AN ANALYTICAL APPROXIMATION TECHNIQUE FOR THE DUFFING OSCILLATOR BASED ON THE ENERGY BALANCE METHOD

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Abstract. In this paper, an analytical approximation technique has been presented of obtaining higher-order approximate solutions for highly nonlinear Duffing oscillator based on the energy balance method (EBM). Higher-order approximate natural frequencies have been obtained in a novel analytical way. The accuracy of the solution method is evaluated within as error analysis. It is highly remarkable that using the presented technique, the approximation solutions produces desired results even for large oscillation as compared with the exact ones. Moreover, the solution method yields much better results than existing solutions after using a suitable truncation formula. The presented technique is applied to well-known Duffing oscillator to illustrate its novelty, reliability and wider applicability.

Keywords: Duffing oscillator; Energy balance method; Analytical approximate technique; Truncation Principle.

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1. Introduction

Considerable attention has been directed towards the study in the field of vibration analysis because this issue is very applicable in dynamics of structures, free and forced vibrations and vibration of elastic plates have been published [1]–[3]. In general, analytical solutions of highly nonlinear oscillators do not always exist and hence most of the researchers have used either approximate analytical techniques or numerical methods to obtain approximate solutions. A few nonlinear systems can be solved explicitly, and numerical methods especially the most well-known Runge-Kutta fourth order method are frequently used to calculated approximate solutions. However, the class of stiff differential equations and chaotic differential equations, the numerical schemes do not always give accurate results, which present big challenge to numerical analysis. In this situation, many researchers have been showed an intensifying interest in the field of analytical approximate techniques. Most of the widely used analytical technique for solving nonlinear equations associated with oscillatory systems is Perturbation Method [4], [5], which is the most versatile tools available in nonlinear analysis of engineering problems, and they are constantly being developed and applied to ever more complex problems. However, the standard perturbation methods have many limitations, and they are not yield for highly nonlinear oscillators. As a result, to overcome the limitations of standard perturbation technique, a large variety of new analytical approximate techniques including Optimal Homotopy Asymptotic Method [6], Homotopy Perturbation Method [7], [8], Modified Homotopy Perturbation Method [9], Modified He's Homotopy Perturbation Method [10], [11], He's Modified Lindsted-Poincare Method [12], Parameterized Perturbation Method [13] commonly used to solve nonlinear systems especially for highly nonlinear oscillators.

Recently, some other approximation techniques such as He's Max-Min Approach Method [14], Elliptic Balance Method [15], Algebraic Method [16], Rational Energy Balance Method [17], He's Frequency-Amplitude Formulation [18]–[20], Iteration Method [21], [22], Variational Approach Method [23]–[25], Harmonic Balance Method [26]–[34], Rational Harmonic Balance Method [35] have been paid much attention to determined periodic solutions of highly nonlinear oscillatory problems. The Energy Balance Method (EBM) and He's Energy Balance Method (HEBM) [36]–[41] are another technique for solving highly nonlinear oscillators. In fact, to the best of our knowledge, there is no clear idea to obtain higher-order approximation solutions in HEBM. Moreover, only first-order approximation has been considered which does not lead much better accuracy. In this study, the higher-order approximate periodic solutions for the well-known Duffing oscillator is studied employing modified energy balance method (MEBM). The presented technique gives much better results than the classical energy balance method, He's energy balance method and other previously existing methods. In adding, a suitable truncation principle has been introduced which produced much better results. Considering the interesting property that the presented technique not only provides accurate results but also it is more convenient and efficient for solving more complex nonlinear problems.

2. Solution approaches

2.1 The basic idea of He's energy balance method

Let us consider a second order nonlinear differential equation is

$$\ddot{x} = -f, \quad \text{with initial condition} \quad x(0) = A_0, \quad \dot{x}(0) = 0, \tag{2.1}$$

in which x and t are represent dimensionless displacement and time variables respectively and $f = f(x, \dot{x})$.

