ON CHARACTERIZATIONS OF BL-ALGEBRAS VIA IMPLICATIVE IDEALS

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Abstract. In the paper, we introduce the concept of implicative ideals in BL-algebras by the pseudo implication operation and show some characterizations of ideals. We prove that implicative ideals coincide with Boolean ideals through analyzing the characterizations of implicative ideals. Finally, we consider the concepts of maximal ideals and investigate the relationships among the introduced ideals.

Keywords: BL-algebra, Boolean ideal, implicative ideal, maximal ideal.

1. Introduction

BL-algebras as the algebraic structures for Hájek's basic logic were raised from the continuous t-norm, familiar in the fuzzy logic framework [4]. The main examples of BL-algebras are from the unit interval endowed with continuous t-norms. BL-algebras arising as Lindenbaum BL-algebras from certain logic axioms have close relationships with the quantum structures. In fact, MV-algebras are BL-algebras while BL-algebras with the double negation are MV-algebras. It has been proved that MV-algebras are categorically equivalent to bounded commutative BCK-algebras [15] and a BL-algebra is a particular case of a reversed left BCK-algebra [6]. The study of these algebras have been carried out from both logic and algebraic standpoints.

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The filter theory plays an important role in studying these algebras. From logic point of view, various filters have natural interpretation as various sets of provable formulas. At present, the filter theory of BL-algebras has been widely studied and some important results are obtained. Hájek [4] introduced the notions of filters and prime filters in BL-algebras and proved the completeness of Basic Logic BL. Turunen studied the filter theory of BL-algebras and proposed the concepts of implicative filters and Boolean filters, which are called deductive systems to emphasize the fact that they correspond to sets of provable formulas and are closed with respect to modus ponens [16], [17]. It turned out that Boolean deductive systems coincide with implicative deductive systems in BL-algebras [17]. In particular, some types of filters such as (positive) implicative filters and fantastic filters [5], integral filters [3] and obstinate filters [1] were introduced and some of their characterizations were presented in [2], [9]. Moreover, based on the fuzzy set theory, the related fuzzy structures of filters in BL-algebras were further investigated [12], [18]. Besides BL-algebras, the filter theories of other algebraic structures such as R_0-algebras [14] and resituated lattices [13], [8], [19] which are closely related to BL-algebras, had been investigated by several researchers.

Since MV-algebras are BL-algebras, it is nature to generalize some notions of MV-algebras to BL-algebras. In MV-algebras, filters and ideals are dual notions, while some papers claimed that the notion of ideals is missing in BL-algebras for lack of a suitable algebraic addition [16], [17]. To fill the gap, Lele and Nganou [11] introduced the notions of ideals, prime ideals and Boolean ideals in BL-algebras and derived some characterizations of them. They also investigated the relationship between ideals and filters by exploiting the set of complements. The results derived from [11] show that Lele and Nganou 's ideals play an important role in the characterizations of BL-algebras, while compared with the filter theory, the notion of ideals has no corresponding deductive types and it is very intricate to introduce other types of ideals, such as implicative ideals etc., based on the original operations. Therefore it is meaningful to introduce new operation and give the deductive type of ideals in BL-algebras.

The aim of the paper is to investigate the ideals further more. We present some characterizations of ideals, prime ideals and Boolean ideals by using pseudo implication operation. It is known that Boolean filters coincide with implicative filters in BL-algebras. In line with the result, we propose the notion of implicative ideals and prove that implicative ideals coincide with Boolean ideals by the derived characterizations of implicative ideals. By introducing maximal ideals, we investigate the relationships among these ideals.

2. Preliminaries

In this section, for purpose of reference, we present some definitions and results about BL-algebras.

Definition 2.1 [4] An algebra $(L, \wedge, \vee, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is called a BL-algebra if it satisfies the following conditions: for all $x, y, z \in L$,

(BL-1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,
(BL-2) $(L, \odot, 1)$ is a commutative monoid,
(BL-3) $x \odot y \leq z$ if and only if $x \leq y \to z$,
(BL-4) $x \odot (x \to y) = x \wedge y$,
(BL-5) $(x \to y) \vee (y \to x) = 1$.

Let L be a BL-algebra. If $x \vee \bar{x} = 1$ for any $x \in L$, then L is called a Boolean algebra, where $\bar{x} = x \to 0$; if L satisfies the double negation, i.e., $\bar{\bar{x}} = x$ for any $x \in L$, then L is called an MV-algebra; if $x^2 = x \odot x = x$ for any $x \in L$, then L is called a Gödel algebra.

The following properties are well known to hold in BL-algebras, we summarize them as follows.

Lemma 2.2 [16], [11] *In any BL-algebra $(L, \wedge, \vee, \odot, \to, 0, 1)$, the following relations hold: for any $x, y, z \in L$,*

(1) $x \leq y$ *if and only if* $x \to y = 1$, $x \odot y = 0$ *if and only if* $x \leq \bar{y}$;

(2) $x \odot (x \to y) \leq y$, $x \odot y \leq x \wedge y$, $x \leq y \to x$;

(3) $x \to y \leq (y \to z) \to (x \to z)$, $y \to x \leq (z \to y) \to (z \to x)$;

(4) $(x \to y) \odot (y \to z) \leq (x \to z)$, $(x \to y) \to (x \to z) \leq x \to (y \to z)$;

(5) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$;

(6) $\bar{0} = 1$, $\bar{1} = 0$, $1 \to x = x$, $x \to 1 = 1$, $x \odot \bar{x} = 0$, $x \leq \bar{\bar{x}}$, $\bar{x} = \bar{\bar{\bar{x}}}$;

