EXISTENCE AND UNIQUENESS SOLUTION OF A BOUNDARY VALUE PROBLEMS FOR INTEGRO-DIFFERENTIAL EQUATION WITH PARAMETER

R.N. Butris

Department of Mathematics
Collage of Basic Education
University of Duhok
Duhok, Kurdistan Region
Iraq
e-mail: raad.khlka@yahoo.com

A.Sh. Rafeeq

Department of Mathematics
Faculty of Science
University of Zakho
Duhok, Kurdistan Region
Iraq
e-mail: ava77h@yahoo.com

Abstract. In this paper, we investigate the existence and uniqueness of the solution to a boundary value problem for integro-differential equation with parameter by using Schauder's fixed point theorem.

Keywords: existence and uniqueness, boundary value problem, integro-differential equation with parameter, Schauder's fixed point theorem.

1. Introduction

For many years, the problems with parameter have been studied and some of them considered as mathematical models of physical systems. Existence of solutions have conditions that are important in analysis theorems and results are often obtained by using fixed-point theorems (Banach, Schauder), by successive approximations, or lower and upper solutions of constructing monotone iterations [6].

The following initial value problem were considered in [12]

$$\dot{x}(t) = \frac{\alpha\tau}{T} + f(t, x(t-\tau), \tau), 0 \leq t \leq T, \underline{\tau} \leq \tau \leq \overline{\tau}, \tag{1.1}$$

with $x(t) = \varphi(t, \tau), (\underline{\tau} > 0), -\overline{\tau} \leq t \leq 0, \underline{\tau} \leq \tau \leq \overline{\tau}$

This boundary value problem used to find those numbers τ in $[\underline{\tau}, \overline{\tau}]$ for which problem (1.1) has a solution which satisfies the condition

$$x(T) = x_T. \tag{1.2}$$

In view of Seidov [13], there are some control parameter in which a certain physical process the initial state, speed and phase coordinate of each point of a controlled object depends on τ value, so we must choice a value of τ such that the object assume a given state at a given time. For related result on the problems with parameters, the reader is referred to [3].

Interesting application of the method also connected with the study of differential equations unresolved with respect to the highest derivative. Furthermore, it suggested that these investigations stimulated the appearance of series of works on the study of such equations see [9]. In [15], the T-periodic boundary problem for the integro-differential equation

$$\frac{dx}{dt} = f(t, x(t), \frac{dx}{dt}, \int_0^t \varphi(t, s, \frac{dx}{ds})ds \tag{1.3}$$

was considered and for some other related concepts and results one can see [1], [10], [14].

In this paper, we consider the initial value problem

$$\frac{dx}{dt} = A\lambda + f\left(t, x(t), \lambda, \sum_{i=1}^{\infty}\left(\int_0^t G(t,s)\dot{x}(s)\ ds\right)^i\right) \tag{1.4}$$

with $x(0) = x_0$, where $\dot{x}(t) = \frac{dx(t)}{dt}$, $0 \le t \le T$, A is $(n \times n)$ matrix and λ is a vector of parameters such that $||\lambda|| \le \rho, (\rho > 0)$. We will consider sufficient conditions for solvability of the problem (1.4) and for convergence of certain iterative processes to a solution, and also we will use Schauder-Tychonoff fixed point theorem to discuss the existence of the solution of the problem (1.4) and (1.2).

2. Preliminaries

Throughout this work we suppose that $f(t, x, \lambda, y)$ is continuous in the domain $[0, T] \times D \times ||\lambda|| \le \rho \times D_1$, where $y = \sum_{i=1}^{\infty}(\int_0^t G(t,s)\dot{x}(s)\ ds)^i$, $D : ||x - x_0|| \le r$, $D_1 : ||y|| \le d$ and D and D_1 are closed bounded subset in Euclidean space R^n. Also assume that $f(t, x, \lambda, y)$ satisfies the inequalities:

$$||f(t, x, \lambda, y)|| \le M \tag{2.1}$$

$$||f(t, x_1, \lambda_1, y_1) - f(t, x_2, \lambda_2, y_2)|| \le K_1||x_1 - x_2|| + K_2||\lambda_1 - \lambda_2|| + K_3||y_1 - y_2|| \tag{2.2}$$

