)\}, \forall x, y, z \in X. \end{aligned}$$

Then (X, G_s) and (X, G_m) are G -metric spaces.

Definition 1.3 [16] Let (X, G) be a G-metric space, and let x_n be a sequence of points of X, a point $x \in X$ is said to be the limit of the sequence x_n, if

$$\lim_{n,m \to \infty} G(x, x_n, x_m) = 0,$$

and one says that the sequence x_n is G-convergent.

Proposition 1.4 [16] *Let (X, G) be a G-metric space, then the following are equivalent.*

(1) *x_n is G-convergent to x.*

(2) $G(x_n, x_n, x) \to 0,$ *as* $n \to \infty$.

(3) $G(x_n, x, x) \to 0,$ *as* $n \to \infty$.

(4) $G(x_m, x_n, x) \to 0,$ *as* $m, n \to \infty$.

Definition 1.5 [13] Let (X, G) be a G-metric space, a sequence x_n is called G-Cauchy if for every $\epsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$, for all $n, m, l \geq N$; that is $G(x_n, x_m, x_l) \to 0$, as $n, m, l \to \infty$.

Proposition 1.6 [17] *If (X, G) is a G-metric space, then the following are equivalent.*

(1) *The sequence x_n is G-Cauchy.*

(2) *For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq N$.*

Definition 1.7 [16] A G-metric space (X, G) is said to be G-complete (or complete G-metric) if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

Definition 1.8 [16] Let (X, G) and (X', G') be two G-metric spaces, and let $f : (X, G) \to (X', G')$ *be a function. Then* f is said to be G-continuous at a point $a \in X$ if and only if, given $\epsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$; and $G(a, x, y) < \delta$ implies that $G'(f(a), f(x), f(y)) < \epsilon$. A function f is G-continuous at X if and only if it is G-continuous at all $a \in X$.

Theorem 1.9 [16] *Let (X, G) and (X', G') be two G-metric spaces. Then a function $f : X \to X'$ is G-continuous at a point $x \in X$ if and only if it is G sequentially continuous at x, that is; whenever x_n is G-convergent to x, $f(x_n)$ is G-convergent to $f(x)$.*

Definition 1.10 [16] *Let (X, G) be a G-metric space. Then for $x_0 \in X$, $r > 0$, the G-ball with center x_0 and radius r is*

$$B_G(x_0, r) = \{y \in X : G(x_0, y, y) < r\}.$$

Theorem 1.11 [15] *Let (X, G) be a G-metric space. The sequence $x_n \subset X$ is G-convergent to x if it converges to x in the G-metric topology, $T(G)$.*

Definition 1.12 [7] The function $\Psi : [0, \infty) \to [0, \infty)$ is called an altering distance function, if the following properties are satisfied.

(i) Ψ is continuous and non-decreasing.

(ii) $\Psi(t) = 0$ if and only if $t = 0$.

Definition 1.13 [7] Let f, g be two self mappings on partially ordered set $(X, \preceq)$. A pair (f, g) is said

(i) weakly increasing if $fx \preceq g(fx)$ and $gx \preceq f(gx)$ for all $x \in X$.

(ii) partially weakly increasing if $fx \preceq g(fx)$ for all $x \in X$.

Definition 1.14 [7] Let $(X, \preceq)$ be partially ordered set and $f, g, h : X \to X$ are given mappings such that $fX \subseteq hX$ and $gX \subseteq hX$. We say that f and g are weakly increasing with respect to h if and only if for all $x \in X$, we have

$$fx \preceq gy, \ \forall y \in h^{-1}(fx),$$

and

$$gx \preceq fy, \ \forall y \in h^{-1}(gx).$$

If $f = g$, we say that f is weakly with respect to h.

Definition 1.15 [7] Let $(X, \preceq)$ be a partially ordered set and $f, g, h : X \to X$ given mappings such that $f(X) \subseteq h(X)$. We say that (f, g) are partially weakly increasing with respect to h if and only if for all $x \in X$, we have

$$fx \preceq gy, \ \forall y \in h^{-1}(fx).$$

Note that, a pairs f and g is weakly increasing with respect to h if and only if the ordered pair (f, g) and (g, f) are partially weakly increasing with respect to h.

Definition 1.16 [7] Let $(X, d, \preceq)$ be an ordered metric space. We say that X is regular if the following hypothesis holds: if $\{x_n\}$ is a non-decreasing in X with respect to $\preceq$ such that $x_n \to x \in X$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Definition 1.17 [21] Let T and S be two self mappings of a metric space (X, d). T and S are said to be weakly compatible if for all $x \in X$ the equality $Tx = Sx$ implies $TSx = STx$.

2. Main results

Theorem 2.18 *Let (X, G) be a complete ordered G-metric space such that X is regular. Let $T : X \to X$ be a self mappings and $\{f_k\}_{k=1}^{\infty}$ a sequence of mappings of X into itself. Suppose that for every $i, j \in \mathbb{N}$ and all $x, y \in X$ with Tx and Ty are comparable, we have*

$$\begin{aligned} &\Psi(G(f_i x, f_j y, f_j y)) \leq \Psi(M_{i,j}(x, y, y)) - \Phi(M_{i,j}(x, y, y)) \\ &+L \min\{G(f_i x, f_i x, Tx), G(f_j y, f_j y, Ty), G(f_i x, Ty, Ty), G(f_j y, f_j y, Tx)\}, \end{aligned} \tag{2.1}$$

where $L \geq 0$,

$$\begin{aligned} M_{i,j}(x, y, y) \;=\; &\max\{G(Tx, Ty, Ty), G(Tx, f_i x, f_i x), G(Ty, f_j y, f_j y), \\ &\alpha[G(Tx, f_j y, f_j y) + G(f_i x, Ty, Ty)]\}, \end{aligned}$$

$0 \leq \alpha \leq \frac{1}{2}$, and $\Psi : [0, \infty) \to [0, \infty)$ is an altering distance function, and $\Phi : [0, \infty) \to [0, \infty)$ is a continuous function with $\Phi(t) = 0$ if and only if $t = 0$. Assume that T and $\{f_k\}_{k=1}^{\infty}$ satisfy the following hypotheses:

(i) $f_kX \subseteq TX$, *for every* $k \in \mathbb{N}$.

(ii) TX *is a closed subset of* (X, G).

(iii) (f_i, f_j) *are partially weakly increasing with respect to* T, *for every* $i, j \in \mathbb{N}$ *and* $j > i$.

Then T *and* $\{f_k\}_{k=1}^{\infty}$ *have a coincidence point* $u \in X$; *that is* $f_1u = f_2u = ... = Tu$.

Proof. Let x_0 be an arbitrary point in X. Since $f_1X \subseteq TX$, there exists $x_1 \in X$ such that $Tx_1 = f_1x_0$. Since $f_2X \subseteq TX$, there exists $x_2 \in X$ such that $Tx_2 = f_2x_1$. Continuing this process, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X defined by

$$y_n = f_{n+1}x_n = Tx_{n+1} \ , \forall n \in \mathbb{N} \cup \{0\}. \tag{2.2}$$

By construction, we have $x_{n+1} \in T^{-1}(f_{n+1}x_n)$. Then, using the fact that (f_{n+1}, f_{n+2}) are partially weakly increasing with respect to T, we obtain

$$Tx_{n+1} = f_{n+1}x_n \preceq f_{n+2}x_{n+1} = Tx_{n+2}, \forall\ n \in \mathbb{N} \cup \{0\}.$$

Therefore, we can write

$$Tx_1 \preceq Tx_2 \preceq ... \preceq Tx_{n+1} \preceq Tx_{n+2} \preceq ...,$$

or

$$y_0 \preceq y_1 \preceq ... \preceq y_n \preceq y_{n+1} \cdots . \tag{2.3}$$

We will prove our result in three steps.