(7) *if* $x \vee \bar{x} = 1$, *then* $x \wedge \bar{x} = 0$;

(8) $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$;

(9) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$, $(x \wedge y) \to z = (x \to z) \vee (y \to z)$;

(10) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$, $x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z)$;

(11) *if* $x \leq y$, *then* $y \to z \leq x \to z$, $z \to x \leq z \to y$, $x \odot z \leq y \odot z$;

(12) $\overline{x \odot y} = x \to \bar{y}$, $\overline{x \vee y} = \bar{x} \wedge \bar{y}$, $\overline{x \wedge y} = \bar{x} \vee \bar{y}$;

(13) $\overline{\overline{x \odot y}} = \bar{\bar{x}} \odot \bar{\bar{y}}$, $\overline{\overline{x \to y}} = \bar{\bar{x}} \to \bar{\bar{y}}$, $\overline{\overline{x \vee y}} = \bar{\bar{x}} \vee \bar{\bar{y}}$, $\overline{\overline{x \wedge y}} = \bar{\bar{x}} \wedge \bar{\bar{y}}$.

In order to introduce the notion of ideals in BL-algebras as a generalization of the existing notion in MV-algebras, Lele and Nganou adopted the pseudo addition operation in a BL-algebra L: $x \oslash y := \bar{x} \to y$, for any $x, y \in L$, and gave the concept of ideals in BL-algebras as follows.

Definition 2.3 [11] Let $(L, \wedge, \vee, \odot, \to, 0, 1)$ be a BL-algebra and I a nonempty subset of L. I is called an ideal if it satisfies: for any $x, y \in L$,

(1) $x, y \in I$ implies $x \oslash y \in I$,

(2) if $x \leq y$ and $y \in I$, then $x \in I$.

From the above definition, it is easy to see that $0 \in I$, and $x \in I$ if and only if $\bar{\bar{x}} \in I$ for any $x \in L$. It is easy to prove that $\{0\}$ is an ideal of L. If a BL-algebra L is not an MV-algebra, then there exists an element $x \in L$ such that $\bar{\bar{x}} \neq x$, i.e., $x \oslash 0 \neq 0 \oslash x$, hence the operation $\oslash$ is not commutative in general.

Let P be an ideal of a BL-algebra L. If $\overline{x \to y} \in I$ or $\overline{y \to x} \in P$ for any $x, y \in L$, then P is called a prime ideal; if $x \wedge \bar{x} \in P$ for any $x \in L$, then P is called a Boolean ideal [11].

Proposition 2.4 [11] *An ideal P of a BL-algebra L is a prime ideal if and only if $x \wedge y \in P$ implies $x \in P$ or $y \in P$, for any $x, y \in L$.*

Theorem 2.5 [11] *Let I be an ideal of a BL-algebra L. Define the relation $\sim_I$ on L by: for any $x, y \in L$,*

$$x \sim_I y \text{ if and only if } \bar{x} \odot y \in I \text{ and } x \odot \bar{y} \in I.$$

Then $\sim_I$ is a congruence relation on L.

Let L be a BL-algebra and I an ideal of L, the set of all congruence classes is denoted by L/I, that is, $L/I := \{[x] | x \in L\}$, where $[x] = \{y \in L | x \sim_I y\}$.

For any $x, y \in L$, the operations $\star$, $\mapsto$, $\sqcap$, $\sqcup$ on L/I are defined as follows: $[x] \sqcap [y] = [x \wedge y]$, $[x] \sqcup [y] = [x \vee y]$, $[x] \star [y] = [x \odot y]$, $[x] \mapsto [y] = [x \to y]$. It follows that $(L/I, \sqcap, \sqcup, \star, \mapsto, [0], [1])$ is a BL-algebra which is called a quotient BL-algebra with respect to I.

Proposition 2.6 [11] *Let L be a BL-algebra and I an ideal of L. Then L/I is an MV-algebra.*

3. Some characterizations of ideals

Compared with filter theory, we observe that the notion of ideals has no corresponding deductive type in BL-algebras. In order to fill the gap, we introduce a new operation and give another description of ideals in BL-algebras.

Let L be a BL-algebra, we define the pseudo implication operation $\rightharpoonup$ by $x \rightharpoonup y := x \odot \bar{y}$, for any $x, y \in L$. It is easy to see that $z \leq x \oslash y$ if and only if $z \rightharpoonup x \leq y$.

We can establish some properties of the operation $\rightharpoonup$ as follows.

Lemma 3.1 *Let L be a BL-algebra, for any $x, y, z \in L$, we have:*

(1) *$x \leq y$ implies $z \rightharpoonup y \leq z \rightharpoonup x$ and $x \rightharpoonup z \leq y \rightharpoonup z$;*

(2) *$(x \rightharpoonup y) \rightharpoonup z = (x \rightharpoonup z) \rightharpoonup y = x \rightharpoonup (y \oslash z)$;*

(3) *$x \rightharpoonup 0 = x$, $0 \rightharpoonup x = 0$, $x \rightharpoonup x = 0$;*

(4) *$(x \rightharpoonup z) \rightharpoonup (y \rightharpoonup z) \leq x \rightharpoonup y$;*

(5) *$(x \rightharpoonup z) \leq (y \rightharpoonup z) \oslash (x \rightharpoonup y)$;*

(6) $x \rightharpoonup y \leq x$, $x \leq y$ *implies* $x \rightharpoonup y = 0$;

(7) $x \rightharpoonup y = 0$ *implies* $\bar{y} \leq \bar{x}$ *and* $x \leq \bar{\bar{y}}$;

(8) $(y \rightharpoonup (y \rightharpoonup x)) \rightharpoonup (x \rightharpoonup (x \rightharpoonup y)) = 0$.