The variational principle of Eq. (2.1) can be easily obtained as follows

$$J(x) = \int_0^t (-\frac{1}{2}\dot{x}^2 + F(x))dt \tag{2.2}$$

where $F(x) = \int f(x, \dot{x})dx$. Its Hamiltonian can be written in the following form

$$H = \frac{1}{2}\dot{x}^2 + F(x) = F(A_0) \tag{2.3}$$

or

$$R(t) = \frac{1}{2}\dot{x}^2 + F(x) - F(A_0) = 0 \tag{2.4}$$

The following trial solution utilized to obtain the natural frequency

$$x = A_0 \cos(\omega t) \tag{2.5}$$

Substituting equation (2.5) into equation (2.4), the following residual equation is reduced as

$$R(t) = \frac{1}{2}A_0^2\omega^2 \sin^2(\omega t) + F(A_0 \cos(\omega t)) - F(A_0) = 0 \tag{2.6}$$

Since equation (2.5) in only an approximation to the exact solution, equation (2.6) cannot be made zero everywhere. Collocation at $\omega t = \frac{\pi}{4}$ gives

$$\omega(A_0) = \frac{2}{A_0}\sqrt{F(A_0) - F(\frac{A_0}{\sqrt{2}})} \tag{2.7}$$

Its period can be determining by using the relation $T = \frac{2\pi}{\omega}$ as

$$T = \frac{2\pi}{\frac{2}{A_0}\sqrt{F(A_0) - F(\frac{A_0}{\sqrt{2}})}} \tag{2.8}$$

2.2 The Modified energy balance method

A general n-th order periodic solution of equation (2.1) is in the form

$$x = A_0(\rho \cos(\omega t) + u \cos(3\omega t) + v \cos(5\omega t) + w \cos(7\omega t) + z \cos(9\omega t) + \cdots), \tag{2.9}$$

where A_0, ρ and ω are constants. If $\rho = 1-u-v-\cdots$, then the solution equation (2.9) readily satisfies the initial conditions given in equation (2.1).

Substituting (2.9) into (2.4) and expanding it in a Fourier series expansion as

$$\frac{1}{2}\dot{x}^2 + F(x) - F(A_0) = b_1\cos(\omega t) + b_3\cos(3\omega t) + b_5\cos(5\omega t) + \cdots \tag{2.10}$$

where $b_1, b_3, \cdots$ will be calculated by using the following integration

$$b_{2n+1} = \frac{4}{\pi}\int_0^{\pi/2}(\frac{1}{2}\dot{x}^2 + F(x) - F(A_0))cos[(2n+1)\varphi]d\varphi; n = 0, 1, 2, 3, \ldots, \tag{2.11}$$

setting $\varphi = \omega t$. Substituting (2.9) into (2.11), the coefficients $b_1, b_3, \ldots$ are determined. Finally, substituting $b_1, b_3, \ldots$ into equation (2.10) and then equating the coefficients of $\cos(\omega t), \cos(3\omega t), \cos(5\omega t), \ldots,$ equal to zero, a set of nonlinear algebraic equations is obtained whose solution provides the unknown natural frequency ω and the others unknown coefficients $u, v, \ldots$ in terms of amplitude A_0. This completes the determination of all related unknowns for the proposed periodic solution given in (2.9).

3. Problem Descriptions

The focused generalized nonlinear oscillator, which is a numerous range of applications in nonlinear sciences and engineering as

$$\ddot{x} + \omega_0^2 x + \varepsilon x^n |\, x \,|^{\alpha-1} + \frac{\gamma x^{m-1}}{\sqrt{x^m+1}} = 0, \quad n = 2k-1, \quad \alpha > 0, \quad m = 2p \tag{3.1}$$

If $\gamma = 0, \alpha = 1$ and $n = 3$, equation (3.1) represents the governing equation of generalized Duffing oscillator is as follows

$$\ddot{x} + \omega_0^2 x + \varepsilon x^3 = 0, \tag{3.2}$$

which is stated in [16]–[18], [36], [37], [39], [41]. Equation (3.2) also represents the free undamped vibration of an orthotropic claimed triangular plate which has been stated in [37].

4. Application of Duffing oscillator

In this section, an example will be presented to illustrate the accuracy, efficiency and its wider applicability of the presented method.