Step1. We show that

$$\lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+1}) = 0,$$

and

$$\lim_{n\to\infty} G(y_n, y_n, y_{n+1}) = 0.$$

First case. For every $n \in \mathbb{N}$, let $y_{n-1} \neq y_{n+1}$. Since Tx_n and Tx_{n+1} are comparable, by inequality (2.1), we have

$$\begin{aligned}
&\Psi(G(y_n, y_{n+1}, y_{n+1})) \\
&\quad = \Psi(G(f_{n+1}x_n, f_{n+2}x_{n+1}, f_{n+2}x_{n+1})) \\
&\quad \leq \Psi(M_{n+1,n+2}(x_n, x_{n+1}, x_{n+1})) - \Phi(M_{n+1,n+2}(x_n, x_{n+1}, x_{n+1})) \\
&\qquad +L \min\{G(f_{n+1}x_n, f_{n+1}x_n, Tx_n), G(f_{n+2}x_{n+1}, f_{n+2}x_{n+1}, Tx_{n+1}), \\
&\qquad G(f_{n+1}x_n, Tx_{n+1}, Tx_{n+1}), G(f_{n+2}x_{n+1}, f_{n+2}x_{n+1}, Tx_n)\} \\
&\quad = \Psi(M_{n+1,n+2}(x_n, x_{n+1}, x_{n+1})) - \Phi(M_{n+1,n+2}(x_n, x_{n+1}, x_{n+1})) \\
&\qquad +L \min \left\{ \begin{array}{c} G(y_n, y_n, y_{n-1}), G(y_{n+1}, y_{n+1}, y_n), \\ G(y_n, y_n, y_n), G(y_{n+1}, y_{n+1}, y_{n-1}) \end{array} \right\} \\
&\quad = \Psi(M_{n+1,n+2}(x_n, x_{n+1}, x_{n+1})) - \Phi(M_{n+1,n+2}(x_n, x_{n+1}, x_{n+1}))
\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}
&M_{n+1,n+2}(x_n, x_{n+1}, x_{n+1})\\
&\quad = \max\left\{\begin{array}{c} G(Tx_n, Tx_{n+1}, Tx_{n+1}), G(Tx_n, f_{n+1}x_n, f_{n+1}x_n), \\ G(Tx_{n+1}, f_{n+2}x_{n+1}, f_{n+2}x_{n+1}), \\ \alpha[G(Tx_n, f_{n+2}x_{n+1}, f_{n+2}x_{n+1}) + G(f_{n+1}x_n, Tx_{n+1}, Tx_{n+1})] \end{array}\right\}\\
&\quad = \max\left\{\begin{array}{c} G(y_{n-1}, y_n, y_n), G(y_{n-1}, y_n, y_n), G(y_n, y_{n+1}, y_{n+1}), \\ \alpha[G(y_{n-1}, y_{n+1}, y_{n+1}) + G(y_n, y_n, y_n)] \end{array}\right\}\\
&\quad = \max\{G(y_{n-1}, y_n, y_n), G(y_n, y_{+1}, y_{n+1}), \alpha G(y_{n-1}, y_{n+1}, y_{n+1})\}.
\end{aligned}$$

Now, we have

$$G(y_{n-1}, y_{n+1}, y_{n+1}) \leq G(y_{n-1}, y_n, y_n) + G(y_n, y_{n+1}, y_{n+1}).$$

Hence, if $G(y_{n-1}, y_n, y_n) \leq G(y_n, y_{n+1}, y_{n+1})$, then we have

$$\begin{aligned}
G(y_{n-1}, y_{n+1}, y_{n+1}) &\leq G(y_{n-1}, y_n, y_n) + G(y_n, y_{n+1}, y_{n+1})\\
&\leq G(y_n, y_{n+1}, y_{n+1}) + G(y_n, y_{n+1}, y_{n+1})\\
&= 2G(y_n, y_{n+1}, y_{n+1}).
\end{aligned}$$

and

$$\frac{1}{2}G(y_{n-1}, y_{n+1}, y_{n+1}) \leq G(y_n, y_{n+1}, y_{n+1}).$$

Therefore, for $0 \leq \alpha \leq \frac{1}{2}$, we have

$$\alpha G(y_{n-1}, y_{n+1}, y_{n+1}) \leq G(y_n, y_{n+1}, y_{n+1}).$$

Similarly, one can do the same when $G(y_n, y_{n+1}, y_{n+1}) \leq G(y_{n-1}, y_n, y_n)$. Therefore, for $0 \leq \alpha \leq \frac{1}{2}$, we have always

$$\alpha G(y_{n-1}, y_{n+1}, y_{n+1}) \leq \max\{G(y_n, y_{n+1}, y_{n+1}), G(y_{n-1}, y_n, y_n)\}.$$

It follows that

$$M_{n+1,n+2}(x_n, x_{n+1}, x_{n+1}) = \max\{G(y_{n-1}, y_n, y_n), G(y_n, y_{n+1}, y_{n+1})\}. \tag{2.5}$$

If $G(y_{n-1}, y_n, y_n) < G(y_n, y_{n+1}, y_{n+1})$, then it follows from (2.5) that

$$M_{n+1,n+2}(x_n, x_{n+1}, x_{n+1}) = G(y_n, y_{n+1}, y_{n+1}).$$

Therefore, (2.4) implies that

$$\Psi(G(y_n, y_{n+1}, y_{n+1})) \leq \Psi(G(y_n, y_{n+1}, y_{n+1})) - \Phi(G(y_n, y_{n+1}, y_{n+1})),$$

which implies that $\Phi(G(y_n, y_{n+1}, y_{n+1})) = 0$, and hence we have

$$G(y_n, y_{n+1}, y_{n+1}) = 0,$$

or that $y_{n-1} = y_{n+1}$. This is a contradiction to our assumption that $y_{n-1} \neq y_{n+1}$. Therefore, for any $n \in \mathbb{N}$,

$$G(y_n, y_{n+1}, y_{n+1}) \leq G(y_{n-1}, y_n, y_n). \tag{2.6}$$

It follows that the sequence $\{G(y_n, y_{n+1}, y_{n+1})\}$ is a monotonic non-increasing sequence. Hence, there exists an $r \geq 0$ such that

$$\lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+1}) = r. \tag{2.7}$$

We now prove that $r = 0$. As Ψ and Φ are continuous, and taking the limit on both sides of (2.4), we get

$$\begin{aligned}
&\Psi\left(\lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+1})\right)\\
&\quad\leq \Psi(\max\{\lim_{n\to\infty} G(y_{n-1}, y_n, y_n), \lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+1})\})\\
&\qquad -\Phi(\max\{\lim_{n\to\infty} G(y_{n-1}, y_n, y_n), \lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+1})\})\\
&\quad\leq \Psi(\max\{\lim_{n\to\infty} G(y_{n-1}, y_n, y_n), \lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+1})\}).
\end{aligned}$$

Then, by (2.7), we deduce that

$$\begin{aligned}
\Psi(r) &\leq \Psi(\max\{r, r\}) - \Phi(\max\{r, r\})\\
&\leq \Psi(\max\{r, r\}) = \Psi(r),
\end{aligned}$$

which implies that $\Phi(\max\{r, r\}) = \Phi(r) = 0$; that is $r = 0$. Thus we have

$$\lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+1}) = 0.$$

Second case. There exists an $n \in \mathbb{N}$ such that $y_{n-1} = y_{n+1}$. If there exists an $n \in \mathbb{N}$ such that $y_{n-1} = y_{n+1}$, then by (2.3), we have

$$y_{n-1} = y_n = y_{n+1}.$$

On the other hand, we have

$$\begin{aligned}
&M_{n+1,n+3}(x_n, x_{n+2}, x_{n+2})\\
&\quad= \max\{G(Tx_n, Tx_{n+2}, Tx_{n+2}), G(Tx_n, f_{n+1}x_n, f_{n+1}x_n),\\
&\qquad G(Tx_{n+2}, f_{n+3}x_{n+2}, f_{n+3}x_{n+2}),\\
&\qquad \alpha[G(Tx_n, f_{n+3}x_{n+2}, f_{n+3}x_{n+2})\\
&\qquad +G(Tx_{n+2}, Tx_{n+2}, f_{n+1}x_n)]\}\\
&\quad= \max\{G(y_{n-1}, y_{n+1}, y_{n+1}), G(y_{n-1}, y_n, y_n),\\
&\qquad G(y_{n+1}, y_{n+2}, y_{n+2}),\\
&\qquad \alpha[G(y_{n-1}, y_{n+2}, y_{n+2}) + G(y_{n+1}, y_{n+1}, y_n)]\}\\
&\quad= \max\{0, 0, G(y_{n+1}, y_{n+2}, y_{n+2}),\\
&\qquad \alpha[G(y_{n-1}, y_{n+2}, y_{n+2}) + 0]\}\\
&\quad= \max\{0, 0, G(y_n, y_{n+2}, y_{n+2}), \alpha G(y_n, y_{n+2}, y_{n+2})\}.
\end{aligned}$$