Proof. By Lemma 2.2, routine computations prove the above results. ∎

We would like to point out that the operation $\rightharpoonup$ is not commutative in general, since $1 \rightharpoonup 0 = 1 \neq 0 \rightharpoonup 1 = 0$.

By using the operation $\rightharpoonup$, we can give a concrete description of ideals in BL-algebras as follows.

Lemma 3.2 [11] *Let I be a nonempty subset of a BL-algebra L. Then I is an ideal of L if and only if it satisfies:*

(1) $0 \in I$,

(2) *for any* $x, y \in L$, *if* $x \rightharpoonup y \in I$ *and* $y \in I$, *then* $x \in I$.

Lemma 3.3 *Let I be an ideal of a BL-algebra L. Then the followings hold: for any $x, y, z \in L$,*

(1) $x \rightharpoonup y \in I$ *if and only if* $\bar{y} \rightharpoonup \bar{x} \in I$;

(2) $x \rightharpoonup y \in I$ *if and only if* $\bar{\bar{x}} \rightharpoonup y \in I$;

(3) $(y \rightharpoonup \bar{x}) \rightharpoonup z \in I$ *if and only if* $(\bar{z} \rightharpoonup \bar{y}) \rightharpoonup \bar{x} \in I$.

Proof. (1) Suppose that $x \rightharpoonup y \in I$. Since $(\bar{y} \rightharpoonup \bar{x}) \rightharpoonup (x \rightharpoonup y) = \bar{y} \odot \bar{\bar{x}} \odot \overline{x \odot \bar{y}} = (\bar{y} \wedge \bar{x}) \odot \bar{\bar{x}} = 0 \in I$ and I is an ideal, thus $\bar{y} \rightharpoonup \bar{x} \in I$.

Conversely, suppose that $\bar{y} \rightharpoonup \bar{x} \in I$. Since $(x \rightharpoonup y) \rightharpoonup (\bar{y} \rightharpoonup \bar{x}) = x \odot \bar{y} \odot \overline{\bar{y} \odot \bar{\bar{x}}} = (\bar{y} \wedge \bar{x}) \odot x = 0 \in I$, therefore $x \rightharpoonup y \in I$.

The proofs of (2) and (3) are similar to that of (1). ∎

Theorem 3.4 *Let L be a BL-algebra. A nonempty subset I of L is an ideal if and only if it satisfies the conditions:*

(1) $0 \in I$,

(2) *for any* $x, y \in L$, *if* $y \in I$ *and* $\overline{\bar{y} \to \bar{x}} \in I$, *then* $x \in I$.

Proof. Suppose that I is an ideal of L. By Lemma 3.2, we get $0 \in I$. For any $x, y \in L$, if $y \in I$ and $\overline{\bar{y} \to \bar{x}} \in I$, that is $y \in I$ and $\overline{\overline{x \rightharpoonup y}} \in I$, then $y \in I$ and $x \rightharpoonup y \in I$. Hence $x \in I$, and so I satisfies the conditions (1) and (2).

Conversely, suppose that the nonempty subset I satisfies the conditions (1) and (2). For any $x, y \in L$, if $x \in I$, then $\bar{\bar{x}} \in I$. In fact, since $\overline{\bar{x} \to \bar{\bar{\bar{x}}}} = \overline{\bar{x} \to \bar{x}} = 0 \in I$ and $x \in I$, hence $\bar{\bar{x}} \in I$. Let $x \rightharpoonup y \in I$ and $y \in I$. Then $\overline{\overline{x \rightharpoonup y}} = \overline{\bar{y} \to \bar{x}} \in I$, and so $x \in I$. Therefore I is an ideal. ∎

Let L be a BL-algebra, we denote by $L(x, y) = \{z \in L | z \rightharpoonup x \leq y\}$ for any $x, y \in L$. Our next aim is to establish further characterizations of ideals.

Proposition 3.5 *Let I be a nonempty subset of a BL-algebra L. Then the following conditions are equivalent:*

(1) *I is an ideal of L,*

(2) *$L(x,y) \subseteq I$, for any $x, y \in I$,*

(3) *$(z \rightharpoonup x) \rightharpoonup y = 0$ implies $z \in I$, for any $x, y \in I$ and $z \in L$.*

Proof. (1) $\Rightarrow$ (2) Let I be an ideal. For any $x, y \in I$, if $z \in L(x,y)$, then $z \rightharpoonup x \leq y$. By Definition 2.3 and Lemma 3.2, we have $z \in I$, hence $L(x,y) \subseteq I$.

(2) $\Rightarrow$ (3) It is obvious.

(3) $\Rightarrow$ (1) Assume that $(z \rightharpoonup x) \rightharpoonup y = 0$ implies $z \in I$, for any $x, y \in I$ and $z \in L$. Since I is a nonempty subset, then there exists $x \in I$. By Lemma 3.1, we get that $(0 \rightharpoonup x) \rightharpoonup x = 0$, therefore $0 \in I$. Let $x \rightharpoonup y \in I$ and $y \in I$. It follows from Lemma 3.1, $(x \rightharpoonup (x \rightharpoonup y)) \rightharpoonup y = (x \rightharpoonup y) \rightharpoonup (x \rightharpoonup y) = 0$, hence $x \in I$, and so I is an ideal. ∎

The following result will display that the operation $\rightharpoonup$ plays an important role in the ideal theory of BL-algebras.