The Variational and Hamiltonian formulations of (3.2) can be obtained as

$$J(x) = \int_0^t \left(-\frac{1}{2}\dot{x}^2 - \frac{\omega_0^2 x^2}{2} - \frac{\varepsilon x^4}{4}\right) dt \tag{4.1}$$

$$H = \frac{1}{2}\dot{x}^2 + \frac{\omega_0^2 x^2}{2} + \frac{\varepsilon x^4}{4} = \frac{\omega_0^2 A_0^2}{2} + \frac{\varepsilon {A_0}^4}{4} \tag{4.2}$$

$$H = \frac{1}{2}\dot{x}^2 + \frac{\omega_0^2 x^2}{2} + \frac{\varepsilon x^4}{4} - \frac{\omega_0^2 A_0^2}{2} - \frac{\varepsilon {A_0}^4}{4} = 0 \tag{4.3}$$

In equation (2.9), the first-order approximation solution is

$$x = A_0 \cos(\omega t) \tag{4.4}$$

Using (4.4) into (4.3) and then substitute into (2.11), $b_1, b_3, \ldots$ are obtained. Finally, substituting $b_1, b_3, \ldots$ into (2.10), then equating the coefficient of $\cos(\omega t)$ equal to zero, the first order natural frequency is obtained as

$$\omega(A_0) = \sqrt{\frac{10\omega_0^2 + 7\varepsilon A_0^2}{10}} \tag{4.5}$$

Therefore, the first-order approximation solution of equation (3.2) is equation (4.4) where ω is given by equation (4.5).

Consider a second-order approximation solution from equation (2.9) is

$$x = A_0 \cos(\omega t) + A_0 u(\cos(3\omega t) - \cos(\omega t)) \tag{4.6}$$

Using (4.6) into (4.3) and then substitute into (2.11), $b_1, b_3, \ldots$ are obtained.

Finally, substituting $b_1, b_3, \ldots$ into (2.10), then equating the coefficient of $\cos(\omega t)$ and $\cos(3\omega t)$ equal to zero, the following nonlinear algebraic equations are

$$\begin{aligned} &\varepsilon A_0^2(-21021 - 54912u + 109824u^2 - 133120u^3 + 71680u^4) \\ &+858((-35 - 112u + 96u^2)\omega_0^2 + (35 + 56u + 368u^2)\omega^2) = 0, \end{aligned} \tag{4.7}$$

$$\begin{aligned} &\varepsilon A_0^2(25311 + 18304u - 89856u^2 + 153600u^3 - 100352u^4) \\ &-286((-147 - 240u + 352u^2)\omega_0^2 + 3(49 - 152u + 208u^2)\omega^2) = 0 \end{aligned} \tag{4.8}$$

The higher order terms of u more than second order terms have no effect on the value of the unknowns u and ω. So, we may ignore more than second order terms of u in equations (4.7)-(4.8); but half of the second order terms are considered. This is called truncation principle (see details in [29]).

Moreover, it is clearly being seen that solution equation (4.6) gives much better results and it saves a lot of calculation. Using the truncation principle, (4.7)-(4.8) take the following form

$$\begin{aligned} \varepsilon A_0^2(-21021 - 54912u + 54912u^2) + 858((-35 - 112u + 48u^2)\omega_0^2 \\ +(35 + 56u + 184u^2)\omega^2) = 0, \end{aligned} \tag{4.9}$$

$$\begin{aligned} \varepsilon A_0^2(25311 + 18304u - 44928u^2) - 286((-147 - 240u + 176u^2)\omega_0^2 \\ +3(49 - 152u + 104u^2)\omega^2) = 0 \end{aligned} \tag{4.10}$$

From (4.9), it can easily written as

$$\omega^2 = \frac{\varepsilon A_0^2(21021 + 54912u - 54912u^2) + 858(35 + 112u - 48u^2)\omega_0^2}{858(35 + 56u + 184u^2)} \tag{4.11}$$

Eliminating ω^2 from equation (4.10) with the help of equation (4.11), the nonlinear algebraic equation of u is as follows

$$\begin{aligned} f(u) : \varepsilon A_0^2(693 - 12320u - 62312u^2 + 63488u^3 + 12288u^4) \\ +176u(-126 - 637u + 176u^2 + 136u^3)\omega_0^2 = 0 \end{aligned} \tag{4.12}$$

Now, applying the iterative homotopy perturbation method (**See Appendix A**) to obtain the value of u from equation (4.12) is

$$u = u_0 + u_1 + u_2 + u_3 + \cdots , \tag{4.13}$$

where u_0 is an initial approximation and the unknowns u_1, u_2, u_3, ... are

$$u_1 = -\frac{f(u_0)}{f'(u_0)}, \tag{4.14}$$

$$u_2 = -\frac{f''(u_0)}{f'(u_0)}\left(\frac{f(u_0)}{f'(u_0)}\right)^2, \tag{4.15}$$

$$u_3 = \frac{1}{f'(u_0)}\left(\frac{1}{6}\left(\frac{f(u_0)}{f'(u_0)}\right)^3\right) f'''(u_0) + \frac{f(u_0)}{f'(u_0)}\left(-\frac{f''(u_0)}{f'(u_0)}\left(\frac{f(u_0)}{f'(u_0)}\right)^2\right), \tag{4.16}$$

and so on.

