OD-CHARACTERIZABILITY OF THE SYMMETRIC GROUP $\mathbb{S}_{27}$

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Abstract. Let G be a finite group with degree pattern $D(G)$. A finite group G is called k-fold OD-characterizable if there are exactly k non-isomorphic groups H such that $|G| = |H|$ and $D(G) = D(H)$. In this paper our purpose is to correct an earlier paper and prove that the symmetric group on 27 letters is 38-OD-characterizable.

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1. Introduction and preliminaries

The set of all the prime divisors of the order of a finite group G is denoted by $\pi(G)$ and the set of elements order in G is denoted by $\pi_e(G)$. The Gruenberg-Kegel graph or the prime graph of G, denoted by $\Gamma(G)$, is a graph with vertex set $\pi(G)$ and two distinct vertices p and q are joined by an edge if and only if $pq \in \pi_e(G)$, and in this case we write $p \sim q$, otherwise we will write $p \nsim q$ which means that G does not have an element of order pq. The number of connected components of $\Gamma(G)$ is denoted by $t(G)$ and these connected components are denoted by $\pi_i(G)$, $1 \leq i \leq t(G)$. If $2 \in \pi(G)$, then $\pi_1(G)$ is the connected component of $\Gamma(G)$ containing 2. If n is a natural number, then $\pi(n)$ is the set of all primes dividing n, hence $|G| = m_1 m_2 \cdots m_{t(G)}$ where m_i is a positive integer with $\pi(m_i) = \pi_i$, $1 \leq i \leq t(G)$. The numbers m_i, $1 \leq i \leq t(G)$ are called the order components of G,

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we write $OC(G) = \{m_1, ..., m_{t(G)}\}$ for the set of order components of G. The set of prime graph components of G is denoted by $T(G) = \{\pi_1(G), \pi_2(G), ..., \pi_{t(G)}(G)\}$.

If the order of G is $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, with p_i's distinct prime numbers and α_i positive integers, then for $p \in \{p_1, ..., p_k\}$, the degree of p in $\Gamma(G)$ is the number of vertices adjacent to p and is denoted by $deg(p)$, i.e., $deg(p) = |\{p_i | p \sim p_i\}|$. We choose an ordering such that $p_1 < p_2 < \cdots < p_k$ and define $D(G) = (deg(p_1), deg(p_2), ..., deg(p_k))$ as the degree pattern of G. Let n be a positive integer. A group H is called n-fold OD-characterizable if there are exactly n non-isomorphic groups G such that $|G| = |H|$ and $D(G) = D(H)$. Usually a 1-fold OD-characterizable group is called an OD-characterizable group.

For the first time the prime graphs of a finite group was defined in [5] and its significance can be found in many recent researches on finite groups. In [1] it is shown that the alternating groups $\mathbb{A}_p$, where p and $p-2$ are prime numbers are OD-characterizable, it is also shown that all the sporadic simple groups, certain groups of Lie type are OD-characterizable, but the projective symplectic group $SP_6(3)$ is 2-fold OD-characterizable. In [10] it is shown that all the simple K_4-groups, except $\mathbb{A}_{10}$ are OD-characterizable, whereas $\mathbb{A}_{10}$ is 2-fold OD-characterizable. In [7] it is proved that all the simple groups of order less than 10^8 except $\mathbb{A}_{10}$ and $U_4(2)$ are OD-characterizable while $\mathbb{A}_{10}$ and $U_4(2)$ are 2-fold OD-characterizable.

