ON EXTENSIONS OF k-SUBADDITIVE LATTICE GROUP-VALUED CAPACITIES

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Abstract. We prove some theorems on extension of lattice group-valued k-subadditive monotone set functions, continuous and (s)-bounded with respect to a single regulator. Furthermore, we pose some open problems.

Key words: lattice group, capacity, k-subadditivity, continuity from above, continuity from below, (s)-boundedness, extension.

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1. Introduction

The non-additive set functions have been the object of several studies. Among the related literature, we quote for instance [1], [19], [31], [35] and their bibliographies. In [17], [18] it is dealt with the so-called *M-measures*, that are increasing set functions, continuous from above and from below and compatible with respect to finite suprema and infima, which have several applications, for example to intuitionistic fuzzy events and observables (see also [1], [33]).

Here we prove some extension results for a continuous k-subadditive lattice-group valued capacity, (s)-bounded with respect to a single regulator, defined on a ring $\mathcal{W}$, to the σ-ring $\sigma(\mathcal{W})$ generated by $\mathcal{W}$, extending earlier results proved in [21] and [31]. We first construct a continuous extension by considering unions of suitable increasing sequences and/or intersections of suitable decreasing sequences of sets, using an approach similar to that in [21], and afterwards we prove (s)-boundedness (and continuity) with respect to a single regulator of the found extension, by "approximating" an element of $\sigma(\mathcal{W})$ with a suitable set of $\mathcal{W}$, by means of a technique similar to that used in [11] in the finitely and countably additive cases. Some other results about extensions of finitely additive or modular real-valued, lattice group- or vector lattice-valued measures can be found, for instance, in [3]-[6], [8], [17], [18], [24]-[30], [32], [36], [37].

We often use the tool of (D)-*convergence* in the lattice group setting, which allows us to apply the Fremlin Lemma, by means of which it is possible to replace a sequence of regulators with a single regulator.

In the literature, the study of extensions of set functions is also related to different kinds of limit theorems. For a recent literature about these topics, see also [10]-[14], [20] and, for a comprehensive overview, see for example [15], [31] and their bibliographies. In [16], some kinds of limit theorems are proved for lattice group-valued k-subadditive capacities. Finally, we pose some open problems.

2. Preliminaries

We begin with recalling the following basic concepts on lattice groups (see also [15]).

Definitions 2.1

(a) An abelian partially ordered group $R = (R, +, \leq)$ with neutral element 0 is called a *lattice group* iff it is a lattice (that is $a \vee b$ and $a \wedge b$, the *supremum* and the *infimum* between a and b, respectively, belong to R for any a, $b \in R$) and $a + c \leq b + c$ whenever a, b, $c \in R$ and $a \leq b$.

(b) For every element x of a lattice group R, set $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$. The elements x^+ and x^- are called the *positive* and *negative part* of x, respectively. Given $x \in R$, the *absolute value* $|x|$ of x is defined by $|x| = x \vee (-x)$. It is not difficult to see that $x = x^+ - x^-$ and $|x| = x^+ + x^-$ for every $x \in R$.

(c) A nonempty set $A \subset R$ is said to be *bounded from above* (*from below*, respectively) iff there exists an element $s \in R$ with $a \leq s$ ($a \geq s$, respectively) for every $a \in A$. We say that A is *bounded* iff it is bounded both from above and from below.

(d) A lattice group R is said to be *Dedekind complete* iff every nonempty subset of R, bounded from above (from below, respectively), has supremum (infimum, respectively) in R.

(e) A Dedekind complete lattice group R is said to be *super Dedekind complete* iff for every nonempty set $A \subset R$, bounded from above, there is a finite or countable subset of A having the same supremum as A.

(f) A sequence $(\sigma_p)_p$ in R is called (O)-*sequence* iff it is decreasing and $\bigwedge_{p=1}^{\infty} \sigma_p = 0$.

(g) A bounded double sequence $(a_{t,r})_{t,r}$ in R is a (D)-*sequence* or a *regulator* iff $(a_{t,r})_r$ is an (O)-sequence for every $t \in \mathbb{N}$.

(h) A lattice group R is said to be *weakly σ-distributive* iff $\displaystyle\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \Big) = 0$, for every (D)-sequence $(a_{t,r})_{t,r}$ in R.