where $y_n = \sum_{i=1}^{\infty}(\int_0^t G(t,s)\dot{x}_n(s)\ ds)^i$, $n = 1, 2, forall\ t \in [0, T]$ and M,K_1,K_2,K_3 are positive constants. Also, $G(t, s)$ is $(n \times n)$ continuous positive matrix such that

$$\int_0^t ||G(t, s)||\ ds \le K,\ (K > 0),\quad S_1 = \sum_{i=1}^{\infty} K^i M^{i-1} < d,$$

$$S_2 = \sum_{i=1}^{\infty}\sum_{j=1}^{i} iK^j M^{j-1} < \infty \tag{2.3}$$

for all $t \in [0, T]$. We define the nonempty sets as follows:

$$D_f = D - r,\ \ D_{1f} = D_1 - r_1, \tag{2.4}$$

where $r = ||A||\rho T + MT$ and $r_1 = ||A||\rho + ||M||$ and suppose that the greatest eigenvalue q_{max} of the matrix

$$Q = \begin{bmatrix} K_1 T & (||A|| + K_2)T & K_3 T S_2 \\ \frac{1}{||A||}K_1 & \frac{1}{||A||}(||A|| + K_2) & \frac{1}{||A||}K_3 S_2 \\ K_1 & ||A|| + K_2 & K_3 S_2 \end{bmatrix}$$

does not exceed unity, that is,

$$q_{max}(Q) = \frac{(a+b+c) + \sqrt{(a+b+c)^2 - 4(ab+ac+bc)}}{2} < 1 \tag{2.5}$$

where $a = K_1 T, b = \frac{1}{||A||}(||A|| + K_2), c = K_3 S_2$.

Furthermore, we prove the existence of a solution $x(t)$ on $[0, T]$ by using Schauder-Tychonoff fixed point theorem for the equations (1.4) and (1.2), where $t \in [0, T]$ and $f(t, x, \lambda, y)$ satisfy the following hypotheses:

(i) It is continuous positive real valued function on $[0, T] \times D \times ||\lambda|| \leq \rho \times D_1$.

(ii) It is a non increasing in $x(t)$ for each fixed point $t \in [0, T]$.

We define $B = C^1[0, T]$ is a space of all continuous and continuous derivative, bounded functions $x(t)$ on $[0, T]$ with the norm $||x|| = max_{t\in[0,T]}|x(t)|$, also define a closed and convex subset X of B as

$$X = \{x \in B; ||x(t) - x_0|| \leq r, t \in [0, T]\}$$

Definition 1 A pair (x^*, λ^*) is called a solution of the problem (1.4) and (1.2), if the function $x^*(t) \in D$ is defined for $t \in [0, T]$ satisfies the system (1.4) and conditions (1.2) for $||\lambda^*|| < \rho$.

Definition 2 [11] A function ϕ defined on an open interval (a, b) is said to be convex if, for each $x, y \in (a, b)$ and each μ, $0 \leq \mu \leq 1$, we have

$$\phi(\mu x + (1-\mu)y \leq \mu\phi(x) + (1-\mu)\phi(y).$$

Definition 3 [11] A sequence $\{f_n\}$ in a normed linear space is said to converge to an element f in the space if given $\epsilon > 0$, there is an N_0 such that for all $n \geq N_0$, we have $|f_n(x) - f(x)| < \epsilon$.

Definition 4 [5] Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on an interval Ω, we say that $\{f_n\}_{n=1}^{\infty}$ is equicontinuous on Ω, if for any positive number ϵ there exists a positive number δ_ϵ (depends on ϵ only) such that $|f_n(x_1) - f_n(x_2)| < \epsilon$, $n \geq 1$ whenever $|x_1 - x_2| < \delta_\epsilon$ for all $x_1, x_2 \in \Omega$.

Definition 5 [4] If T maps V into itself and x_0 is an element of V such that $T(x_0) = x_0$, the we say that x_0 is a fixed point of T.

Definition 6 [5] The set Ω is said to be compact if every open cover of Ω has a finite subcover.

Definition 7 [5] The closure of Ω is the intersection of all close sets which contain Ω.