Proposition 3.6 *Let P be a proper ideal of a BL-algebra L. Then P is a prime ideal if and only if $x \rightharpoonup y \in P$ or $y \rightharpoonup x \in P$, for any $x, y \in L$.*

Proof. Indeed, P is prime if and only if $\overline{x \to y} \in P$ or $\overline{y \to x} \in P$, for all $x, y \in L$. Now, it follows from Lemma 2.2 (13) that $\overline{x \to y} \in P$ if and only if $\overline{\bar{x} \to \bar{y}} \in P$ for all $x, y \in L$. The result is now clear since $a \in P$ if and only if $\bar{\bar{a}} \in P$, and $\overline{\bar{x} \to \bar{y}} = \overline{\overline{y \rightharpoonup x}}$. ∎

Notice the fact that $x \wedge \bar{x} \leq \bar{\bar{x}} \wedge \bar{x}$ for any $x \in L$, we can obtain a condition under which an ideal is a Boolean ideal in BL-algebras.

Remark 3.7 Let I be an ideal of a BL-algebra L. Then I is a Boolean ideal if and only if $\bar{x} \wedge \bar{\bar{x}} \in I$ for any $x \in L$.

4. Implicative ideals

In this section, we introduce the notion of implicative ideals in BL-algebras and investigate some of their properties. Here we obtain that an ideal is Boolean if and only if it is an implicative ideal.

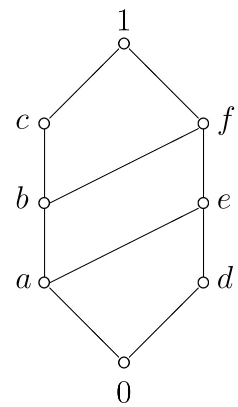
Definition 4.1 A nonempty subset I of a BL-algebra L is called an implicative ideal if it satisfies:

(1) $0 \in I$,

(2) $(x \rightharpoonup y) \rightharpoonup z \in I$ and $y \rightharpoonup z \in I$ imply $x \rightharpoonup z \in I$, for any $x, y, z \in L$.

For better understanding of the above definition, we illustrate it by the following example.

Example 4.2 Let $L = \{0, a, b, c, d, e, f, 1\}$ be a set with Hasse diagram and Cayley tables as follows.

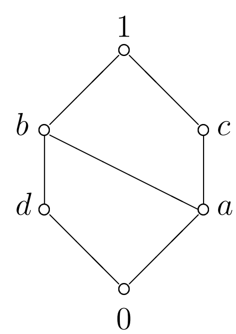


$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	a	a	0	a	a	a
b	0	a	a	b	0	a	a	b
c	0	a	b	c	0	a	b	c
d	0	0	0	0	d	d	d	d
e	0	a	a	a	d	e	e	e
f	0	a	a	b	d	e	e	f
1	0	a	b	c	d	e	f	1

$\to$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	d	1	1	1	d	1	1	1
b	d	f	1	1	d	f	1	1
c	d	e	f	1	d	e	f	1
d	c	c	c	c	1	1	1	1
e	0	c	c	c	d	1	1	1
f	0	b	c	c	d	f	1	1
1	0	a	b	c	d	e	f	1

Then $(L, \wedge, \vee, \otimes, \to, 0, 1)$ is a BL-algebra that is not an MV-algebra. It is easy to check that $I = \{0, a, b, c\}$ is an implicative ideal of L.

Example 4.3 Let $L = \{0, a, b, c, d, 1\}$ be a set with Hasse diagram and Cayley tables as follows.



$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	d	1	1
b	c	d	1	c	d	1
c	b	b	b	1	1	1
d	a	b	b	d	1	1
1	0	a	b	c	d	1

According to [7], $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra. It is easy to check that $I = \{0, c, d\}$ is an implicative ideal of L.

The following proposition describes the relationship between implicative ideals and ideals.

Proposition 4.4 *Let I be an implicative ideal of a BL-algebra L. Then I is an ideal, but the converse is not true in general.*

Proof. Suppose that I is an implicative ideal of a BL-algebra L. Let $x, y \in L$. If $x \rightharpoonup y \in I$ and $y \in I$, then $(x \rightharpoonup y) \rightharpoonup 0 = x \rightharpoonup y \in I$ and $y \rightharpoonup 0 = y \in I$. By hypothesis we get $x = x \rightharpoonup 0 \in I$, hence I is an ideal.

For the converse we consider the BL-algebra L of Example 4.3, it is easy to see that $J = \{0, b\}$ is an ideal but not an implicative ideal since $(c \rightharpoonup a) \rightharpoonup a = 0 \in J$ and $a \rightharpoonup a = 0 \in I$ but $c \notin J$. ∎

In the following, we give some characterizations of implicative ideals for further discussion.

Theorem 4.5 *Let I be an ideal of a BL-algebra L. Then I is an implicative ideal of L if and only if it satisfies the condition* (PI)*: $(y \rightharpoonup \bar{x}) \rightharpoonup z \in I$ and $x \rightharpoonup y \in I$ imply $x \rightharpoonup z \in I$, for any $x, y, z \in L$.*

Proof. Suppose that I is an implicative ideal. For any $x, y \in L$, let $(y \rightharpoonup \bar{x}) \rightharpoonup z \in I$ and $x \rightharpoonup y \in I$. By Lemma 3.3, we have $(\bar{z} \rightharpoonup \bar{y}) \rightharpoonup \bar{x} \in I$ and $\bar{y} \rightharpoonup \bar{x} \in I$. Hence $\bar{z} \rightharpoonup \bar{x} \in I$, and so $x \rightharpoonup z \in I$.