In [9] it is proved that $\mathbb{A}ut(L_2(49))$ is 9-fold OD-characterizable, and in [8] the authors prove that the alternating group of degree 16 is OD-characterizable. In [3] the authors studied OD-characterizability of the alternating and symmetric groups on 27 letters. Although their results for the alternating group is correct but there is an error in the results stated for the symmetric group. OD-characterizability of $\mathbb{S}_{27}$ is also studied in [2] but the conclusion is in error. In [4] we found some results on OD-characterizability of the symmetric group on 27 letters. In this paper our aim is to give a complete proof of OD-characterizability of the group $\mathbb{S}_{27}$. Our main result is:

Theorem 1.1. *Let G be a finite group such that $|G| = |\mathbb{S}_{27}|$ and $D(G) = D(\mathbb{S}_{27})$. Then G is 38 OD-characterizable.*

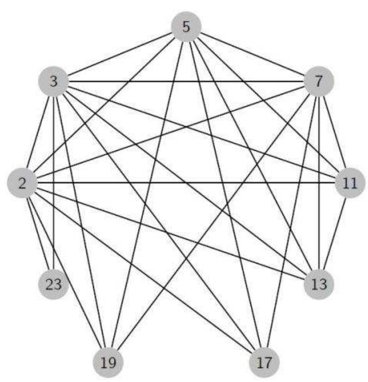

Figure 1: The prime graph of $\mathbb{S}_{27}$

For the proof we need to know about non-abelian simple groups with a prime divisor at most 19. Using [6] we listed these groups in the Table 1 of [3] which is used frequently in this paper.

2. Proof of the main result

The order and degree pattern of $\mathbb{S}_{27}$ are as follows:

$$|\mathbb{S}_{27}| = 2^{23}.3^{13}.5^6.7^3.11^2.13^2.17.19.23 \text{ and } D(\mathbb{S}_{27}) = (8,8,7,7,5,5,4,4,2).$$

Let G be a finite group such that $|G| = |\mathbb{S}_{27}|$ and $D(G) = D(\mathbb{S}_{27})$. Since deg(2)=8, the vertex 2 is joined to all other vertices of $\Gamma(G)$, hence $\Gamma(G)$ is a connected graph. Obviously $\Gamma(G) = \Gamma(\mathbb{S}_{27})$ and that $\pi_e(G) \supseteq \{2,3,5,7,11,13,17,19,23,6,10,14,22,$ $26,34,38,46,15,27,33,39,51,57,69,35,55,65,85,95,77,91,$ $119,133,143\}$. We also have $\Gamma(\mathbb{S}_{27}) = \Gamma(\mathbb{A}_{27}) = \Gamma(\mathbb{S}_{26})$.

Lemma 2.1. *Let K be the maximal normal solvable subgroup of G. Then K is a $\{2,3\}$-group. In particular G is a non-solvable group.*