Theorem 1 [4] (The Arzela Ascoli Theorem) *Let F be equicontinuous, uniformly bounded family of real valued function f on the interval $\Omega = [0, T]$. Then F contains a uniformly convergent sequence function f_n, converging to a function $f \in C(\Omega)$ where $C(\Omega)$ denotes the space of all continuous bounded functions on Ω. Thus any sequence in F contains a uniformly bounded convergent subsequence on Ω and consequently F has a compact closure in $C(\Omega)$.*

Theorem 2 [2] (The Mean Value Theorem) *Let a and b be real numbers such that $a < b$. If $f : [a, b] \to R$ is continuous on $[a, b]$, and f is differentiable at each point in (a, b), then there is a number c in (a, b) such that $f(b) - f(a) = f'(c)(b - a)$.*

Theorem 3 [7] (Schauder-Tyconoff Fixed Point Theorem) *Let B be a locally convex topological vector space. Let Y be a compact, convex subset of B and T^* a continuous map of Y into itself. Then T^* has a fixed point $y \in Y$, i.e., $T^*_y = y$.*

Lemma 1 [8] *Let $v(x)$ and $y(x)$ be non negative continuous functions on I=$[0, \infty)$ for which the inequality $y(x) \leq c + \int_{x_0}^{x} v(t)y(t)dt$, for all $x \in I$, holds, where c is a non negative constant. Then, $y(x) \leq ce^{\int_{x_0}^{x} v(t)dt}$, for all $x \in I$.*

3. Main results

Firstly, we start from $x_0 \in D_f$ and $||\lambda_0|| \leq \rho$, and constructed by the following iteration for equation (1.4) and (1.2):

$$x_{n+1}(t) = x_0 + \int_0^t \left(A\lambda_n + f\left(s, x_n(s), \lambda_n, \sum_{i=1}^{\infty} \left(\int_0^s G(s,\tau)\dot{x}_n(\tau)\, d\tau \right)^i \right) \right) ds \tag{2.6}$$

$$\lambda_{n+1} = \frac{1}{||A||T} \left[x_T - x_0 - \int_0^T f\left(t, x_n(t), \lambda_n, \sum_{i=1}^{\infty} \left(\int_0^t G(t,s)\dot{x}_n(s)\, ds \right)^i \right) dt \right] \tag{2.7}$$

$$\dot{x}_{n+1}(t) = A\lambda_n + f(t, x_n(t), \lambda_n, \sum_{i=1}^{\infty} \left(\int_0^t G(t,s)\dot{x}_n(s)\, ds)^i \right) \tag{2.8}$$

for all $0 \leq t \leq T$, $||\lambda_{n+1}|| \leq \rho$, $y_n = \sum_{i=1}^{\infty} \left(\int_0^t G(t,s)\dot{x}_n(s)\, ds \right)^i$, $n = 0, 1, 2, ...$, we indeed prove the following theorems:

Theorem 4 *Suppose that the function $f(t,x,\lambda,y)$ is continuous in the domain $[0,T]\times D\times ||\lambda||\le\rho\times D_1$, and satisfy inequalities (2.1) and (2.2), finally suppose that for all $||\lambda||\le\rho$ and all continuous functions $x(t)$ on $[0,T]$ with $||x(t)-x_0||\le r$, $||\dot{x}(t)||\le r_1$, we have*

$$\frac{1}{||A||T}[||x_T-x_0||+MT]\le\rho$$

and $(||A||\rho+M)T=r$, $||A||\rho+M=r_1$, and condition (2.5).

Then the solution of problem (1.4) and (1.2) exists and the successive approximations (2.6), (2.7) and (2.8) converge to a solution rapidly enough so that

$$\begin{pmatrix} ||x_{n+1}(t)-x^*(t)|| \\ ||\lambda_{n+1}-\lambda^*|| \\ ||\dot{x}_{n+1}(t)-\dot{x}^*(t)|| \end{pmatrix}\le Q^n(E-Q)^{-1}\begin{pmatrix} r \\ 2\rho \\ r_1\end{pmatrix}, \tag{2.9}$$

where E is an $n\times n$ identity matrix.