(i) A sequence $(x_n)_n$ in R is said to be *order convergent* (or (O)-*convergent*) to x iff there exists an (O)-sequence $(\sigma_p)_p$ in R such that for every $p \in \mathbb{N}$ there is a positive integer n_0 with $|x_n - x| \le \sigma_p$ for each $n \ge n_0$, and in this case we write $(O)\lim\limits_n x_n = x$.

(j) If $(x_n)_n$ is a bounded sequence in R, then set

$$\limsup_n x_n = \bigwedge_{s=1}^{\infty} \Big(\bigvee_{n=s}^{\infty} x_n \Big), \quad \liminf_n x_n = \bigvee_{s=1}^{\infty} \Big(\bigwedge_{n=s}^{\infty} x_n \Big).$$

Note that $(O)\lim\limits_n x_n = x$ if and only if $\limsup\limits_n x_n = \liminf\limits_n x_n = x$ (see also [15]).

(k) A sequence $(x_n)_n$ in R is (D)-*convergent* to x iff there is a (D)-sequence $(a_{t,r})_{t,r}$ in R such that, for every $\varphi \in \mathbb{N}^{\mathbb{N}}$, there is $n^* \in \mathbb{N}$, with $|x_n - x| \le \displaystyle\bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$ whenever $n \ge n^*$, and we write $(D)\lim\limits_n x_n = x$.

(l) We call *sum* of a series $\displaystyle\sum_{n=1}^{\infty} x_n$ in R the limit $(O)\lim\limits_n \displaystyle\sum_{j=1}^{n} x_j$, if it exists in R.

Remarks 2.2

(a) Observe that in every Dedekind complete lattice group R any (O)-convergent sequence is also (D)-convergent, while the converse is true if and only if R is weakly σ-distributive.

(b) Some examples of super Dedekind complete and weakly σ-distributive lattice groups are the space $\mathbb{N}^{\mathbb{N}}$ endowed with the usual componentwise order and the space $L^0(X, \mathcal{B}, \nu)$ of all ν-measurable functions defined on a set function space $(X, \mathcal{B}, \nu)$ with the identification up to ν-null sets endowed with almost everywhere convergence, where ν is a positive, countably additive and σ-finite extended real-valued set function (see also [15]).

We now recall the Fremlin Lemma, which has a fundamental importance in the setting of (D)-convergence, because it allows us to replace a sequence of regulators with a single (D)-sequence.

Lemma 2.3 (see also [23, Lemma 1C], [33, Theorem 3.2.3]) *Let R be any Dedekind complete lattice group and $(a^{(n)}_{t,r})_{t,r}$, $n \in \mathbb{N}$, be a sequence of regulators in R. Then for every $u \in R$, $u \ge 0$ there is a (D)-sequence $(a_{t,r})_{t,r}$ in R with*

$$u \wedge \Big(\sum_{n=1}^{\infty} \Big(\bigvee_{t=1}^{\infty} a^{(n)}_{t,\varphi(t+n)} \Big) \Big) \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \quad \textit{for every } \varphi \in \mathbb{N}^{\mathbb{N}}.$$

We now recall the following version of the Fremlin lemma in the setting of (O)-convergence, which allows to replace a countable family of (O)-sequences with a single (O)-sequence, and will be useful in the sequel.

Lemma 2.4 (see also [13, Lemma 2.8]) *Let R be a super Dedekind complete and weakly σ-distributive lattice group, and $\{(\sigma_p^{(n)})_p : n \in \mathbb{N}\}$ be a countable family of (O)-sequences in R, such that the set $\{(\sigma_p^{(n)})_p\colon\, n,\, p \in \mathbb{N}\}$ is bounded in R. Then there exists an (O)-sequence $(b_j)_j$, such that for every $j, n \in \mathbb{N}$ there is $p = p(j,n) \in \mathbb{N}$ with $\sigma_p^{(n)} \le b_j$.*

We now recall some fundamental properties of lattice group-valued capacities (see also [16], [19], [31]). From now on, R is a super Dedekind complete and weakly σ-distributive lattice group, G is any infinite set, $\mathcal{P}(G)$ is the family of all subsets of G, $\mathcal{W} \subset \mathcal{P}(G)$ is a ring, $\sigma(\mathcal{W})$ is the smallest sub-σ-ring of $\mathcal{P}(G)$ containing $\mathcal{W}$, and k is a fixed positive integer.