Conversely, suppose that I satisfies the condition (PI). Let $(x \rightharpoonup y) \rightharpoonup z \in I$ and $y \rightharpoonup z \in I$, that is $x \rightharpoonup (y \oslash z) \in I$ and $y \rightharpoonup z \in I$. By Lemma 3.3 we get $(\bar{y} \rightharpoonup \bar{\bar{z}}) \rightharpoonup \bar{x} \in I$ and $\bar{z} \rightharpoonup \bar{y} \in I$. Therefore $\bar{z} \rightharpoonup \bar{x} \in I$, and so $x \rightharpoonup z \in I$. Hence I is an implicative ideal of L. ∎

Theorem 4.6 *Let I be an ideal of a BL-algebra L. Then the following conditions are equivalent:*

(1) *I is an implicative ideal of L;*

(2) *for any $a \in L$, the set $I_a := \{x \in L | x \rightharpoonup a \in I\}$ is an ideal of L.*

Proof. Suppose that I is an implicative ideal. For any $x, y \in L$, if $x \rightharpoonup y \in I_a$ and $y \in I_a$, then $(x \rightharpoonup y) \rightharpoonup a \in I$ and $y \rightharpoonup a \in I$, hence $x \rightharpoonup a \in I$, that is, $x \in I_a$. By Lemma 3.1, $0 \rightharpoonup a = 0 \in I$, then $0 \in I_a$, hence I_a is an ideal.

Conversely, suppose that I_a is an ideal of L for any $a \in L$. For any $x, y, z \in L$, if $(x \rightharpoonup y) \rightharpoonup z \in I$ and $y \rightharpoonup z \in I$, then $x \rightharpoonup y \in I_z$ and $y \in I_z$. Since I_z is an ideal, we obtain $x \in I_z$, that is, $x \rightharpoonup z \in I$. Therefore I is an implicative ideal. ∎

As a consequence of Theorem 4.6, we have the following result.

Proposition 4.7 *Let I be an implicative ideal of a BL-algebra L. Then for any $a \in L$, I_a is the least ideal of L containing I and a.*

Proof. Assume that I is an implicative ideal and $a \in L$. By Theorem 4.6, we obtain I_a an ideal. For any $x \in I$, then $x \rightharpoonup a \leq x$. From Proposition 4.4, it follows that I is an ideal of L, hence $x \rightharpoonup a \in I$, that is, $x \in I_a$, and so $I \subseteq I_a$. Notice that $a \rightharpoonup a = 0 \in I$, we obtain $a \in I_a$. If J is an ideal containing I and a, then for any $x \in I_a$, we have $x \rightharpoonup a \in I \subseteq J$. Since J is an ideal of L and $a \in J$, we get $x \in J$, that is $I_a \subseteq J$. Therefore I_a is the least ideal containing I and a. ∎

Next, we discuss some properties of I_a.

Proposition 4.8 *Let I, J be ideals of a BL-algebra L. Then the following statements hold: for any $a, b \in L$,*

(1) $I_a = I$ *if and only if* $a \in I$,

(2) $a \leq b$ *implies* $I_a \subseteq I_b$,

(3) $I \subseteq J$ *implies* $I_a \subseteq J_a$,

(4) $(I \cap J)_a = I_a \cap J_a$, $(I \cup J)_a = I_a \cup J_a$,

(5) *if I is an implicative ideal, then* $I_{a \oslash b} = (I_a)_b$

Proof. (1) Suppose that $I_a = I$. Since $a \rightharpoonup a = 0 \in I$, we have $a \in I_a$ by the definition of I_a, hence $a \in I$.

Conversely, assume that $a \in I$. For any $x \in I$, since $x \rightharpoonup a \leq x \in I$ and I is ideal, we have $x \rightharpoonup a \in I$. Hence $x \in I_a$, and so $I \subseteq I_a$. On the other hand, for any $x \in I_a$, then $x \rightharpoonup a \in I$. Since $a \in I$, we get $x \in I$. Therefore $I_a \subseteq I$ and so $I_a = I$.

(2) Suppose that $a \leq b$. For any $x \in I_a$, then $x \rightharpoonup a \in I$. Since $x \rightharpoonup b \leq x \rightharpoonup a$ and I is an ideal, we have $x \rightharpoonup b \in I$. Therefore $x \in I_b$, and so $I_a \subseteq I_b$.

(3) Assume that $I \subseteq J$. For any $x \in I_a$, we have $x \rightharpoonup a \in I \subseteq J$, then $x \in J_a$, it follows that $I_a \subseteq J_a$.

(4) Since $I \cap J \subseteq I, J$, we have $(I \cap J)_a \subseteq I_a \cap J_a$ by (3). On the other hand, for any $x \in I_a \cap J_a$, we have $x \rightharpoonup a \in I$ and $x \rightharpoonup a \in J$. Hence $x \rightharpoonup a \in I \cap J$, and so $I_a \cap J_a \subseteq (I \cap J)_a$. Thus $(I \cap J)_a = I_a \cap J_a$. $(I \cup J)_a = I_a \cup J_a$ can be proved in the similar way.

(5) Suppose that I is an implicative ideal. By Theorem 4.6, I_a is an ideal. $x \in I_{a \oslash b}$ if and only if $x \rightharpoonup (a \oslash b) = (x \rightharpoonup a) \rightharpoonup b = (x \rightharpoonup b) \rightharpoonup a$ if and only if $x \in (I_a)_b$. Hence $I_{a \oslash b} = (I_a)_b$. ∎

The following results are the characterizations of implicative ideals.