Proof. By assumption, we have $||\lambda_0||\le\rho$, $x_0\in D_f$, and from (2.6), we have

$$\begin{aligned}
&||x_{n+1}(t)-x_n(t)|| \\
&=\left\|x_0+\int_0^t\left(A\lambda_n+f\left(s,x_n(s),\lambda_n,\sum_{i=1}^{\infty}\left(\int_0^s G(s,\tau)\dot{x}_n(\tau)\,d\tau\right)^i\right)\right)ds\right. \\
&\left.-x_0-\int_0^t\left(A\lambda_{n-1}+f\left(s,x_{n-1}(s),\lambda_{n-1},\sum_{i=1}^{\infty}\left(\int_0^s G(s,\tau)\dot{x}_{n-1}(\tau)\,d\tau\right)^i\right)\right)ds\right\| \\
&\le ||A||T||\lambda_n-\lambda_{n-1}||+TK_1||x_n(t)-x_{n-1}(t)||+TK_2||\lambda_n-\lambda_{n-1}|| \\
&+TK_3\left\|\sum_{i=1}^{\infty}\left(\int_0^t G(t,s)\dot{x}_n(s)\,ds\right)^i-\sum_{i=1}^{\infty}\left(\int_0^t G(t,s)\dot{x}_{n-1}(s\,ds)^i\right)\right\| \\
&\le TK_1||x_n(t)-x_{n-1}(t)||+(||A||+K_2)T||\lambda_n-\lambda_{n-1}|| \\
&+TK_3S_2||\dot{x}_n(t)-\dot{x}_{n-1}(t)||.
\end{aligned}$$

From the difference $\lambda_{n+1}-\lambda_n$ and $\dot{x}_{n+1}(t)-\dot{x}_n(t)$, we have

$$\begin{aligned}
||\lambda_{n+1}-\lambda_n|| &\le \frac{1}{||A||}K_1||x_n(t)-x_{n-1}(t)||+\left(\frac{1}{||A||}K_2+1\right)||\lambda_n-\lambda_{n-1}|| \\
&+\frac{1}{||A||}K_3S_2||\dot{x}_n(t)-\dot{x}_{n-1}(t)|| \\
||\dot{x}_{n+1}(t)-\dot{x}_n(t)|| &\le K_1||x_n(t)-x_{n-1}(t)||+(||A||+K_2)||\lambda_n-\lambda_{n-1}|| \\
&+K_3S_2||\dot{x}_n(t)-\dot{x}_{n-1}(t)||
\end{aligned}$$

so that

$$\begin{pmatrix} ||x_{n+1}(t)-x_n(t)|| \\ ||\lambda_{n+1}-\lambda_n|| \\ ||\dot{x}_{n+1}(t)-\dot{x}_n(t)|| \end{pmatrix}\le\begin{pmatrix} K_1T & (||A||+K_2)T & K_3TS_2 \\ \frac{1}{||A||}K_1 & \frac{1}{||A||}(||A||+K_2) & \frac{1}{||A||}K_3S_2 \\ K_1 & ||A||+K_2 & K_3S_2\end{pmatrix}\begin{pmatrix} ||x_n(t)-x_{n-1}(t)|| \\ ||\lambda_n-\lambda_{n-1}|| \\ ||\dot{x}_n(t)-\dot{x}_{n-1}(t)||\end{pmatrix}$$

If let $V_{n+1} = \begin{pmatrix} ||x_{n+1}(t) - x_n(t)|| \\ ||\lambda_{n+1} - \lambda_n|| \\ ||\dot{x}_{n+1}(t) - \dot{x}_n(t)|| \end{pmatrix}$, $n = 0, 1, 2, ...$, we have

$$V_{n+1} \leq QV_n \leq ... \leq Q^n V_1, \text{ where } V_1 = \begin{pmatrix} r \\ 2\rho \\ r_1 \end{pmatrix}. \tag{2.10}$$

From (2.10), the following inequality

$$\begin{pmatrix} ||x_{n+k}(t) - x_n(t)|| \\ ||\lambda_{n+k} - \lambda_n|| \\ ||\dot{x}_{n+k}(t) - \dot{x}_n(t)|| \end{pmatrix} \leq \sum_{i=0}^{k-1} Q^{n+i} V_1,$$

holds for all $k > 1$ and $t \in [0, T]$. But the maximum eigenvalue of the matrix Q is assumed to lie within the circle of a unit radius, i.e., $q_{max} < 1$, which implies that

$$\sum_{i=0}^{k-1} Q^{n+i} \leq Q^m \sum_{i=0}^{\infty} Q^i \leq Q^n (E - Q)^{-1}$$