Proof. Let p be a prime divisor of $|K|$ and S_p be a Sylow p-subgroup of K. By the Frattini argument, we have $G = KN_G(S_p)$. We consider the following cases:
Case (i) $p = 23$. Since S_{23} is a cyclic 23-group, $\overline{N} = N_G(S_{23})/C_G(S_{23})$ is isomorphic to a subgroup of $\mathbb{Z}_{22}$, implying that $|\overline{N}|$ is a divisor of 22. Since G contains elements of order 46 and 69, we deduce that $C_G(S_{23})$ is a $\{2,3,23\}$-group, and from $G = KN_G(S_{23})$ we obtain $19 \mid |K|$. By assumption K is a solvable group, hence by considering a $\{19,23\}$-Hall subgroup of K we obtain $19 \sim 23$, a contradiction.
Case (ii) $p = 17$ or 19. Let S_p be a Sylow p-subgroup of K, $p = 17$ or 19. Then, similar to case (i), $G = KN_G(S_p)$ and $\overline{N} = N_G(S_p)/C_G(S_p)$ is a subgroup of $\mathbb{Z}_{16}$ or $\mathbb{Z}_{18}$. But using the prime graph $\Gamma(G)$, we see that only $2,3,5,7$ are joined to 17 and 19. We deduce that in any case $23 \mid |K|$ implying that K contains an element of order $23p$, $p = 17$ or 19, a contradiction.
Case (iii) $p = 11$ or 13. In this case, a Sylow p-subgroup S_p of K may have order p or p^2. If S_p is cyclic, then $|\mathbb{A}ut(S_p)| = 11.10$ or 13.12, for the respective cases $p = 11$ or 13. Since $G = KN_G(S_p)$ and $|\overline{N}| = |N_G(S_p)/C_G(S_p)|$ divides 11.10 or 13.12, we deduce that in any case $23 \mid |K|$. Therefore K contains element of order 23.11 or 23.13, a contradiction. Next suppose that a Sylow p-subgroup S_p of K is not cyclic, hence it is of the form $\mathbb{Z}_p \times \mathbb{Z}_p$, $p = 11$ or 13. But $\mathbb{A}ut(S_{13}) = GL_2(13)$ is a group of order $2^5.3^2.7.13$ and $\mathbb{A}ut(S_{11}) = GL_2(11)$ is a group of order $2^4.3.5^2.11$. But $deg(13) = deg(11) = 5$ and 13 is joined to the vertices $2,3,5,7,11$, whereas 11 is joined to the vertices $2,3,5,7,13$. Therefore, $N_G(S_p)$ is a $\{2,3,5,7,11,13\}$-group, $p = 11$ or 13. Hence, $19 \mid |K|$, a contradiction.
Case (iv) $p = 7$. In this case, a Sylow 7-subgroup of K has order 7, 7^2 or 7^3. If $|S_7| = 7$ or 7^2, then using the same techniques as in case (i) and (iii) we obtain a contradiction. Therefore, suppose $|K| = 7^3$. From $G = KN_G(S_7)$ and the fact that $23 \nmid |K|$, we deduce that $23 \mid |N_G(S_7)|$, so S_7 is normalized by an element σ

of order 23. Since G has no element of order 23.7, $\langle\sigma\rangle$ should act fixed-point freely on S_7, implying $23 \mid 7^3 - 1$, a contradiction.

Case (v) $p = 5$. In this case, $|S_5| = 5^\alpha$, $1 \leq \alpha \leq 6$. By case (i), we have $23 \nmid |K|$ and from $G = KN_G(S_5)$ we deduce that S_5 is normalized by an element of order 23, hence $23 \mid 5^\alpha - 1$. But considering all the numbers $1 \leq \alpha \leq 6$, we obtain a contradiction.

Therefore, we have proved that $|K|$ can only be divisible by 2 and 3 and the Lemma is proved. ∎

Lemma 2.2. *G/K is an almost simple group, $S \leq G/K \leq \mathbb{A}ut(S)$ where S is a simple group.*