Definitions 2.5

(a) A *capacity* $m : \mathcal{W} \to R$ is a bounded increasing set function with $m(\emptyset) = 0$.

(b) We say that a capacity m is *k-subadditive* on $\mathcal{W}$ iff

(2.1) $$m(A \cup B) \le m(A) + k\, m(B) \quad \text{whenever } A, B \in \mathcal{W},\ A \cap B = \emptyset.$$

(c) When $R = \mathbb{R}$, a 1-subadditive capacity is called also a *submeasure* (see also [15], [21], [22]).

We now recall the following result.

Proposition 2.6 (see [16, Proposition 3.2]) *A capacity m is k-subadditive on $\mathcal{W}$ if and only if*

$$m\Big(\bigcup_{q=1}^{n} E_q\Big) \le m(E_1) + k \sum_{q=2}^{n} m(E_q) \tag{2.2}$$

for each $n \in \mathbb{N}$, $n \ge 2$, and whenever $E_1, E_2, \dots, E_n \in \mathcal{W}$.

Definitions 2.7

(a) Let $E \in \mathcal{W}$. We say that a capacity m is *continuous from above* (*from below*, respectively) *at* E iff

$$\bigwedge_{n=1}^{\infty} m(E_n) = (D)\lim_n m(E_n) = m(E) \tag{2.3}$$

$$\Big(\bigvee_{n=1}^{\infty} m(E_n) = (D)\lim_n m(E_n) = m(E), \text{ respectively}\Big) \tag{2.4}$$

for every decreasing (increasing, respectively) sequence $(E_n)_n$ in $\mathcal{W}$ such that $\bigcap_{n=1}^{\infty} E_n = E \in \mathcal{W}$ $\quad$ ($\bigcup_{n=1}^{\infty} E_n = E \in \mathcal{W}$, respectively).

(b) A capacity m is *continuous from above* (*from below*, respectively) *on* $\mathcal{W}$ iff it is continuous from above (from below, respectively) at every $E \in \mathcal{W}$.

(c) If in (2.3) ((2.4), respectively) it is possible to take the involved (D)-limits with respect to a single regulator, then we say that m is *globally continuous from above* (*globally continuous from below*) *at* E, respectively. Similarly as above, the concepts of *global continuity from above* and *from below on* $\mathcal{W}$ can be formulated.

Note that, when $R = \mathbb{R}$, the concepts of continuity and global continuity are equivalent (see also [15]).

(d) A capacity $m : \mathcal{W} \to R$ is said to be (s)-*bounded on* $\mathcal{W}$ iff

$$(D) \lim_n m(C_n) = 0 \tag{2.5}$$

for every disjoint sequence $(C_n)_n$ in $\mathcal{W}$.

(e) If the (D)-limit in (2.5) can be taken with respect to a single (D)-sequence, then m is said to be *globally* (s)-*bounded on* $\mathcal{W}$.

3. The main results

We begin with giving the following

Proposition 3.1 *Let $m : \mathcal{W} \to R$ be a k-subadditive capacity, continuous from above at $\emptyset$. Then m is continuous from above and from below.*

Proof. We first prove continuity from above. Let $(A_n)_n$ be a decreasing sequence in $\mathcal{W}$, $A := \bigcap_{n=1}^{\infty} A_n$, $A \in \mathcal{W}$, and let $B_n := A_n \setminus A$. We get $B_n \in \mathcal{W}$ for each $n \in \mathbb{N}$, $\bigcap_{n=1}^{\infty} B_n = \emptyset$, and hence

$$(D) \lim_n m(B_n) = \bigwedge_{n=1}^{\infty} m(B_n) = 0.$$

Taking into account monotonicity and k-subadditivity of m (see also Proposition 2.6), we obtain

$$0 \le m(A_n) - m(A) \le k\, m(A_n \setminus A) = k\, m(B_n),$$

and so

$$0 \le \limsup_n (m(A_n) - m(A)) \le k \bigwedge_{n=1}^{\infty} m(B_n) = 0.$$

Therefore $(D) \lim_n (m(A_n) - m(A)) = 0$, namely $(D) \lim_n (m(A_n)) = m(A)$, that is

$$m(A) = (D) \lim_n m(A_n) = \bigwedge_{n=1}^{\infty} m(A_n).$$

Thus, we obtain continuity from above of m.