Since $0 \leq \alpha \leq \frac{1}{2}$, we have

$$M_{n+1,n+3}(x_n, x_{n+2}, x_{n+2}) = G(y_n, y_{n+2}, y_{n+2}).$$

Since Tx_n and Tx_{n+2} are comparable. By inequality (2.1), we have

$$\begin{aligned}
&\Psi(G(y_n, y_{n+2}, y_{n+2})) \\
&\quad= \Psi(G(f_{n+1}x_n, f_{n+3}x_{n+2}, f_{n+3}x_{n+2})) \\
&\quad\leq \Psi(M_{n+1,n+3}(x_n, x_{n+2}, x_{n+2})) - \Phi(M_{n+1,n+3}(x_n, x_{n+2}, x_{n+2})) \\
&\qquad +L\min\{G(f_{n+1}x_n, f_{n+1}x_n, Tx_n), G(f_{n+3}x_{n+2}, f_{n+3}x_{n+2}, Tx_{n+2}), \\
&\qquad G(f_{n+1}x_n, Tx_{n+2}, Tx_{n+2}), G(f_{n+3}x_{n+2}, f_{n+3}x_{n+2}, Tx_n)\} \\
&\quad= \Psi(G(y_n, y_{n+2}, y_{n+2})) - \Phi(G(y_n, y_{n+2}, y_{n+2})) \\
&\qquad +L\min\{G(y_n, y_n, y_{n-1}), G(y_{n+2}, y_{n+2}, y_{n+1}), \\
&\qquad G(y_n, y_{n+1}, y_{n+1}), G(y_{n+2}, y_{n+2}, y_{n-1})\} \\
&\quad= \Psi(G(y_n, y_{n+2}, y_{n+2})) - \Phi(G(y_n, y_{n+2}, y_{n+2})) + 0 \\
&\quad\leq \Psi(G(y_n, y_{n+2}, y_{n+2})),
\end{aligned}$$

which implies that $\Phi(G(y_n, y_{n+2}, y_{n+2})) \leq 0$; that is $G(y_n, y_{n+2}, y_{n+2}) = 0$, and hence $y_n = y_{n+2}$. Thus, for $k \geq n$, we have $y_k = y_{n-1}$. This implies that

$$\lim_{n\to\infty} G(y_n, y_{n+1}, y_{n+1}) = 0. \tag{2.8a}$$

Now,

$$\begin{aligned}
G(y_n, y_n, y_{n+1}) &= G(y_n, y_{n+1}, y_n) \\
&\leq G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+1}, y_n) \\
&= 2G(y_n, y_{n+1}, y_{n+1}),
\end{aligned}$$

and by letting $n \to \infty$ in the above inequality and using (2.8a), we get

$$\lim_{n\to\infty} G(y_n, y_n, y_{n+1}) = 0. \tag{2.8b}$$

Step 2. We claim that $\{y_n\}$ is a Cauchy sequence in X. Suppose the contrary, i.e., $\{y_n\}$ is not a Cauchy sequence. Then, there exists an $\epsilon > 0$ for which we can find two subsequences $\{y_{m(k)}\}$ and $\{y_{n(k)}\}$ of $\{y_n\}$ such that $n(k)$ is the smallest index for which

$$n(k) > m(k) > k, G(y_{m(k)}, y_{n(k)}, y_{n(k)}) \geq \epsilon. \tag{2.9}$$

This means that

$$G(y_{m(k)}, y_{n(k)-1}, y_{n(k)-1}) < \epsilon. \tag{2.10}$$

Therefore, we use (2.9), (2.10), and the rectangle inequality to get

$$\begin{aligned}
\epsilon &\leq G(y_{m(k)}, y_{n(k)}, y_{n(k)}) \\
&\leq G(y_{m(k)}, y_{n(k)-1}, y_{n(k)-1}) + G(y_{n(k)-1}, y_{n(k)}, y_{n(k)}) \\
&< \epsilon + G(y_{n(k)-1}, y_{n(k)}, y_{n(k)}).
\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (2.8a), we obtain

$$\lim_{k\to\infty} G(y_{m(k)}, y_{n(k)}, y_{n(k)}) = \epsilon. \tag{2.11}$$

Again, using the rectangle inequality we have

$$\left|G(y_{m(k)-1}, y_{n(k)}, y_{n(k)}) - G(y_{m(k)}, y_{n(k)}, y_{n(k)})\right| \le G(y_{m(k)-1}, y_{m(k)}, y_{m(k)})$$

Letting $k \to \infty$ in the above inequality and using (2.8a), (2.11)

$$\lim_{k\to\infty} G(y_{m(k)-1}, y_{n(k)}, y_{n(k)}) = \epsilon. \tag{2.12}$$

On the other hand, we have

$$G(y_{m(k)}, y_{n(k)}, y_{n(k)}) \le G(y_{m(k)}, y_{n(k)+1}, y_{n(k)+1}) + G(y_{n(k)+1}, y_{n(k)}, y_{n(k)}).$$

Letting $k \to \infty$, we have from the above inequality that

$$\epsilon \le \lim_{k\to\infty} G(y_{m(k)}, y_{n(k)+1}, y_{n(k)+1}). \tag{2.13}$$

Also, by rectangle inequality, we have

$$\begin{aligned} G(y_{m(k)}, y_{n(k)}, y_{n(k)}) &\le G(y_{m(k)}, y_{m(k)-1}, y_{m(k)-1}) + G(y_{m(k)-1}, y_{n(k)}, y_{n(k)}) \\ &\le G(y_{m(k)}, y_{m(k)-1}, y_{m(k)-1}) + G(y_{m(k)-1}, y_{n(k)+1}, y_{n(k)+1}) \\ &\quad + G(y_{n(k)+1}, y_{n(k)}, y_{n(k)}). \end{aligned}$$

Letting $k \to \infty$ in the above inequality, and using (2.8b) and (2.11), we obtain

$$\epsilon \le \lim_{k\to\infty} G(y_{m(k)-1}, y_{n(k)+1}, y_{n(k)+1}).$$

Now,

$$\begin{aligned} G(y_{m(k)-1}, y_{n(k)+1}, y_{n(k)+1}) &\le G(y_{m(k)-1}, y_{m(k)}, y_{m(k)}) + G(y_{m(k)}, y_{n(k)+1}, y_{n(k)+1}) \\ &\le G(y_{m(k)-1}, y_{m(k)}, y_{m(k)}) + G(y_{m(k)}, y_{n(k)}, y_{n(k)}) \\ &\quad + G(y_{n(k)}, y_{n(k)+1}, y_{n(k)+1}). \end{aligned}$$

Letting $k \to \infty$ in the above inequality, and using (2.8a) and (2.13), we obtain

$$\lim_{k\to\infty} G(y_{m(k)-1}, y_{n(k)+1}, y_{n(k)+1}) \le \epsilon.$$

So,

$$\lim_{k\to\infty} G(y_{m(k)-1}, y_{n(k)+1}, y_{n(k)+1}) = \epsilon. \tag{2.14}$$

Now, by the rectangle inequality, we have

$$\begin{aligned} G(y_{m(k)}, y_{n(k)+1}, y_{n(k)+1}) &\le G(y_{m(k)}, y_{n(k)}, y_{n(k)}) + G(y_{n(k)}, y_{n(k)+1}, y_{n(k)+1}) \\ &\le G(y_{m(k)}, y_{m(k)-1}, y_{m(k)-1}) + G(y_{m(k)-1}, y_{n(k)}, y_{n(k)}) \\ &\quad + G(y_{n(k)}, y_{n(k)+1}, y_{n(k)+1}). \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (2.8a) and (2.12) we obtain

$$\lim_{k\to\infty} G(y_{m(k)}, y_{n(k)+1}, y_{n(k)+1}) \leq \epsilon. \tag{2.15}$$