Theorem 4.9 *Let I be a nonempty subset of a BL-algebra L. Then the following conditions are equivalent:*

(1) *I is an implicative ideal of L,*

(2) *I is an ideal and for any $x, y \in L$, $(x \rightharpoonup y) \rightharpoonup y \in I$ implies $x \rightharpoonup y \in I$,*

(3) *I is an ideal and for any $x, y, z \in L$, $(x \rightharpoonup y) \rightharpoonup z \in I$ implies $(x \rightharpoonup z) \rightharpoonup (y \rightharpoonup z) \in L$,*

(4) $0 \in I$, *and if* $((x \rightharpoonup y) \rightharpoonup y) \rightharpoonup z \in I$ *and* $z \in I$, *then* $x \rightharpoonup y \in I$, *for any* $x, y, z \in L$.

Proof. (1) $\Rightarrow$ (2) Let I be an implicative ideal and $(x \rightharpoonup y) \rightharpoonup y \in I$. It follows from Proposition 4.4 that I is an ideal. Observe that $y \rightharpoonup y = 0 \in I$, we get $x \rightharpoonup y \in I$.

(2) $\Rightarrow$ (3) Assume that (2) holds. For any $x, y, z \in L$, let $(x \rightharpoonup y) \rightharpoonup z \in I$. According to Lemma 3.1, we have $((x \rightharpoonup (y \rightharpoonup z)) \rightharpoonup z) \rightharpoonup z = ((x \rightharpoonup z) \rightharpoonup (y \rightharpoonup z)) \rightharpoonup z \leq (x \rightharpoonup y) \rightharpoonup z$. It follows that $((x \rightharpoonup (y \rightharpoonup z)) \rightharpoonup z) \rightharpoonup z \in I$. Therefore $(x \rightharpoonup z) \rightharpoonup (y \rightharpoonup z) = (x \rightharpoonup (y \rightharpoonup z)) \rightharpoonup z \in I$ by hypothesis, and so (3) is valid.

(3) $\Rightarrow$ (4) Suppose that (3) holds. Obviously, $0 \in I$. For any $x, y, z \in L$, if $((x \rightharpoonup y) \rightharpoonup y) \rightharpoonup z \in I$ and $z \in I$, then $(x \rightharpoonup y) \rightharpoonup y \in I$. It follows from hypothesis and Lemma 3.1 that $x \rightharpoonup y = (x \rightharpoonup y) \rightharpoonup (y \rightharpoonup y) \in I$, therefore (4) is valid.

(4) $\Rightarrow$ (1) Assume that (4) is valid. We assert that I is an ideal of L. In fact, for any $x, y \in L$, if $x \rightharpoonup y \in I$ and $y \in I$, then $((x \rightharpoonup 0) \rightharpoonup 0) \rightharpoonup y = x \rightharpoonup y \in I$. By hypothesis, we get $x = x \rightharpoonup 0 \in I$, therefore I is an ideal. Let $(x \rightharpoonup y) \rightharpoonup z \in I$ and $y \rightharpoonup z \in I$. From Lemma 3.1, it follows that $((x \rightharpoonup z) \rightharpoonup z) \rightharpoonup (y \rightharpoonup z) \leq (x \rightharpoonup z) \rightharpoonup y = (x \rightharpoonup y) \rightharpoonup z$. Due to the fact that I an ideal of L, we have $((x \rightharpoonup z) \rightharpoonup z) \rightharpoonup (y \rightharpoonup z) \in I$, therefore $x \rightharpoonup z \in I$, and thus (1) holds. ∎

The extension property for implicative ideals is obtained from the following proposition.

Proposition 4.10 *Let I and J be ideals of a BL-algebra L such that $I \subseteq J$. If I is an implicative ideal, then so is J.*

Proof. Let $x, y, z \in L$ such that $(x \rightharpoonup y) \rightharpoonup z \in J$. Denote $u = (x \rightharpoonup y) \rightharpoonup z$, it follows that $((x \rightharpoonup u) \rightharpoonup y) \rightharpoonup z = ((x \rightharpoonup y) \rightharpoonup z) \rightharpoonup u = 0 \in I$. Since I is an implicative ideal of L, we have $((x \rightharpoonup u) \rightharpoonup z) \rightharpoonup (y \rightharpoonup z) = ((x \rightharpoonup z) \rightharpoonup (y \rightharpoonup z)) \rightharpoonup u \in I \subseteq J$ by Theorem 4.9. Consider that J is an ideal and $u \in J$, we obtain that $(x \rightharpoonup z) \rightharpoonup (y \rightharpoonup z) \in J$. Hence J is an implicative ideal of L. ∎

Now we continue to study the characterizations of implicative ideals.

Theorem 4.11 *Let I be a nonempty subset of a BL-algebra L. Then the following conditions are equivalent:*

(1) *I is an implicative ideal of L,*

(2) $0 \in I$, *and if* $(x \rightharpoonup (y \rightharpoonup x)) \rightharpoonup z \in I$ *and* $z \in I$, *then* $x \in I$, *for any* $x, y, z \in L$,

(3) *I is an ideal and for any $x, y \in L$, $x \rightharpoonup (y \rightharpoonup x) \in I$ implies $x \in I$.*

Proof. (1) $\Rightarrow$ (2) Suppose that I is an implicative ideal. Obviously, $0 \in I$. For any $x, y, z \in L$, if $(x \rightharpoonup (y \rightharpoonup x)) \rightharpoonup z \in I$ and $z \in I$, then $x \rightharpoonup (y \rightharpoonup x) \in I$. Since $((y \rightharpoonup (y \rightharpoonup x)) \rightharpoonup (y \rightharpoonup x)) \rightharpoonup (x \rightharpoonup (y \rightharpoonup x)) = ((y \rightharpoonup (y \rightharpoonup x)) \rightharpoonup (x \rightharpoonup