and $\lim_{n\to\infty} Q^n = 0$, then the sequences $\{x_n(t)\}_{n=1}^{\infty}$, $\{\lambda_n\}_{n=1}^{\infty}$, $\{\dot{x}_n(t)\}_{n=1}^{\infty}$ converge. Then let $\lim_{n\to\infty} x_n(t) = x^*(t)$, $\lim_{n\to\infty} \lambda_n = \lambda^*$ and $\lim_{n\to\infty} \dot{x}_n(t) = \dot{x}^*(t)$. If we take the limit in (2.6) and (2.7) we obtain

$$x^*(t) = x_0 + \int_0^t \left(A\lambda^* + f\left(s, x^*(s), \lambda^*, \sum_{i=1}^{\infty} \left(\int_0^s G(s,\tau)\dot{x}^*(\tau)\, d\tau \right)^i \right) \right) ds \tag{2.11}$$

$$\lambda^* = \frac{1}{||A||T} \left(x_T - x_0 - \int_0^T f\left(t, x^*(t), \lambda^*, \sum_{i=1}^{\infty} \left(\int_0^t G(t,s)\dot{x}^*(s)\, ds \right)^i \right) dt \right). \tag{2.12}$$

Mean that the pair $(x^*(t), \lambda^*)$ is a solution of problem (1.4) and (1.2). It is easy to see that the speed of convergence of (2.6) and (2.7) is validly described by (2.5), the theorem is proved. ∎

Remark 1 By analogous arguments, we call shaw that the following iteration process converges to a solution of (1.4) and (1.2)

$$x_{n+1}(t) = x_0 + \int_0^t \left(A\lambda_n + f\left(s, x_n(s), \lambda_n, \sum_{i=1}^{\infty} \left(\int_0^s G(s,\tau)\dot{x}_n(\tau)\, d\tau \right)^i \right) \right) ds,$$

$$\lambda_{n+1} = \frac{1}{||A||T} \left(x_T - x_0 - \int_0^T f(t, x_n(t), \lambda_n, \sum_{i=1}^{\infty} \left(\int_0^t G(t,s)\dot{x}_n(s)\, ds \right)^i)dt \right),$$

for all $0 \leq t \leq T$, $x_0 \in D_f$, $||\lambda|| \leq \rho$, $n = 0, 1, 2, ...$

Secondly, the solution of the initial value problem (1.4) for each $||\lambda|| \leq \rho$ will be denoted by $x(t,\lambda)$, suppose that $||x(t) - x_0|| \leq r$, then we have $||\lambda|| \leq \rho$ and all continuous functions $x(t)$ on $[0,T]$ with $||x(t) - x_0|| \leq r$, $||\dot{x}(t)|| \leq r_1$, we have

$$x(t,\lambda) = x_0 + \int_0^t \left(A\lambda + f\left(s, x(s,\lambda), \lambda, \sum_{i=1}^{\infty} \left(\int_0^s G(s,\tau)\dot{x}(\tau,\lambda)\, d\tau \right)^i \right) \right) ds \tag{2.13}$$

with $x(0) = x_0$, and for all $0 \leq t \leq T$, $x_0 \in D_f$.

Lemma 2 *Suppose that the continuous function* $f(t,x,\lambda,y)$ *satisfies conditions* (2.1) *and* (2.2). *Then the solution of* (2.13) *satisfies the bound.*

$$||x(t,\lambda_1) - x(t,\lambda_2)|| \leq e^{K_1(1+K_3S_2L)}[T(||A|| + K_2)(1 + K_3S_2L)]||\lambda_1 - \lambda_2|| \tag{2.14}$$

where $L = \frac{1}{1-K_3S_2}$.