Therefore, from (2.12) and (2.15), we have

$$\lim_{k\to\infty} G(y_{m(k)}, y_{n(k)+1}, y_{n(k)+1}) = \epsilon. \tag{2.16}$$

From (2.1), we have

$$\begin{aligned}
&\Psi(G(y_{m(k)}, y_{n(k)+1}, y_{n(k)+1}))\\
&\quad= \Psi(G(f_{m(k)+1}x_{m(k)}, f_{n(k)+2}x_{n(k)+1}, f_{n(k)+2}x_{n(k)+1}))\\
&\quad\leq \Psi(M_{m(k)+1,n(k)+2}(x_{m(k)}, x_{n(k)+1}, x_{n(k)+1})\\
&\qquad -\Phi(M_{m(k)+1,n(k)+2}(x_{m(k)}, x_{n(k)+1}, x_{n(k)+1})\\
&\qquad +L\min\{G(Tx_{m(k)}, f_{m(k)+1}x_{m(k)}, f_{m(k)+1}x_{m(k)}),\\
&\qquad G(Tx_{n(k)+1}, f_{n(k)+2}x_{n(k)+1}, f_{n(k)+2}x_{n(k)+1}),\\
&\qquad G(f_{m(k)+1}x_{m(k)}, Tx_{n(k)+1}, Tx_{n(k)+1}),\\
&\qquad G(f_{n(k)+2}x_{n(k)+1}, f_{n(k)+2}x_{n(k)+1}, Tx_{m(k)})\},
\end{aligned} \tag{2.17}$$

where

$$\begin{aligned}
&M_{m(k)+1,n(k)+2}(x_{m(k)}, x_{n(k)+1}, x_{n(k)+1})\\
&\quad= \max\{G(Tx_{m(k)}, Tx_{n(k)+1}, Tx_{n(k)+1}),\\
&\qquad G(Tx_{m(k)}, f_{m(k)+1}x_{m(k)}, f_{m(k)+1}x_{m(k)}),\\
&\qquad G(Tx_{n(k)+1}, f_{n(k)+2}x_{n(k)+1}, f_{n(k)+2}x_{n(k)+1}),\\
&\qquad \alpha[G(Tx_{m(k)}, f_{n(k)+2}x_{n(k)+1}, f_{n(k)+2}x_{n(k)+1})\\
&\qquad +G(f_{m(k)+1}x_{m(k)}, Tx_{n(k)+1}, Tx_{n(k)+1})]\}\\
&\quad= \max\{G(y_{m(k)-1}, y_{n(k)}, y_{n(k)}), G(y_{m(k)-1}, y_{m(k)}, y_{m(k)}),\\
&\qquad G(y_{n(k)}, y_{n(k)+1}, y_{n(k)+1}), \alpha[G(y_{m(k)-1}, y_{n(k)+1}, y_{n(k)+1})\\
&\qquad +G(y_{m(k)}, y_{n(k)}, y_{n(k)})]\},
\end{aligned}$$

and

$$\begin{aligned}
&L\min\{G(Tx_{m(k)}, f_{m(k)+1}x_{m(k)}, f_{m(k)+1}x_{m(k)}),\\
&\qquad G(Tx_{n(k)+2}, f_{n(k)+2}x_{n(k)+1}, f_{n(k)+2}x_{n(k)+1}),\\
&\qquad G(f_{m(k)+1}x_{m(k)}, Tx_{n(k)+1}, Tx_{n(k)+1}),\\
&\qquad G((f_{n(k)+2}x_{n(k)+1}, f_{n(k)+2}x_{n(k)+1}, Tx_{m(k)})\}\\
&= L\min\{G(y_{m(k)-1}, y_{m(k)}, y_{m(k)}), G(y_{n(k)+1}, y_{n(k)+1}, y_{n(k)+1}),\\
&\qquad G(y_{m(k)}, y_{n(k)}, y_{n(k)}), G(y_{n(k)+1}, y_{n(k)+1}, y_{m(k)-1})\}.
\end{aligned}$$

Letting $k \to \infty$ in (2.17) and using (2.8a), (2.8b), (2.11), (2.12), (2.14), (2.16), and the continuity of Ψ and Φ , we get that

$$\begin{aligned}
\Psi(\epsilon) &\leq \Psi(\max\{\epsilon, 0, 0, \alpha(\epsilon+\epsilon)\}) - \Phi(\max\{\epsilon, 0, 0, \alpha(\epsilon+\epsilon)\})\\
&\leq \Psi(\max\{\epsilon, 0, 0, \alpha(\epsilon+\epsilon)\},
\end{aligned}$$

or

$$\Psi(\epsilon) \leq \Psi(\epsilon) - \Phi(\epsilon) \leq \Psi(\epsilon).$$

Which implies that $\Psi(\epsilon) = 0$ and hence, $\epsilon = 0$, which is a contradiction. Thus, $\{y_n\}$ is a Cauchy sequence in X.

Step 3. In this step, we show the existence of a coincidence point for $\{f_k\}_{k=1}^{\infty}$ and T. From the completeness of (X, G), there exists $v \in X$ such that

$$\lim_{n\to\infty} y_n = v. \tag{2.18}$$

Since TX is a closed subset of (X, G), there exists $u \in X$ such that

$$y_n = Tx_{n+1} \to Tu.$$

Therefore, $v = Tu$. Since $\{y_n\}$ is a non-decreasing sequence and X is regular, it follows from (2.18) that $y_n \preceq v$ for all $n \in \mathbb{N} \cup \{0\}$. Hence,

$$Tx_n = y_{n-1} \preceq v = Tu.$$

Applying inequality (2.1), we get

$$\begin{aligned} \Psi(G(y_n, f_k u, f_k u)) &= \Psi(G(f_{n+1}x_n, f_k u, f_k u)) \\ &\leq \Psi(M_{n+1,k}(x_n, u, u)) - \Phi(M_{n+1,k}(x_n, u, u)) + \\ &\quad L \min\{ G(f_{n+1}x_n, f_{n+1}x_n, Tx_n), G(f_k u, f_k u, Tu), \\ &\quad G(f_{n+1}x_n, Tu, Tu), G(f_k u, f_k u, Tx_n)\}, \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} M_{n+1,k}(x_n, u, u) &= \max\{G(Tx_n, Tu, Tu), G(Tx_n, f_{n+1}x_n, f_{n+1}x_n), G(Tu, f_k u, f_k u) \\ &\quad \alpha[G(Tx_n, f_k u, f_k u) + G(f_{n+1}x_n, Tu, Tu)]\} \\ &= \max\{G(y_{n-1}, v, v), G(y_{n-1}, y_n, y_n), G(v, f_k u, f_k u), \\ &\quad \alpha[G(y_{n-1}, f_k u, f_k u) + G(y_n, v, v)]\}, \end{aligned}$$

and

$$\begin{aligned} &L \min \left\{ \begin{array}{c} G(f_{n+1}x_n, f_{n+1}x_n, Tx_n), G(f_k u, f_k u, Tu), \\ G(f_{n+1}x_n, Tu, Tu), G(Tu, Tu, f_{n+1}x_n) \end{array} \right\} \\ &= L \min \left\{ \begin{array}{c} G(y_{n-1}, y_{n-1}, y_n), G(f_k u, f_k u, v), \\ G(y_n, v, v), G(v, v, y_{n-1}) \end{array} \right\}. \end{aligned}$$

Letting $n \to \infty$ in (2.19) and using (2.18), we obtain

$$\begin{aligned} \Psi(G(v, f_k u, f_k u)) &\leq \Psi(\max\{0, 0, G(v, f_k u, f_k u), \alpha[G(v, f_k u, f_k u) + 0]\}) \\ &\quad -\Phi(\max 0, 0, G(v, f_k u, f_k u), \alpha[G(v, f_k u, f_k u) + 0]\}) \\ &\quad +L \min\{0, G(f_k u, f_k u, v), 0, 0\} \end{aligned}$$

Since, $0 \leq \alpha \leq \frac{1}{2}$, we have

$$\Psi(G(v, f_k u, f_k u)) = \Psi(G(v, f_k u, f_k u)) - \Phi(G(v, f_k u, f_k u)),$$

which implies that $\Phi(G(v, f_k u, f_k u)) = 0$; that is $G(v, f_k u, f_k u) = 0$, and hence

$$Tu = v = f_k u, \forall k \in \mathbb{N}.$$

Therefore, u is a coincidence point of $\{f_k\}_{k=1}^{\infty}$ and T. ∎

The proof of the following theorem is omitted. It is similar to that of Theorem 2.18.