Substituting the value of u from equation (4.13) into equation (4.11), the second order approximate natural frequency is determined the following

$$\omega(A_0) = \sqrt{\frac{\varepsilon A_0^2(21021 + 54912u - 54912u^2) + 858(35 + 112u - 48u^2)\omega_0^2}{858(35 + 56u + 184u^2)}} \tag{4.17}$$

The third-order approximate solution is in the form as

$$x = A_0\cos(\omega t) + A_0 u(\cos(3\omega t) - \cos(\omega t)) + A_0 v(\cos(5\omega t) - \cos(\omega t)) \tag{4.18}$$

which is easily apply for obtaining third-order approximate solutions in the presented method.

5. Results and discussion

Comparison the first- and second-order approximation solutions of equation (3.2) for $\omega_0 = \varepsilon = 1$ and initial amplitude $A_0 = 10$ corresponding with exact solutions have been shown in Figure 1.

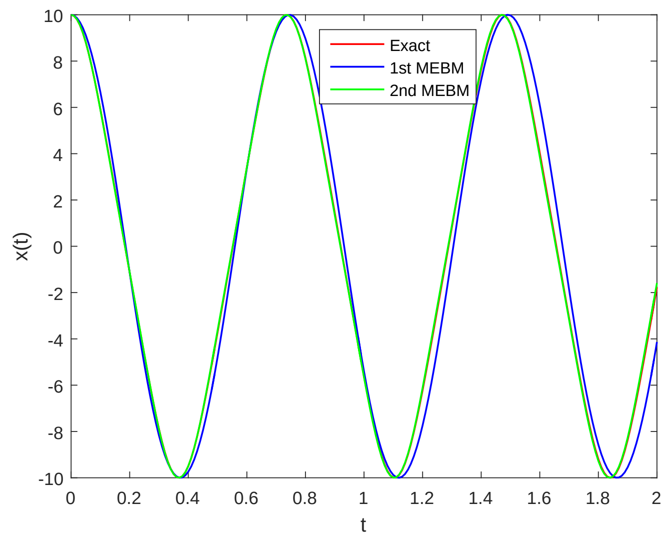

Figure 1: Time history of dynamic response, equation (3.2) for $\omega_0 = \varepsilon = 1$ and $A_0 = 10$.

Using different values of A_0, the approximate natural frequencies has been compared with corresponding exact and previously existing frequencies are listed in Table 1.

Table 1: Comparison the obtained natural frequencies and previously existing results with corresponding exact frequency of equation (3.2) for $\omega_0 = \varepsilon = 1$:

εA_0^2	ω_{ex}	$\omega_{1stEBM}^{[36]}$ $Er(\%)$	$\omega_{2ndEBM}^{[36]}$ $Er(\%)$	$\omega_{1stEBM}^{[39]}$ $Er(\%)$	$\omega_{2ndEBM}^{[39]}$ $Er(\%)$	$\omega_{1stMEBM}$ $Er(\%)$	$\omega_{2ndMEBM}$ $Er(\%)$
0.5	1.1708	1.1726 *0.1537*	1.1702 *0.0512*	1.1619 *0.7601*	1.1702 *0.0512*	1.1619 *0.7601*	1.1708 *0.0000*
1	1.3178	1.3229 *0.3870*	1.3161 *0.1290*	1.3038 *1.0623*	1.3163 *0.1138*	1.3038 *1.0623*	1.3180 *0.0151*
5	2.1504	2.1795 *1.3532*	2.1406 *0.4557*	2.1213 *1.3532*	2.1426 *0.3627*	2.1213 *1.3532*	2.1518 *0.0651*
10	2.8666	2.9155 *1.7058*	2.8500 *0.5790*	2.8284 *1.3325*	2.8535 *0.4569*	2.8284 *1.3325*	2.8690 *0.0837*
100	8.5336	8.7178 *2.1585*	8.4700 *0.7452*	8.4261 *1.2597*	8.4842 *0.5788*	8.4261 *1.2597*	8.5425 *0.1042*
1000	26.8107	27.4044 *2.2144*	26.6055 *0.7653*	26.4764 *1.2468*	26.6519 *0.5923*	26.4764 *1.2468*	26.8394 *0.1070*
5000	59.9157	61.2454 *2.2192*	59.4559 *0.7674*	59.1692 *1.2459*	59.5599 *0.5938*	59.1692 *1.2459*	59.9799 *0.1071*