Proof. From (2.13), we have

$$\begin{aligned} &||x(t,\lambda_1) - x(t,\lambda_2)|| \\ &= \Bigg\| x_0 + \int_0^t \left(A\lambda_1 + f\left(s, x(s,\lambda_1), \lambda_1, \sum_{i=1}^{\infty} \left(\int_0^s G(s,\tau)\dot{x}(\tau,\lambda_1)\, d\tau \right)^i \right) \right) ds \\ &\quad - x_0 - \int_0^t \left(A\lambda_2 + f\left(s, x(s,\lambda_2), \lambda_2, \sum_{i=1}^{\infty} \left(\int_0^s G(s,\tau)\dot{x}(\tau,\lambda_2)\, d\tau \right)^i \right) \right) ds \Bigg\| \end{aligned}$$

so

$$\begin{aligned} ||x(t,\lambda_1) - x(t,\lambda_2)|| &\leq (||A|| + K_2)T||\lambda_1 - \lambda_2|| \\ &+ K_1 \int_0^t ||x(s,\lambda_1) - x(s,\lambda_2)||ds \\ &+ K_3S_2 \int_0^t ||\dot{x}(s,\lambda_1) - \dot{x}(s,\lambda_2)||ds. \end{aligned} \tag{2.15}$$

Differentiate equation (2.13), then we have

$$\begin{aligned} ||\dot{x}(t,\lambda_1) - \dot{x}(t,\lambda_2)|| &\leq (||A|| + K_2)||\lambda_1 - \lambda_2|| + K_1||x(t,\lambda_1) - x(t,\lambda_2)|| \\ &+ K_3S_2||\dot{x}(t,\lambda_1) - \dot{x}(t,\lambda_2)||, \end{aligned}$$

then

$$||\dot{x}(t,\lambda_1) - \dot{x}(t,\lambda_2)|| \leq (||A|| + K_2)L||\lambda_1 - \lambda_2|| + K_1L||x(t,\lambda_1) - x(t,\lambda_2)||ds. \tag{2.16}$$

Substituting (2.16) in (2.15), we get

$$\begin{aligned} ||x(t,\lambda_1) - x(t,\lambda_2)|| &\leq T(||A|| + K_2)(1 + K_3S_2L)||\lambda_1 - \lambda_2|| \\ &+ K_1(1 + K_3S_2L) \int_0^t ||x(s,\lambda_1) - x(s,\lambda_2)||ds. \end{aligned}$$

Therefore, from the lemma 2.10, we obtain

$$||x(t,\lambda_1) - x(t,\lambda_2)|| \leq e^{K_1(1+K_3S_2L)}[T(||A|| + K_2)(1 + K_3S_2L)]||\lambda_1 - \lambda_2||.$$

The lemma is proved. ∎

Remark 2 Solvability of problem (1.4) and (1.2) is equivalent to solvability of the equation

$$\lambda = \frac{1}{||A||T}\left(x_T - x_0 - \int_0^T f\left(t, x(t), \lambda, \sum_{i=1}^{\infty}\left(\int_{-\infty}^{t} G(t,s)\dot{x}(s)\ ds\right)^i\right) dt\right).$$

Here $x(t,\lambda)$ is the solution of (2.13) corresponding to the value $||\lambda|| \leq \rho$.

Theorem 5 *Let the function* $f(t,x,\lambda,y)$ *of problem* (1.4) *satisfy* (2.1) *and* (2.2) *and hypotheses* (i) *and* (ii) *and all continuous function* $x(t)$ *with* $||x(t) - x_0|| \leq r$, *we have*

$$\left\|\frac{1}{||A||T}\left[x_T - x_0 - \int_0^T f\left(t, x(t), \lambda, \sum_{i=1}^{\infty}\left(\int_0^t G(t,s)\dot{x}(s)\ ds\right)^i\right)dt\right]\right\| \leq \rho.$$

Then, problem (1.4) *and* (1.2) *has at least one solution.*

Proof. Let X be a subset of B, for $x(t) \in X$, we define the norm $||x|| = \max\limits_{t\in[0,T]} |x(t)|$ and the map $T^* : B \to B$ defined by

$$T^*_{x(t)} = x_0 + \int_0^t \left(A\lambda + f\left(s, x(s), \lambda, \sum_{i=1}^{\infty}\left(\int_0^s G(s,\tau)\dot{x}(\tau)\ d\tau\right)^i\right)\right) ds$$

for all $0 \leq t \leq T$, $||\lambda|| \leq \rho$ in order to apply Schauder-Tychonoff fixed point theorem, we should prove the following steps:

Step 1. T^* maps X into itself. Since $f(t,x,\lambda,y)$ is continuous on $[0,T]\times D\times||\lambda|| \leq \rho\times D_1$ then $\int_0^t f(t,x,\lambda,y)ds$ is also continuous, so $T^*_{x(t)}$ is continuous, then

$$\begin{aligned} ||x_0|| - r \leq{}& ||T_{x(t)}|| \\ ={}& \left|x_0 + \int_0^t \left(A\lambda + f\left(s, x(s), \lambda, \sum_{i=1}^{\infty}\left(\int_0^s G(s,\tau)\dot{x}(\tau)\ d\tau\right)^i\right)\right) ds\right\| . \\ \leq{}& ||x_0|| + ||A||\rho T + ||M||T = ||x_0|| + r \end{aligned}$$

Thus, $T^*_{x(t)} \in X$, that is, T^* maps X into itself.

Step 2. T^* is continuous mapping on X. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X that converges to $x(t) \in B$, but X is closed subset of B, so $x(t) \in X$ consider

$$\begin{aligned} ||T^*_{x_n(t)} - T^*_{x(t)}|| \leq{}& (||A|| + K_2)T||\lambda_n - \lambda|| + K_1T||x_n(t) - x(t)||ds \\ &+ K_3S_2T||\dot{x}_n(t) - \dot{x}(t)||ds. \end{aligned}$$

Since $f(t,x,\lambda,y)$ is continuous function and the sequences $\{x_n(t)\}_{n=1}^{\infty}$, $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\dot{x}_n(t)\}_{n=1}^{\infty}$ converge to $x(t)$, λ and $\dot{x}(t)$ respectively, meaning that

$\max\limits_{t\in[0,T]} ||x_n(t) - x_(t)|| \to 0$, $||\lambda_n - \lambda|| \to 0$ and $\max\limits_{t\in[0,T]} ||\dot{x}_n(t) - \dot{x}(t)|| \to 0$, as $n \to \infty$, then

$$\lim_{n\to\infty} ||T^*_{x_n(t)} - T^*_{x(t)}|| = 0.$$

Therefore, T^* is a continuous mapping on X.

Step 3. The closure of $T^*X = \{T^*_{x(t)}; x(t) \in X\}$ is compact.

To prove Step 3, we will prove that the family T^*X is uniformly bounded and equicontinuous, T^*X is uniformly bounded as shown in Step 1. For proving the equicontinuous, since $||x(t) - x_0|| \leq r$, and $f(t, x, \lambda, y)$ is non increasing in $x(t)$, hence we have

$$||T^{*'}_{x(t)}|| = ||A\lambda + f\left(t, x(t), \lambda, \sum_{i=1}^{\infty}\left(\int_0^t G(t,s)\dot{x}(s)\ ds\right)^i\right)|| \leq ||A||\rho + ||M|| = r_1.$$

Choose $\delta = \frac{\epsilon}{r_1}$, $\epsilon > 0$, for $t_1, t_2 \in [0, T]$, such that $t_1 < t_2$, with $|t_1 - t_2| < \delta$, then by using the mean value theorem, there exists a number $b \in (t_1, t_2)$ such that

$$||T^*_{x(t_1)} - T^*_{x(t_2)}|| = ||z'(b)|||t_1 - t_2| < r_1\delta = \epsilon.$$

This inequality proves that the family T^*X is equicontinuous, since δ is independent of t_1, t_2 and $x(t) \in X$, thus by Ascoli-Arzelà theorem, T^*X has compact closure. In view of Step 1, Step 2 and Step 3, the Schauder-Tychonoff fixed point theorem shows that T^* has at least one fixed $x(t) \in X$ that is $T^*_{x(t)} = x(t)$ for all $t \in [0, T]$ so $x(t)$ is a solution of the problem (1.4) and (1.2). ■

Theorem 6 *Suppose that the function $f(t, x, \lambda, y)$ satisfies all conditions of Theorem 3.1 and condition(2.5). Then, problem (1.4) and (1.2) has at most one solution in the region $0 \leq t \leq T$, $||\lambda|| \leq \rho$, $||x(t) - x_0|| \leq r$.*