Theorem 2.19 *Let (X, G) be a complete ordered G-metric space such that X is regular. Let $T : X \to X$ be a self mappings and $\{f_k\}_{k=1}^{\infty}$ a sequence of mappings of X into itself. Suppose that for every $i, j \in \mathbb{N}$ and all $x, y \in X$ with Tx and Ty are comparable, we have*

$$\Psi(G(f_i x, f_j y, f_j y)) \leq \Psi(M_{i,j}(x, y, y)) - \Phi((N_{i,j}(x, y, y)),$$

where

$$\begin{aligned} M_{i,j}(x, y, y) &= \max\{G(Tx, Ty, Ty), G(Tx, f_i x, f_i x), G(Ty, f_j y, f_j y), \\ &\quad \alpha[G(Tx, f_j y, f_j y) + G(Ty, Ty, f_i x)]\}, \end{aligned}$$

and $0 \leq \alpha \leq \frac{1}{2}$,

$$N_{i,j}(x, y, y) = \max\{G(Tx, Ty, Ty), G(Tx, f_j y, f_j y), G(Ty, Ty, f_i x)\},$$

and $\Psi : [0, \infty) \to [0, \infty)$ is an altering distance function, and $\Phi : [0, \infty) \to [0, \infty)$ is a continuous function with $\Phi(t) = 0$ if and only if $t = 0$. Assume that T and $\{f_k\}_{k=1}^{\infty}$ satisfy the following hypotheses:

(i) *$f_k X \subseteq TX$, for every $k \in \mathbb{N}$.*

(ii) *TX is a closed subset of (X, G).*

(iii) *(f_i, f_j) are partially weakly increasing with respect to T, for every $i, j \in \mathbb{N}$ and $j > i$.*

Then T and $\{f_k\}_{k=1}^{\infty}$ have a coincidence point $u \in X$; that is $f_1 u = f_2 u = ... = Tu$.

Corollary 2.20 *Let (X, G) be a complete ordered G-metric space such that X is regular. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of mappings of X into itself. Suppose that for every $i, j \in \mathbb{N}$ and all $x, y \in X$ are comparable, we have*

$$\begin{aligned} \Psi(G(f_i x, f_j y, f_j y)) &\leq \Psi(M_{i,j}(x, y, y)) - \Phi((M_{i,j}(x, y, y)) \\ &\quad + L \min \left\{ \begin{array}{c} G(f_i x, f_i x, x), G(f_j y, f_j y, y), \\ G(f_i x, y, y), G(f_j y, f_j y, x) \end{array} \right\} \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} M_{i,j}(x, y, y) &= \max\{G(x, y, y), G(x, f_i x, f_i x), G(y, f_j y, f_j y), \\ &\quad \alpha[G(x, f_j y, f_j y) + G(y, y, f_i x)]\}, \end{aligned}$$

$0 \leq \alpha \leq \frac{1}{2}$, and Ψ and Φ are as in Theorem 2.18.

If (f_i, f_j) is partially weakly increasing, for every $i, j \in \mathbb{N}$, then the sequence of mappings $\{f_k\}_{k=1}^{\infty}$ has a common fixed point $u \in X$; that is, $f_1 u = f_2 u = ... = u$.

Proof. It follows straightforwardly from Theorem 2.18, by taking the mapping $Tx = x$. ∎

Corollary 2.21 *Let (X, G) be a complete ordered G-metric space such that X is regular. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of mappings of X into itself. Suppose that for every $i, j \in \mathbb{N}$ and all $x, y \in X$ are comparable, we have*

$$\Psi(G(f_i x, f_j y, f_j y)) \leq \Psi(M_{i,j}(x, y, y)) - \Phi((N_{i,j}(x, y, y)) \tag{2.21}$$

where

$$\begin{aligned} M_{i,j}(x, y, y) &= \max\{G(x, y, y), G(x, f_i x, f_i x), G(y, f_j y, f_j y), \\ &\quad \alpha[G(x, f_j y, f_j y) + G(y, y, f_i x)]\}, \end{aligned}$$

and $0 \leq \alpha \leq \frac{1}{2}$,

$$N_{i,j}(x, y, y) = \max\{G(x, y, y), G(x, f_j y, f_j y), G(y, y, f_i x)\},$$

and Ψ and Φ are as in Theorem 2.19.

If (f_i, f_j) is partially weakly increasing, for every $i, j \in \mathbb{N}$, then the sequence of mappings $\{f_k\}_{k=1}^{\infty}$ has a common fixed point $u \in X$; that is, $f_1 u = f_2 u = ... = u$.

Proof. It follows straightforwardly from Theorem 2.19, by taking the mapping $Tx = x$. ∎

The following examples illustrate Corollary 2.20 and 2.21.

Example 2.22 Let $X = \{0, 1, 2, 3, ...\}$. We define a partial order $\preceq$ on X as $x \preceq y$ if $x \geq y$ for all $x, y \in X$. Let $G : X \times X \times X \to [0, \infty)$ be given by:

$$G(x, y, z) = \begin{cases} 0, & x = y = z, \\ x + y + z, & x \neq y \neq z, \\ x + y, & z = x \text{ or } z = y. \end{cases}$$

Let $f_k : X \to X$ be defined by:

$$f_k(x) = \begin{cases} 0\ , & 0 \leq x < k, \\ x - 1, & x \geq k. \end{cases}$$

For every $k \in \mathbb{N}$, define $\Psi, \Phi : [0, \infty) \to [0, \infty)$ by $\Psi(t) = t^2$ and $\Phi(t) = \ln(1 + t)$. Then we have the following.

(i) $(X, G, \preceq)$ is a complete partially ordered G-metric space.
(ii)$(X, G, \preceq)$ is regular.
(iii) (f_i, f_j) are partially weakly increasing for every $i, j \in \mathbb{N}$ such that $j > i$.
(iv) For any $i, j \in \mathbb{N}$, f_i and f_j satisfy (21), for every $x, y \in X$ with $x \preceq y$.

Thus, by Corollary 2.21, $\{f_k\}_{k=1}^{\infty}$ have a common fixed point. Moreover, 0 is the unique common fixed point of $\{f_k\}_{k=1}^{\infty}$.

Solution. The proof of (i) is clear. We need to show that X is regular, let $\{x_n\}$ be a non-decreasing sequence in X with respect to $\preceq$ such that $x_n \to x$, then there exists $k \in \mathbb{N}$ such that $x_n = x$ for all $n \geq k$. Hence, $(X, G, \preceq)$ is regular.

To prove (iii), let $x \in X$ and $i, j \in \mathbb{N}$ such that $j > i$.If $x < i$, then $f_i x = 0$ and $f_j(f_i x) = 0$. So, $f_i x \preceq f_j(f_i x)$.

If $i \leq x \leq j$, then $f_i x = x - 1$ and $f_j(f_i x) = 0$, so, $f_i x = x - 1 \geq 0 = f_j(f_i x)$ or $f_i x \preceq f_j(f_i x)$.

Finally, if $x \geq j$, then $f_i x = x - 1$ and $f_j(f_i x) = x - 2$, so, $f_i x = x - 1 \geq x - 2 = f_j(f_i x)$ or $f_i x \preceq f_j(f_i x)$. Therefore, (f_i, f_j) are partially weakly increasing.

Now, we prove (iv). By its symmetry and without loss of generality, it suffices to assume that $i \leq j$. Let $x, y \in X$ with $x \preceq y$, so $x \geq y$. Then, the following cases are possible.

Case 1. $x < i$ and $y < j$. Then $f_i x = 0$ and $f_j y = 0$ and hence,

$$G(f_i x, f_j y, f_j y) = 0,$$

$$\begin{aligned} M_{i,j}(x, y, y) &= \max\{G(x, y, y), G(x, 0, 0), G(y, 0, 0), \alpha[G(x, 0, 0) + G(0, y, y)]\} \\ &= \max\{G(x, y, y), x, y\}, \end{aligned}$$

and

$$\begin{aligned} N_{i,j}(x, y, y) &= \max\{G(x, y, y), G(x, 0, 0), G(0, y, y)\} \\ &= \max\{G(x, y, y), x, y\}. \end{aligned}$$

Since, by elementary calculus, $\Psi(t) - \Phi(t) \geq 0$, for every $t \geq 0$, we have

$$\Psi(0) = 0 \leq \Psi(\max\{G(x, y, y), x, y\}) - \Phi(\max\{G(x, y, y), x, y\}),$$

or

$$\Psi(G(f_i x, f_j y, f_j y)) \leq \Psi(M_{i,j}(x, y, y)) - \Phi(G(N_{i,j}(x, y, y)).$$

Case 2. $x \geq i$ and $y < j$. Then $f_i x = x - 1$ and $f_j y = 0$ and hence,

$$G(f_i x, f_j y, f_j y) = x - 1,$$

$$\begin{aligned} M_{i,j}(x, y, y) &= \max\{G(x, y, y), G(x, x - 1, x - 1), G(y, 0, 0), \\ &\quad \alpha[G(x, 0, 0) + G(x - 1, y, y)]\} \\ &= \max\{G(x, y, y), 2x - 1, y, \alpha[x + G(x - 1, y, y)]\} \\ &= 2x - 1, \end{aligned}$$

and

$$\begin{aligned} N_{i,j}(x, y, y) &= \max\{G(x, y, y), G(x, 0, 0), G(x - 1, y, y)\} \\ &= \max\{G(x, y, y), x, G(x - 1, y, y) \\ &= \begin{cases} 2x - 1, & x = y \text{ or } x - 1 = y, \\ x + y, & x \neq y \text{ and } x - 1 \neq y. \end{cases} \end{aligned}$$