$(y \rightharpoonup x)) \rightharpoonup (y \rightharpoonup x) \leq (y \rightharpoonup x) \rightharpoonup (y \rightharpoonup x) = 0 \in I$, we get that $(y \rightharpoonup (y \rightharpoonup x)) \rightharpoonup (y \rightharpoonup x) \in I$. Using Theorem 4.9, we obtain that $y \rightharpoonup (y \rightharpoonup x) \in I$. Since $(x \rightharpoonup (x \rightharpoonup y)) \rightharpoonup (y \rightharpoonup (y \rightharpoonup x)) = 0 \in I$, then $x \rightharpoonup (x \rightharpoonup y) \in I$. By Lemma 2.2, it is easy to obtain that $((x \rightharpoonup y) \rightharpoonup z) \rightharpoonup (x \rightharpoonup (y \rightharpoonup x)) = 0$, therefore $(x \rightharpoonup y) \rightharpoonup z \in I$. Notice that $z \in I$, we have $x \rightharpoonup y \in I$. Follows the result $x \rightharpoonup (x \rightharpoonup y) \in I$, we get $x \in I$, and so (2) holds.

(2) $\Rightarrow$ (3) For any $x, y \in L$, if $x \rightharpoonup y \in I$ and $y \in I$, then $(x \rightharpoonup (x \rightharpoonup x)) \rightharpoonup y = x \rightharpoonup y \in I$. By hypothesis, we have $x \in I$, hence I is an ideal. Suppose that $x \rightharpoonup (y \rightharpoonup x) \in I$, then $(x \rightharpoonup (y \rightharpoonup x)) \rightharpoonup 0 \in I$ and $0 \in I$. Therefore $x \in I$, and thus (3) is valid.

(3) $\Rightarrow$ (1) Assume that (3) holds. For any $x, y \in L$, let $(x \rightharpoonup y) \rightharpoonup y \in I$. Routine calculation shows that $((x \rightharpoonup y) \rightharpoonup (x \rightharpoonup (x \rightharpoonup y))) \rightharpoonup ((x \rightharpoonup y) \rightharpoonup y) = 0$, then $(x \rightharpoonup y) \rightharpoonup (x \rightharpoonup (x \rightharpoonup y)) \in I$. By hypothesis, $x \rightharpoonup y \in I$. From Theorem 4.9, it follows that I is an implicative ideal. ∎

To show the relationship between implicative ideals and Boolean ideals in BL-algebras, we give a characterization of implicative ideals as follows.

Theorem 4.12 *Let I be an ideal of a BL-algebra L. Then I is an implicative ideal of L if and only if $(x \oslash x) \rightharpoonup x \in I$ for any $x \in L$.*

Proof. For any $x \in L$, we have $x \rightharpoonup x = 0 \in I$ and $((x \oslash x) \rightharpoonup x) \rightharpoonup x = (\bar{x} \to x) \odot \bar{x} \odot \bar{x} = (\bar{x} \wedge x) \odot \bar{x} = 0 \in I$. Due to the fact that I is an implicative ideal, we get $(x \oslash x) \rightharpoonup x \in I$.

Conversely, suppose that for any $x \in L$, $(x \oslash x) \rightharpoonup x \in I$ holds. Let $x, y, z \in L$ such that $(x \rightharpoonup y) \rightharpoonup z \in I$ and $y \rightharpoonup z \in I$. By Lemma 3.1, it not difficult to obtain that $(x \rightharpoonup (z \oslash z)) \rightharpoonup ((x \rightharpoonup y) \rightharpoonup z) \rightharpoonup (y \rightharpoonup z) = 0$. From Definition 2.3 and Lemma 3.2, it follows that $x \rightharpoonup (z \oslash z) \in I$. Since $(z \oslash z) \rightharpoonup z \in I$ and $x \rightharpoonup z \leq ((z \oslash z) \rightharpoonup z) \oslash (x \rightharpoonup (z \oslash z))$, we get $x \rightharpoonup z \in I$. Hence I is an implicative ideal of L. ∎

Let L be a BL-algebra. For any $x \in L$, we have $(x \oslash x) \rightharpoonup x = (\bar{x} \to x) \odot \bar{x} = \bar{x} \wedge x$. As an application of the above theorem, we have the following result.

Corollary 4.13 *Let I be an ideal of a BL-algebra L. Then I is an implicative ideal if and only if I is a Boolean ideal of L.*

Summarizing the above results, we have the following proposition.

Proposition 4.14 *In a BL-algebra L, the following conditions are equivalent:*

(1) *any ideal I of L is implicative;*

(2) $\{0\}$ *is an implicative ideal;*

(3) *for any $a \in L$, the set $L(a) := \{x \in L | x \rightharpoonup a = 0\}$ is an ideal of L.*

Proof. (1) $\Leftrightarrow$ (2) It is clear by Proposition 4.10.

(2) $\Rightarrow$ (3) For any $a, x, y \in L$, if $x \rightharpoonup y \in L(a)$ and $y \in L(a)$, then $(x \rightharpoonup y) \rightharpoonup a = 0$ and $y \rightharpoonup a = 0$. Since $\{0\}$ is an implicative ideal, we have $x \rightharpoonup a = 0$, that is, $x \in L(a)$. Hence $L(a)$ is an ideal of L.

(3) $\Rightarrow$ (2) For any $x, y \in L$, if $(x \rightharpoonup y) \rightharpoonup y = 0$, then $x \rightharpoonup y \in L(y)$. Since $L(y)$ is an ideal and $y \in L(y)$, then $x \in L(y)$, that is $x \rightharpoonup y = 0$. By Theorem 4.9, $\{0\}$ is an implicative ideal. ■

Proposition 4.15 *If L is a Boolean algebra or a Gödel algebra, then any ideal of L is an implicative ideal.*

Proof. To prove the proposition, we consider two cases.