Proof. Suppose that the exist two solutions of (1.4) and (1.2) called (u_1, λ_1) and (u_2, λ_2). Then, we have

$$\frac{du_1(t)}{dt} = A\lambda + f\left(t, u_1(t), \lambda, \sum_{i=1}^{\infty}\left(\int_0^t G(t,s)\dot{u}_1(s)\ ds\right)^i\right),\ u_1(0) = u_0,\ 0 \leq t \leq T$$

$$\frac{du_2(t)}{dt} = A\lambda + f\left(t, u_2(t), \lambda, \sum_{i=1}^{\infty}\left(\int_0^t G(t,s)\dot{u}_2(s)\ ds\right)^i\right),\ u_2(0) = u_0,\ 0 \leq t \leq T.$$

From the hypotheses, we have

$$\begin{pmatrix} ||u_1(t) - u_2(t)|| \\ ||\lambda_1 - \lambda_2|| \\ ||\dot{u}_1(t) - \dot{u}_2(t)|| \end{pmatrix} \leq \begin{pmatrix} K_1T & (||A|| + K_2)T & K_3TS_2 \\ \frac{1}{||A||}K_1 & \frac{1}{||A||}(||A|| + K_2) & \frac{1}{||A||}K_3S_2 \\ K_1 & ||A|| + K_2 & K_3S_2 \end{pmatrix}\begin{pmatrix} ||u_1(t) - u_2(t)|| \\ ||\lambda_1 - \lambda_2|| \\ ||\dot{u}_1(t) - \dot{u}_2(t)|| \end{pmatrix}.$$

Let $W = \begin{pmatrix} ||u_1(t) - u_2(t)|| \\ ||\lambda_1 - \lambda_2|| \\ ||\dot{u}_1(t) - \dot{u}_2(t)|| \end{pmatrix}$, then we have $W \leq QW \leq ... \leq Q^nW$. Since $q_{max} < 1$, we conclude that $\lambda_1 = \lambda_2$ and $u_1(t) = u_2(t)$, which means that the solution of (1.4) and (1.2) is unique. ■

References

[1] Butris, R.N., *Periodic solutions for nonlinear system of differential equations with pulse action of parameters*, International Journal of Engineering Invention, 1 (12) (2012), 75-87.

[2] Cohn, D.L., *Measure Theory*, Birkhauser, Boston, 1980.

[3] Goma, I.A., *Method of successive approximation in a two-point boundary value problem with parameter*, Ukrainskii Matematicheskii Zhurnal, 29 (6) (1977), 800-807.

[4] Goldberg, R.R., *Method of Real Analysis*, John Wiley and Sons, Inc., USA, 1976.

[5] Hoffman, K., *Analysis in Euclidean Space*, Prentice-Hall, Inc., Englewood, New Jersey, USA, 1975.

[6] Jankowski, T., *Application of the numerical-analytic method to systems of differential equations with parameter*, Ukrainian Mathematical Journal, 54 (4) (2002).

[7] Jenab, B., *Asymptotic Theory of Second Order non linear Ordinary Differential Equation,* M.Sc., Thesis, British Culombia University, Canada, 1985.

[8] Oguntuase, J.A., *On an inequality of Gronwall*, Journal of Inequalities Pure and Applied Mathematics, 2 (1) (2001).

[9] Petrishyn, W.W., Yu, Z.S., *Periodic solutions of non-linear second-order differential equations which are not solvable for the highest derivative*, J. Math. Anal. Appl., 89 (1982), 462-488.

[10] Ronto, M.I., Samoilenko, A.M., Trofimchuk, S.I., *The theory of the numerical-analytic method: Achievements and new trends of development VII*, Ukrainian Mathematical Journal, 51 (9) (1999).

[11] Roydne, H.L., *Real Analysis*, Prentice-Hall of India Private Limited, New Delhi, 110001, 2005.

[12] Seidov, Z.B., *A boundary-value problem for differential equation with a retarded argument*, Ukrainskii Matematicheskii Zhurnal, 25 (6) (1973), 830-834.

[13] Seidov, Z.B., *A Boundary-value problem with control parameters for second-order differential equations with retarded argument*, Ukrainskii Matematicheskii Zhurnal, 28 (5) (1977), 690-695.

[14] Stanek, S., *On functional boundary value problem systems of functional differential equations depending on parameters*, Arch. Math. 64 (1995), 434-443.

[15] Turbaev, B.E., *Periodic solutions to system of differential equations unsolved with respects to the derivative*, Visn, Kyiv Univ. Ser. Mat. Mekh., 27 (1985), 98-104.

Accepted: 11.07.2016