If $x = y$ or $x - 1 = y$, then, since by elementary calculus,

$$(x-1)^2 \leq (2x-1)^2 - \ln(1+2x-1)$$

for every $x \geq 1$, we have

$$\Psi(G(f_i x, f_j y, f_j y) \leq \Psi(M_{i,j}(x,y,y)) - \Phi((N_{i,j}(x,y,y)).$$

If $x \neq y$ and $x - 1 \neq y$, then, since $y \leq x - 1$, we have

$$\ln(1+x+y) \leq \ln(1+x+x-1) = \ln(2x) \leq (2x-1)^2 - (x-1)^2,$$

or

$$(x-1)^2 \leq (2x-1)^2 - \ln(1+x+y).$$

Therefore,

$$\Psi(G(f_i x, f_j y, f_j y)) \leq \Psi(M_{i,j}(x,y,y)) - \Phi(G(N_{i,j}(x,y,y)).$$

Case 3. $x \geq i$ and $y \geq j$. Then $f_i x = x - 1$ and $f_j y = y - 1$ and hence

$$G(f_i x, f_j y, f_j y) = \begin{cases} 0, & x = y, \\ x+y-2, & x \neq y, \end{cases}$$

$$\begin{aligned} M_{i,j}(x,y,y) &= \max\{G(x,y,y), G(x,x-1,x-1), G(y,y-1,y-1), \\ &\qquad \alpha[G(x,y-1,y-1) + G(x-1,y,y)]\} \\ &= \max\{G(x,y,y), 2x-1, 2y-1, \alpha[x+y-1+G(x-1,y,y)]\} \\ &= 2x-1, \end{aligned}$$

and

$$\begin{aligned} N_{i,j}(x,y,y) &= \max\{G(x,y,y), G(x,y-1,y-1), G(x-1,y,y)\} \\ &= \max\{G(x,y,y), x+y-1, G(x-1,y,y)\} \\ &= \begin{cases} 2x-1 & x = y \text{ or } x-1 = y, \\ x+y & x \neq y \text{ and } x-1 \neq y. \end{cases} \end{aligned}$$

If $x = y$, then, since $\Psi(2x-1) - \Phi(2x-1) \geq 0$, for every $x \geq 1$, we have

$$\Psi(0) = 0 \leq \Psi(2x-1) - \Phi(2x-1),$$

or

$$\Psi(G(f_i x, f_j y, f_j y)) \leq \Psi(M_{i,j}(x,y,y)) - \Phi((N_{i,j}(x,y,y)).$$

By elementary calculus, we get $\ln(2x) \leq 2(4x-5)$, for every $x \geq 2$.

If $x - 1 = y$, then we have

$$(2x-3)^2 \leq (2x-1)^2 - \ln(1+2x-1),$$

and, therefore,

$$\Psi(G(f_i x, f_j y, f_j y)) \leq \Psi(M_{i,j}(x,y,y)) - \Phi((N_{i,j}(x,y,y)).$$

If $x \neq y$ and $x - 1 \neq y$, then, since $y \leq x - 1$, we have

$$(x + y - 1)^2 \leq (2x - 1)^2 - \ln(1 + x + y),$$

and, therefore,

$$\Psi(G(f_i x, f_j y, f_j y)) \leq \Psi(M_{i,j}(x, y, y)) - \Phi((N_{i,j}(x, y, y)).$$

Thus, Ψ, Φ and $\{f_k\}_{k=1}^{\infty}$ satisfy all hypotheses of Corollary 2.21 and hence $\{f_k\}_{k=1}^{\infty}$ have a common fixed point. Indeed, 0 is the unique common fixed point of $\{f_k\}_{k=1}^{\infty}$.

Using the calculation above, in all cases one can show that

$$\begin{aligned}\Psi(G(f_i x, f_j y, f_j y)) \leq\ & \Psi(M_{i,j}(x, y, y)) - \Phi((M_{i,j}(x, y, y)). \\ & +L \min\{G(f_i x, f_i x, x), G(f_j y, f_j y, y), \\ & G(f_i x, y, y), G(f_j y, f_j y, x)\}.\end{aligned}$$

hence, by Corollary 2.20, $\{f_k\}_{k=1}^{\infty}$ have a common fixed point. Indeed, 0 is the unique common fixed point of $\{f_k\}_{k=1}^{\infty}$.

3. Cyclic contractions

In this section, we investigate the existence of common fixed point for a sequence of mappings satisfying the almost generalized cyclic weak contractive condition in G-metric space.

Definition 3.1 [8] Let X be a non-empty set, p be a positive integer, and $T : X \to X$ be a mapping. $X = \cup_{i=1}^{p} A_i$ is said to be a cyclic representation of X with respect to T if

(i) $A_i, i = 1, 2, ..., p$ are non-empty closed sets,

(ii) $T(A_1) \subseteq A_2, ..., T(A_{p-1}) \subseteq A_p, T(A_p) \subseteq A_1$.

Theorem 3.2 *Let (X, G) be a complete G-metric space, and let $A_1, A_2, ..., A_p$ be a non-empty closed subset of X and $X = \cup_{i=1}^{p} A_i$. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of mappings of X into itself. Suppose that, for every $i, j \in \mathbb{N}$ and all $x, y \in X$, we have*

$$\begin{aligned}\Psi(G(f_i x, f_j y, f_j y)) \leq\ & \Psi(M_{i,j}(x, y, y)) - \Phi(M_{i,j}(x, y, y)) \\ & +L \min\{G(f_i x, f_i x, x), G(f_j y, f_j y, y), \\ & G(f_i x, y, y), G(f_j y, f_j y, x)\}, \end{aligned} \tag{3.1}$$

where $L \geq 0$, and

$$\begin{aligned}M_{i,j}(x, y, y) =\ & \max\{G(x, y, y), G(x, f_i x, f_i x), G(y, f_j y, f_j y), \\ & \alpha[G(x, f_j y, f_j y) + G(f_i x, y, y)]\},\end{aligned}$$

$0 \le \alpha \le \frac{1}{2}$, $\Psi : [0,\infty) \to [0,\infty)$ *is an altering distance function, and* $\Phi : [0,\infty) \to [0,\infty)$ *is a continuous function with* $\Phi(t) = 0$ *if and only if* $t = 0$.

Assume that $f_k(A_i) \subseteq A_{i+1}$, *for every* $k \in \mathbb{N}$. *Then,* $\{f_k\}_{k=1}^{\infty}$ *have a common fixed point* $u \in \cap_{i=1}^{p} A_i$; *that is,*

$$f_1 u = f_2 u = ... = u.$$

Proof. Let $x_0 \in A_1$, then there exists $x_1 \in A_2$ such that $x_1 = f_1 x_0$, and there exists $x_2 \in A_3$ such that $x_2 = f_2 x_1$. Continuing this process we can construct a sequence $\{x_n\}$ in X defined by

$$x_{n+1} = f_{n+1} x_n. \tag{3.2}$$

We will prove that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{3.3}$$

If, for some l, we have $x_{l+1} = x_l$, then (3.3) follows immediately. So, we can assume that $G(x_n, x_{n+1}, x_{n+1}) > 0$ for all n. Now, for all n, there exists $i = i(n) \in \{1, 2, ..., p\}$ such that $(x_n, x_{n+1}) \in A_i \times A_{i+1}$. Then, from (3.1), we have

$$\begin{aligned} &\Psi(G(f_n x_{n-1}, f_{n+1} x_n, f_{n+1} x_n) \\ &\quad = \Psi(G(x_n, x_{n+1}, x_{n+1})) \\ &\quad \le \Psi(M_{n,n+1}(x_{n-1}, x_n, x_n)) - \Phi(M_{n,n+1}(x_{n-1}, x_n, x_n)) \\ &\qquad +L \min\{G(f_n x_{n-1}, f_n x_{n-1}, x_{n-1}), G(f_{n+1} x_n, f_{n+1} x_n, x_n), \\ &\qquad G(f_n x_{n-1}, x_n, x_n), G(f_{n+1} x_n, f_{n+1} x_n, x_{n-1})\} \\ &\quad = \Psi(M_{n,n+1}(x_{n-1}, x_n, x_n)) - \Phi(M_{n,n+1}(x_{n-1}, x_n, x_n)) \\ &\qquad +L \min\{G(x_n, x_n, x_{n-1}), G(x_{n+1}, x_{n+1}, x_n), \\ &\qquad G(x_n, x_n, x_n), G(x_{n+1}, x_{n+1}, x_{n-1})\} \\ &\quad \le \Psi(M_{n,n+1}(x_{n-1}, x_n, x_n)), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} M_{n,n+1}(x_{n-1}, x_n, x_n) &= \max\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, f_n x_{n-1}, f_n x_{n-1}), \\ &\quad G(x_n, f_{n+1} x_n, f_{n+1} x_n), \\ &\quad \alpha[G(x_{n-1}, f_{n+1} x_n, f_{n+1} x_n) + G(f_n x_{n-1}, x_n, x_n)]\} \\ &= \max\{G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), \\ &\quad \alpha[G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_n, x_n)]\} \\ &= \max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{+1}, x_{n+1}), \\ &\quad \alpha G(x_{n-1}, x_{n+1}, x_{n+1})\}. \end{aligned}$$