Note: In Table 1, $\omega_{1stEBM}^{[36]}$, $\omega_{2ndEBM}^{[36]}$, $\omega_{1stEBM}^{[39]}$ and $\omega_{2ndEBM}^{[39]}$ represent first- and second-order approximate natural frequencies previously obtained in [36, 39]. $\omega_{1stMEBM}$ and $\omega_{2ndMEBM}$ denote first- and second-order approximate natural frequencies obtained in present study. ω_{ex} represents the exact frequency which is stated in [36]. $Er(\%)$ denotes the percentage error which has been calculated by the relation $|\frac{\omega_{MEBM}(A_0)-\omega_{ex}(A_0)}{\omega_{ex}(A_0)}| \times 100$.

In regard to the above Figure 1, it can clearly be seen that the presented method is a better applicable and reliable for solving strongly nonlinear oscillators with high precision like the presented problem in Section 3. Seeing in Table 1, comparing the relative errors with previously existing different methods, it is observed that the accuracy of the presented method is much better in the whole range of initial amplitude A_0. It is highly remarkable that the approximate solutions (second-order approximation) obtained by the presented technique is very close to the exact solutions and better than those obtained previously by several authors. The advantages of this method include its analytical simplicity and computational efficiency, and the ability to objectively find better results for many other oscillatory problems arising in nonlinear sciences and engineering.

6. Conclusion

An analytical approximation technique based on the EBM has been presented to determine approximate solutions of the Duffing oscillator. In this problem, the approximate solutions show much better agreement compared with the corresponding exact solutions and previously existing results. High accuracy of the approximate solutions reveals the versatility of the presented technique in solving highly nonlinear problems. Analytical simplicity, computational efficiency and the ability to objectively find better results are the advantages of this method. It can be concluded that the presented technique is a better and efficient alternative than the existing ones for approximating solutions for highly nonlinear oscillatory problems.

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7. APPENDIX A

A higher-order nonlinear algebraic equation is of the form

$$f(x) = 0 \tag{7.1}$$

Consider the nonlinear algebraic equation equation (7.1), and we construct a homotopy $H : R \times [0, 1] \to R$ which satisfies

$$H(x,p) = f(x) - f(x_0) + pf(x_0) = 0, \quad x \in R, \quad p \in [0,1] \tag{7.2}$$

where p is embedding parameter, x_0 is an initial approximation of equation (7.1). Hence, it is obvious that

$$H(x,0) = f(x) - f(x_0) = 0 \tag{7.3}$$

$$H(x,1) = f(x) = 0 \tag{7.4}$$

and the changing process of p from 0 to 1, refers to $H(x,p)$ from $H(x,0)$ to $H(x,1)$. Applying the perturbation technique (**See details in [22]**), due to the fact that $0 \leq p \leq 1$ can be considered as a small parameter, we can assume that the solution of equation (7.2) can be express as a series in p

$$x = x_0 + x_1 p + x_2 p^2 + x_3 p^3 + \cdots \tag{7.5}$$

Where $p \to 1$, equation (7.2) corresponds to equation (7.1) and equation (7.5) becomes the approximate solution of equation (7.1), that is [22].

$$x = \lim_{p \to 1} = x_0 + x_1 + x_2 + x_3 + \cdots \tag{7.6}$$

and in [22] the unknowns are

$$x_1 = -\frac{f(x_0)}{f'(x_0)}, \tag{7.7}$$

$$x_2 = -\frac{f''(x_0)}{f'(x_0)} \left(\frac{f(x_0)}{f'(x_0)} \right)^2, \tag{7.8}$$

$$x_3 = \frac{1}{f'(x_0)} \left(\frac{1}{6} \left(\frac{f(x_0)}{f'(x_0)} \right)^3 \right) f'''(x_0) + \frac{f(x_0)}{f'(x_0)} \left(-\frac{f''(x_0)}{f'(x_0)} \left(\frac{f(x_0)}{f'(x_0)} \right)^2 \right), \tag{7.9}$$

and so on.

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