Proof. We set $\overline{G} = G/K$ and $\overline{S} = soc(\overline{G})$ where $soc(\overline{G})$ denotes the socle of $\overline{G}$, i.e the subgroup of $\overline{G}$ generated by the set of all the minimal normal subgroups of $\overline{G}$. We have $\overline{S} \leq \overline{G} \leq \mathbb{A}ut(\overline{S})$ and $\overline{S} \cong S_1 \times S_2 \times \cdots \times S_n$; where S_i's are finite non-abelian simple groups. We show that $n = 1$. Assume on the contrary $n \geq 2$. If $23 \mid |\overline{S}|$, then the order of one S_i is divisible by 23. We assume $23 \mid |S_1|$. Since $23 \sim 2$ and $23 \sim 3$, the S_i's, $i \geq 2$ must be $\{2, 3\}$-groups contradicting the simplicity of S_i. Therefore $23 \nmid |\overline{S}|$ and by Lemma 2.1, $23 \mid |\overline{G}|$. From $N_G(\overline{S})/C_G(\overline{S}) = \overline{G}/C_G(\overline{S}) \leq \mathbb{A}ut(\overline{S})$ we deduce that $23 \in \pi(\overline{G}) \subseteq \pi(\mathbb{A}ut(\overline{S}))$. But $|\mathbb{A}ut(\overline{S})| = |\overline{S}|.|Out(\overline{S})|$, hence $23 \mid |Out(\overline{S})|$. Let $P_1, \cdots, P_k$ be non-isomorphic simple groups among $S_1, \cdots, S_n$ such that $\overline{S} \cong P_1^{t_1} \times \cdots \times P_k^{t_k}$, $t_1 + \cdots + t_k = n$. We have $Out(S) = Out(P_1^{t_1}) \times \cdots \times Out(P_k^{t_k})$ and from $23 \nmid |\overline{S}|$ we obtain $23 \nmid |P_i|$ for all $1 \leq i \leq k$, so P_i's are simple $\{2, 3, 5, 7, 11, 13, 17, 19\}$-group and by Table 1 in [2] we obtain $23 \nmid |Out(P_i)|, 1 \leq i \leq k$. But $\mathbb{A}ut(P_i^{t_i}) \cong P_i wr \mathbb{S}_{t_i}$ and from $23 \mid |\mathbb{A}ut(P_1^{t_1})| = |P_1|^{t_1}.t_1!$ we obtain $23 \mid t_1!$ which implies $t_1 \geq 23$. Therefore $(23!)_2 \times 2^{23} = 2^{46}$ must divide the order of G, a contradiction. Hence $n = 1$, and $\overline{S}$ is a simple group with $\overline{S} \leq \overline{G} \leq \mathbb{A}ut(\overline{S})$, the Lemma is proved. ∎

Lemma 2.3. *$\overline{S}$ is isomorphic to $\mathbb{A}_{26}$ or $\mathbb{A}_{27}$.*

Proof. By Lemma 2.2, $\overline{S} \leq \overline{G} \leq \mathbb{A}ut(\overline{S})$, where $\overline{S}$ is a simple group such that the largest prime factor of $|\overline{S}|$ is 23. Hence by $|G| = |\mathbb{S}_{27}|$ and the fact that $\{13, 17, 19, 23\} \cap \pi(Out(\overline{S})) = \emptyset$, we obtain $13^2.17.19.23 \mid |\overline{S}|$. Now by [5], $\overline{S} \cong \mathbb{A}_{26}$ or $\mathbb{A}_{27}$. ∎

Lemma 2.4. *If $\overline{S} \cong \mathbb{A}_{27}$, then there are 3 possibilities for G such that $|G| = |\mathbb{S}_{27}|$ and $OD(G) = OD(\mathbb{S}_{27})$.*

Proof. If $\overline{S} \cong \mathbb{A}_{27}$ then $\mathbb{A}_{27} \leq G/K \leq \mathbb{S}_{27}$. If $G/K \cong \mathbb{S}_{27}$, then $K = 1$ and $G \cong \mathbb{S}_{27}$. If $G/K \cong \mathbb{A}_{27}$, then $|K| = 2$, hence $G \cong \mathbb{Z}_2.\mathbb{A}_{27}$, which gives us 2 possibilities for G. The Lemma is proved. ∎

Lemma 2.5. *If $\overline{S} \cong \mathbb{A}_{26}$, then there are 35 non-isomorphic groups G with $|G| = |\mathbb{S}_{27}|$ and $OD(G) = OD(\mathbb{S}_{27})$.*

Proof. If $\overline{S} \cong \mathbb{A}_{26}$, then from $\mathbb{A}_{26} \leq G/K \leq \mathbb{S}_{26}$ we obtain $G/K \cong \mathbb{A}_{26}$ or $\mathbb{S}_{26}$.

Case (i) $G/K \cong \mathbb{A}_{26}$. In this case $|K| = 54$. It can be shown that $G \cong K \times \mathbb{A}_{26}$ or $G \cong P \times (2.\mathbb{A}_{26})$ where P is the unique subgroup of order 27 in K. Since there are 15 non-isomorphic groups of order 54 and 5 non-isomorphic groups of order 27, we obtain 20 possibilities for G.