Now, we have

$$G(x_{n-1}, x_{n+1}, x_{n+1}) \le G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}),$$

Hence, if $G(x_{n-1}, x_n, x_n) \leq G(x_n, x_{n+1}, x_{n+1})$, then we have

$$\begin{aligned} G(x_{n-1}, x_{n+1}, x_{n+1}) &\leq G(x_{n-1}, x_n, x_n) + G(x_n, x_{n+1}, x_{n+1}) \\ &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_n, x_{n+1}, x_{n+1}) \\ &= 2G(x_n, x_{n+1}, x_{n+1}), \end{aligned}$$

and

$$\frac{1}{2} G(x_{n-1}, x_{n+1}, x_{n+1}) \leq G(x_n, x_{n+1}, x_{n+1}).$$

Therefore, for $0 \leq \alpha \leq \frac{1}{2}$, we have

$$\alpha G(x_{n-1}, x_{n+1}, x_{n+1}) \leq G(x_n, x_{n+1}, x_{n+1}).$$

Similarly, one can do the same when $G(x_n, x_{n+1}, x_{n+1}) \leq G(x_{n-1}, x_n, x_n)$. Therefore, for $0 \leq \alpha \leq \frac{1}{2}$, we always have

$$\alpha G(x_{n-1}, x_{n+1}, x_{n+1}) \leq \max\{G(x_n, x_{n+1}, x_{n+1}), G(x_{n-1}, x_n, x_n)\}.$$

Consequently,

$$M_{n,n+1}(x_{n-1}, x_n, x_n) = \max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\}, \tag{3.5}$$

If $G(x_{n-1}, x_n, x_n) < G(x_n, x_{n+1}, x_{n+1})$, then it follows from (3.5) that

$$M_{n,n+1}(x_{n-1}, x_n, x_n) = G(x_n, x_{n+1}, x_{n+1}).$$

Therefore, equation (3.4) implies that

$$\Psi(G(x_n, x_{n+1}, x_{n+1})) \leq \Psi(G(x_n, x_{n+1}, x_{n+1})) - \Phi(G(x_n, x_{n+1}, x_{n+1})),$$

which implies that $\Phi(G(x_n, x_{n+1}, x_{n+1}))=0$, and hence we have $G(x_n, x_{n+1}, x_{n+1})=0$. This is contradicts our assumption that $G(x_n, x_{n+1}, x_{n+1}) > 0$. Therefore, for any $n \in \mathbb{N}$,

$$G(x_n, x_{n+1}, x_{n+1}) \leq G(x_{n-1}, x_n, x_n) \tag{3.6}$$

which implies that $\{G(x_n, x_{n+1}, x_{n+1})\}$ is a monotonic non-increasing. Then there exists $r \geq 0$ such that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = r. \tag{3.7}$$

We now prove that $r = 0$. As Ψ and Φ are continuous, and taking the limit on both sides of equation (3.4), we get

$$\begin{aligned} &\Psi(\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1})) \\ &\quad\leq \Psi(\max\{\lim_{n\to\infty} G(x_{n-1}, x_n, x_n), \lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1})\}) \\ &\qquad -\Phi(\max\{\lim_{n\to\infty} G(x_{n-1}, x_n, x_n), \lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1})\}) \\ &\quad\leq \Psi(\max\{\lim_{n\to\infty} G(x_{n-1}, x_n, x_n), \lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1})\}). \end{aligned}$$

Then, by (3.7), we deduce that

$$\begin{aligned}\Psi(r) &\leq \Psi(\max\{r,r\}) - \Phi(\max\{r,r\}) \\ &\leq \Psi(\max\{r,r\}) = \Psi(r),\end{aligned}$$

which implies that $\Phi(\max\{r,r\}) = \Psi(r) = 0$, $r = 0$. Thus, we have

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0.$$

Now, we will prove that $\{x_n\}$ is a Cauchy sequence in (X, G). Suppose the contrary, i.e., $\{x_n\}$ is not a Cauchy sequence. Then, there exists an $\epsilon > 0$ for which we can find two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that $n(k)$ is the smallest index for which $n(k) > m(k) > k$,

$$G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \geq \epsilon, G(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}) < \epsilon. \tag{3.8}$$

Using equation (3.8) and the rectangle inequality, we get

$$\begin{aligned}\epsilon &\leq G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\ &\leq G(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) \\ &< \epsilon + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}).\end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (3.3), we obtain

$$\lim_{n\to\infty} G(x_{m(k)}, x_{n(k)}, x_{n(k)}) = \epsilon. \tag{3.9}$$

On the other hand, for all k, there exists $j(k) \epsilon \{1, ..., p\}$ such that

$$n(k) - m(k) + j(k) \equiv 1[p].$$

Then, $x_{m(k)-j(k)}$ (for k large enough; $m(k) > j(k)$) and $x_{n(k)}$ lie in different adjacently labelled sets A_i and A_{i+1} for certain $i\epsilon\{1, ..., p\}$. Using (3.1), we obtain

$$\begin{aligned}&\Psi(G(x_{m(k)-j(k)+1}, x_{n(k)+1}, x_{n(k)+1})) \\ &\quad= \Psi(G(f_{m(k)-j(k)+1}x_{m(k)-j(k)}, f_{n(k)+1}x_{n(k)}, f_{n(k)+1}x_{n(k)})) \\ &\quad\leq \Psi(M_{m(k)-j(k)+1,n(k)+1}(x_{m(k)-j(k)}, x_{n(k)}, x_{n(k)})) \\ &\qquad -\Phi(M_{m(k)-j(k)+1,n(k)+1}(x_{m(k)-j(k)}, x_{n(k)}, x_{n(k)})) \\ &\qquad +L\min\{G(x_{m(k)-j(k)+1}, x_{m(k)-j(k)+1}, x_{m(k)-j(k)}), \\ &\qquad G(x_{n(k)+1}, x_{n(k)+1}, x_{n(k)}), G(x_{m(k)-j(k)+1}, x_{n(k)}, x_{n(k)}), \\ &\qquad G(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)-j(k)}))\}\end{aligned} \tag{3.10}$$

for all k. Now, we have

$$\begin{aligned}&M_{m(k)-j(k)+1,n(k)+1}(x_{m(k)-j(k)}, x_{n(k)}, x_{n(k)}) \\ &\quad= \max\{G(x_{m(k)-j(k)}, x_{n(k)}, x_{n(k)}), \\ &\qquad G(x_{m(k)-j(k)}, x_{m(k)-j(k)+1}, x_{m(k)-j(k)+1}), G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}), \\ &\qquad \alpha[G(x_{m(k)-j(k)}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{m(k)-j(k)+1}, x_{n(k)}, x_{n(k)})]\}\end{aligned}$$

for all k. Using the rectangle inequality and (3.3), we get

$$\begin{aligned}
&\left|G(x_{m(k)-j(k)}, x_{n(k)}, x_{n(k)}) - G(x_{m(k)}, x_{n(k)}, x_{n(k)})\right| \\
&\quad \leq G(x_{m(k)-j(k)}, x_{m(k)}, x_{m(k)}) \\
&\quad \leq \sum_{l=0}^{j(k)-1} G(x_{m(k)-j(k)+l}, x_{m(k)-j(k)+l+1}, x_{m(k)-j(k)+l+1}) \\
&\quad \leq \sum_{l=0}^{p} G(x_{m(k)-j(k)+l}, x_{m(k)-j(k)+l+1}, x_{m(k)-j(k)+l+1}) \to 0, \text{as } k \to \infty,
\end{aligned}$$

which implies from (3.9) that

$$\lim_{k\to\infty} G(x_{m(k)-j(k)}, x_{n(k)}, x_{n(k)}) = \epsilon. \tag{3.11}$$