Case 1. If L is a Boolean algebra, then $x \vee \bar{x} = 1$ for any $x \in L$. By Lemma 2.2, we have $x \wedge \bar{x} = 0$. It follows that $\{0\}$ is a Boolean ideal. By Corollary 4.13 and Proposition 4.14, any ideal of L is an implicative ideal.

Case 2. Assume that L is a Gödel algebra and I is an ideal of L. For any $x \in L$, we have $(x \oslash x) \rightharpoonup x = (\bar{x} \to x) \odot \bar{x} = (\bar{x} \to x) \odot (\bar{x} \odot \bar{x}) = (\bar{x} \wedge x) \odot \bar{x} = 0$. Notice that I is an ideal, we have $x \oslash x \rightharpoonup x \in I$. From Theorem 4.12, it follows that I is an implicative ideal of L. ■

Next, we introduce the notion of maximal ideals in BL-algebras and discuss the relations among various ideals.

Definition 4.16 Let L be a BL-algebra. A proper ideal I of L is called a maximal ideal if I is not a proper subset of any proper ideal of L.

It is easy to prove the following result.

Proposition 4.17 *Any maximal ideal of a BL-algebra L is a prime ideal.*

The following example shows that maximal ideals exist.

Example 4.18 Let $(L, \wedge, \vee, \odot, \to, 0, 1)$ be a BL-algebra of Example 4.3. It can be verified that $I = \{0, c, d\}$ is a maximal ideal, and also a prime ideal.

For the purpose of investigating some relationships among various ideals of BL-algebras, we prepare the following results.

Lemma 4.19 *Let M be an ideal of a BL-algebra. Then ideal M is maximal if and only if L/M is simple, i.e, has no proper ideal other than $\{0\}$.*

Lemma 4.20 [11] *An ideal P of a BL-algebra L is a prime ideal if and only if the BL-algebra quotient L/P is an MV-chain.*

Lemma 4.21 [11] *Let L be a BL-algebra, then an ideal I is Boolean if and only if L/I is a Boolean algebra.*

Theorem 4.22 *Let I be a proper ideal of a BL-algebra L. Then the following statements are equivalent:*

(1) *I is a maximal and implicative ideal;*

(2) *$x \notin I$ and $y \notin I$ imply $x \rightharpoonup y \in I$ and $y \rightharpoonup x \in I$ for any $x, y \in L$;*

(3) *if $x \notin I$, then exists some $n > 0$ such that $x^n_{\oslash} := \underbrace{\bar{x} \oslash \cdots \oslash \bar{x}}_{n \text{ times}} \in I$;*

(4) *$x \in I$ or $\bar{x} \in I$ for any $x \in L$;*

(5) *I is a prime and implicative ideal.*

Proof. (1) $\Rightarrow$ (2) Let $x, y \notin I$. It follows from Proposition 4.7 that $I_y = \{z \in L \mid z \rightharpoonup y \in I\}$ is the least ideal containing I and y. Since I is maximal and $y \notin I$, we get $I_y = L$. Thus $x \in I_y$, and so $x \rightharpoonup y \in I$. $y \rightharpoonup x \in I$ can be proved similarly.

(2) $\Rightarrow$ (3) Suppose that $x \notin I$. Since I is a proper ideal, then $1 \notin I$. By hypothesis, we have $x \rightharpoonup 1 = 0 \in I$ and $1 \rightharpoonup x = \bar{x} \in I$. It follows from Definition 2.3 that $x^n_{\oslash} \in I$ for any $n > 0$.

(3) $\Rightarrow$ (4) Assume that (3) holds. For any $x \in L$, if $x \in I$, it is true. Assume $x \notin I$, then there exists some $n > 0$ such that $x^n_{\oslash} \in I$. Since $\bar{x} \leq x^n_{\oslash}$ and I is a proper ideal of L, we have $\bar{x} \in I$. Thus (4) is valid.

(4) $\Rightarrow$ (5) For any $x \in L$, $x \wedge \bar{x} \leq x, \bar{x}$, it follows from hypothesis and Corollary 4.13 that I is an implicative ideal. For any $y \in L$, we have $(y \rightharpoonup x) \rightharpoonup \bar{x} = 0 \in I$ and $(x \rightharpoonup y) \rightharpoonup x = 0 \in I$. Since I is a proper ideal, and $x \in I$ or $\bar{x} \in I$, we have $x \rightharpoonup y \in I$ or $y \rightharpoonup x \in I$. It follows from Proposition 3.6 that I is prime. Hence I is a prime and implicative ideal.

(5) $\Rightarrow$ (1) Let I be a prime and implicative ideal. From Lemmas 4.20 and 4.21, it follows that L/I would be an MV-chain that is also a Boolean algebra. While the only Boolean chain is the two-element Boolean algebra, hence $L/I \cong 2$. According to Lemma 4.19, (1) is clear. ∎

5. Conclusions

In this paper, we gave some characterizations of ideals, prime ideals and Boolean ideals by the pseudo implication operation. Then we introduced the notions of implicative ideals and investigated some characterizations of them. For future work, we could use the pseudo implication operation to investigate the relationships among BL-algebras and other logic algebras, and obtain some logic results.

Acknowledgements. The authors are grateful to the referees for useful suggestions and comments which led to an improvement of the paper. The works described in this paper are partially supported by the Higher Education Key Scientific Research Program Funded by Henan Province (No. 16A110028, 16A130004) and National Natural Science Foundation of China (No. 11571281, 11401128).

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Accepted: 19.07.2016