Case (ii) $G/K \cong \mathbb{S}_{26}$. In this case, $|K| = 27$. It can be shown that G has a normal subgroup H isomorphic to $K \times \mathbb{A}_{26}$ such that $G/H \cong \mathbb{Z}_2$. This will give us 15 possibilities for G. Hence, altogether, we obtain 35 groups with the property mentioned in the theorem. ∎

Main theorem. *There are* 38 *non-isomorphic finite groups* G *with* $|G| = |\mathbb{S}_{27}|$ *and* $OD(G) = OD(\mathbb{S}_{27})$.

Proof. This follows from Lemmas 2.1-2.5. ∎

Remark 2.1. The structures of the 35 groups mentioned in Lemma 2.5 may be described in terms of the semidirect product.

Case(i) $G/K \cong \mathbb{A}_{26}, |K| = 54$.

Consider the semidirect product $K \rtimes \mathbb{A}_{26}$. Then, we have a homomorphism $\varphi : \mathbb{A}_{26} \to \mathbb{A}ut(K)$. Since $\mathbb{A}_{26}$ is a simple group, we obtain $Ker\varphi = \mathbb{A}_{26}$ implying that $G \cong K \times \mathbb{A}_{26}$. There are 15 groups of order 54, hence there are 15 possibilities for G.

Case (ii) $G/K \cong \mathbb{S}_{26}, |K| = 27$.

Let us consider the semidirect product $K \rtimes \mathbb{S}_{26}$ corresponding to the homomorphism $\varphi : \mathbb{S}_{26} \to \mathbb{A}ut(K)$. Considering the normal subgroups of $\mathbb{S}_{26}$ we have $ker\varphi = \mathbb{S}_{26}$ or $\mathbb{A}_{26}$. If $Ker\varphi = \mathbb{S}_{26}$, then $G \cong K \times \mathbb{S}_{26}$ and we obtain 5 possibilities for G. If $Ker\varphi = \mathbb{A}_{26}$, then $\mathbb{S}_{26}/\mathbb{A}_{26} \cong \varphi(\mathbb{S}_{26}) \leq \mathbb{A}ut(K)$. But $\mathbb{S}_{26}/\mathbb{A}_{26} \cong \mathbb{Z}_2$, implying that $\varphi(\mathbb{S}_{26})$ corresponds to an involution (an element of order 2) in $\mathbb{A}ut(K)$. But there are 5 non-isomorphic groups of order 27.

1) $K \cong \mathbb{Z}_{27}$, then $|\mathbb{A}ut(K)| = 18$ has only one involution.
2) $K \cong \mathbb{Z}_3 \times \mathbb{Z}_9$, then $\mathbb{A}ut(K) = \mathbb{Z}_2 \times \mathbb{Z}_2$, which has 3 involutions.
3) $K \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, then $\mathbb{A}ut(K) = GL_3(3)$ and it can be verified that this group has 7 conjugacy classes of involutions.
4) K is a group of order 27 with element of order 9, then $\mathbb{A}ut(K) \cong V_3(3) \rtimes GL_2(3)$, where $V_3(3)$ is the vector space of dimension 3 over the field with 3 elements. It can be verified that $V_3(3) \rtimes GL_2(3)$ has 3 conjugacy classes of involutions.
5) K is a group of order 27 with elements of order 9, then $\mathbb{A}ut(K) = V_3(3) \rtimes AGL_1(3)$, where $AGL_1(3)$ is the affine group in dimension 1 over the field with 3 elements. This group has only one conjugacy classes of involutions. Considering the non-isomorphic groups obtained from cases $1-4$, we obtain 15 groups. Hence altogether we obtain 35 non-isomorphic groups from the semidirect product $K \rtimes \mathbb{A}_{26}$, $|K| = 54$ and $K \rtimes \mathbb{S}_{26}, |K| = 27$.

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