Using (3.3), we have

$$\lim_{k\to\infty} G(x_{m(k)-j(k)}, x_{m(k)-j(k)+1}, x_{m(k)-j(k)+1}) = 0. \tag{3.12}$$

$$\lim_{k\to\infty} G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) = 0. \tag{3.13}$$

Again, using the rectangle inequality, we get

$$\begin{aligned}
&\left|G(x_{m(k)-j(k)}, x_{n(k)+1}, x_{n(k)+1}) - G(x_{m(k)-j(k)}, x_{n(k)}, x_{n(k)})\right| \\
&\qquad \leq G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}).
\end{aligned} \tag{3.14}$$

Letting $k \to \infty$ in the above inequality, using (3.13) and (3.11), we get

$$\lim_{k\to\infty} G(x_{m(k)-j(k)}, x_{n(k)+1}, x_{n(k)+1}) = \epsilon. \tag{3.15}$$

Similarly, we have

$$\begin{aligned}
&\left|G(x_{m(k)-j(k)+1}, x_{n(k)}, x_{n(k)}) - G(x_{m(k)-j(k)}, x_{n(k)}, x_{n(k)})\right| \\
&\qquad \leq G(x_{m(k)-j(k)+1}, x_{m(k)-j(k)}, x_{m(k)-j(k)}).
\end{aligned}$$

Letting $k \to \infty$, using (3.3) and (3.11), we obtain

$$\lim_{k\to\infty} G(x_{m(k)-j(k)+1}, x_{n(k)}, x_{n(k)}) = \epsilon. \tag{3.16}$$

Letting again $k \to \infty$ in (3.12) and using (3.3), (3.11), (3.12), (3.13), (3.15), (3.16) and the continuity of Ψ and Φ, we get that

$$\begin{aligned}
\Psi(\epsilon) &\leq \Psi(\max\{\epsilon, 0, 0, \alpha(\epsilon+\epsilon)\}) - \Phi(\max\{\epsilon, 0, 0, \alpha(\epsilon+\epsilon\}) + L\min\{0, 0, \epsilon, \epsilon\} \\
&\leq \Psi(\max\{\epsilon, 0, 0, \alpha(\epsilon+\epsilon)\}.
\end{aligned}$$

Since $0 \leq \alpha \leq \frac{1}{2}$, then

$$\Psi(\epsilon) \leq \Psi(\epsilon) - \Phi(\epsilon) + 0 \leq \Psi(\epsilon),$$

which implies that $\Phi(\epsilon) = 0$, and hence that $\epsilon = 0$, which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence in X.

Finally, we need to show the existence of a common fixed point for $\{f_k\}_{k=1}^{\infty}$. Now, since (X, G) is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u. \tag{3.17}$$

We will prove that

$$u \in \bigcap_{i=1}^{p} A_i. \tag{3.18}$$

Since $X = \bigcup_{i=1}^{p} A_i$, and since $x_0 \in A_1$, we have $\{x_{np}\}_{n\geq 0} \subseteq A_1$. Since A_1 is closed, from (3.17), we get that $u \in A_1$. Again, we have $\{x_{np+1}\}_{n\geq 0} \subseteq A_2$. Since A_2 is closed, from (3.17), we get that $u \in A_2$. Continuing this process, we obtain (3.18). Now, we will prove that u is a common fixed point of $\{f_k\}_{k=1}^{\infty}$. Indeed, from (3.18), since for all n, there exists $i(n) \in \{1, 2, ..., p\}$ such that $x_n \in A_{i(n)}$. Applying (3.1), with $x = u$ and $y = x_n$, we obtain

$$\begin{aligned} \Psi(G(f_k u, x_{n+1}, x_{n+1})) = {} & \Psi(G(f_k u, f_{n+1}x_n, f_{n+1}x_n)) \\ \leq {} & \Psi(M_{k,n+1}(u, x_n, x_n)) - \Phi(M_{k,n+1}(u, x_n, x_n)), \\ & +L \min\{G(f_k u, f_k u, u), G(f_{n+1}x_n, f_{n+1}x_n, x_n), \\ & G(f_k u, x_n, x_n), G(f_{n+1}x_n, f_{n+1}x_n, u)\} \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} M_{k,n+1}(u, x_n, x_n) &= \max\{G(u, x_n, x_n), G(u, f_k u, f_k u), G(x_n, f_{n+1}x_n, f_{n+1}x_n), \\ &\quad \alpha[G(u, f_{n+1}x_n, f_{n+1}x_n) + G(f_k u, x_n, x_n)]\} \\ &= \max\{G(u, x_n, x_n), G(u, f_k u, f_k u), G(x_n, x_{n+1}, x_{n+1}), \\ &\quad \alpha[G(u, x_{n+1}, x_{n+1}) + G(f_k u, x_n, x_n)]\}, \end{aligned}$$

and

$$\begin{aligned} & L \min\left\{ \begin{array}{c} G(f_k u, f_k u, u), G(f_{n+1}x_n, f_{n+1}x_n, x_n), \\ G(f_k u, x_n, x_n), G(f_{n+1}x_n, f_{n+1}x_n, u) \end{array} \right\} \\ & = L \min\left\{ \begin{array}{c} G(f_k u, f_k u, u), G(x_{n+1}, x_{n+1}, x_n), \\ G(f_k u, x_n, x_n), G(x_{n+1}, x_{n+1}, u) \end{array} \right\}. \end{aligned}$$

Letting $k \to \infty$ in (3.19) and using (3.17), we obtain

$$\begin{aligned} \Psi(G(f_k u, u, u)) \leq {} & \Psi(\max\{0, G(u, f_k u, f_k u), 0, \alpha[0 + G(f_k u, u, u)]\}) \\ & -\Phi(\max\{0, G(u, f_k u, f_k u), 0, \alpha[0 + G(f_k u, u, u)]\}), \\ & +L \min\{G(f_k u, f_k u, u), 0, G(f_k u, u, u), 0\}, \end{aligned}$$

or

$$\begin{aligned} \Psi(G(f_k u, u, u)) &\leq \Psi(G(f_k u, f_k u, u)) - \Phi(G(f_k u, u, u)) \\ &\leq \Psi(G(f_k u, f_k u, u)), \end{aligned}$$

which implies that $\Phi(G(f_k u, u, u)) \leq 0$, that is, $G(f_k u, u, u) = 0$ and hence $u = f_k u$, $\forall k \in \mathbb{N}$. Therefore, u is a common fixed point of $\{f_k\}_{k=1}^{\infty}$. ■

The proof of the following theorem is omitted because it is similar to that of Theorem 3.2.

Theorem 3.3 *Let (X, G) be a complete G-metric space, and let $A_1, A_2, ..., A_p$ be a non-empty closed subset of X and $X = \bigcup_{i=1}^{p} A_i$. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of mappings of X into itself. Suppose that for every $i, j \in \mathbb{N}$ and all $x, y \in X$, we have*

$$\Psi(G(f_i x, f_j y, f_j y)) \leq \Psi(M_{i,j}(x, y, y)) - \Phi(N_{i,j}(x, y, y))$$

where

$$\begin{aligned} M_{i,j}(x, y, y) &= \max\{G(x, y, y), G(x, f_i x, f_i x), G(y, f_j y, f_j y), \\ &\quad \alpha[G(x, f_j y, f_j y) + G(f_i x, y, y)]\}, \end{aligned}$$

$$N_{i,j}(x, y, y) = \max\{G(x, y, y), G(x, f_j y, f_j y), G(f_i x, y, y)\},$$

where $0 \leq \alpha \leq \frac{1}{2}$. Let $\Psi : [0, \infty) \to [0, \infty)$ be an altering distance function, and $\Phi : [0, \infty) \to [0, \infty)$ be a continuous function with $\Phi(t) = 0$ if and only if $t = 0$.

Assume that $f_k(A_i) \subseteq A_{i+1}$, for every $k \in \mathbb{N}$. Then, $\{f_k\}_{k=1}^{\infty}$ have a common fixed point $u \in \cap_{i=1}^{p} A_i$; that is,

$$f_1 u = f_2 u = ... = u.$$

4. Conclusions

We have proved some results on coincidence and common fixed points for a sequence of mappings satisfying generalized $(\Psi - \Phi\ )$ contractive conditions in G-metric space. The existence of common fixed point for a sequence of mappings satisfying the almost generalized cyclic weak contractive condition in G-metric space is investigated and an example supporting our results is included.

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