We now prove continuity from below. Let $(E_n)_n$ be an increasing sequence of elements of $\mathcal{W}$, $E := \bigcup_{n=1}^{\infty} E_n$, $E \in \mathcal{W}$. Let $F_n := E \setminus E_n$, $n \in \mathbb{N}$. Note that $F_n \in \mathcal{W}$ for every $n \in \mathbb{N}$ and that $\bigcap_{n=1}^{\infty} F_n = \emptyset$. Hence, by hypothesis, we get $(D)\lim_n m(F_n) = \bigwedge_{n=1}^{\infty} m(F_n) = 0$. By monotonicity and k-subadditivity of m, we have

$$0 \le m(E) - m(E_n) \le k\, m(E \setminus E_n) = k\, m(F_n),$$

and hence

$$0 \le \limsup_n (m(E) - m(E_n)) \le k \bigwedge_{n=1}^{\infty} m(F_n) = 0.$$

Thus, $(D)\lim_n (m(E) - m(E_n)) = 0$, that is $m(E) = (D)\lim_n m(E_n) = \bigvee_{n=1}^{\infty} m(E_n)$. So, we get that m is continuous from below. ∎

Set now $\mathcal{W}^+ := \{E \subset G$: there exists an (increasing) sequence $(E_n)_n$ in $\mathcal{W}$ with $E = \bigcup_{n=1}^{\infty} E_n\}$, $\mathcal{W}^- := \{E \subset G$: there is a (decreasing) sequence $(E_n)_n$ in $\mathcal{W}$ with $E = \bigcap_{n=1}^{\infty} E_n\}$, and similarly let us define $\mathcal{W}^{+-}$ and $\mathcal{W}^{-+}$. It is not difficult to see that $\mathcal{W}^+$, $\mathcal{W}^-$, $\mathcal{W}^{+-}$ and $\mathcal{W}^{-+}$ are four lattices, $\mathcal{W}^+$ and $\mathcal{W}^{-+}$ are closed under countable (increasing) unions, $\mathcal{W}^-$ and $\mathcal{W}^{+-}$ are closed under countable (decreasing) intersections. We will give extension results for globally (s)-bounded, k-subadditive and continuous R-valued capacities from a ring $\mathcal{W}$ to $\sigma(\mathcal{W})$, extending [11, Theorem 4.4], [21, Theorem 18], [26, Theorem 2.5], [31, Theorems 5.11 and 5.12]. To this aim, we proceed in several steps.

Theorem 3.2 *Let $m_0 : \mathcal{W} \to R$ be a k-subadditive capacity, globally (s)-bounded and continuous from above at $\emptyset$ on $\mathcal{W}$, and define $m_0^+ : \mathcal{W}^+ \to R$ as*

$$m_0^+\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \bigvee_{n=1}^{\infty} m_0(E_n), \tag{3.1}$$

whenever $E \in \mathcal{W}^+$ and $(E_n)_n$ is any increasing sequence in $\mathcal{W}$ with $E = \bigcup_{n=1}^{\infty} E_n$. Then $m_0^+ : \mathcal{W}^+ \to R$ is a k-subadditive capacity, continuous from above and from below on $\mathcal{W}^+$.

Proof. First of all, we prove that the set function m_0^+ in (3.1) is well-defined. Let $E \in \mathcal{W}^+$, $E = \bigcup_{n=1}^{\infty} E_n = \bigcup_{q=1}^{\infty} F_q$, where $(E_n)_n$ and $(F_q)_q$ are any two increasing sequences in $\mathcal{W}$. Since, by Proposition 3.1, m_0 is continuous from below on $\mathcal{W}$, then we get

$$\begin{aligned}\bigvee_{n=1}^{\infty} m_0(E_n) &= \bigvee_{n=1}^{\infty} m_0(E_n \cap E) = \bigvee_{n=1}^{\infty} m_0\Big(\bigcup_{q=1}^{\infty}(E_n \cap F_q)\Big) \\ &= \bigvee_{n=1}^{\infty}\Big(\bigvee_{q=1}^{\infty} m_0(E_n \cap F_q)\Big) = \bigvee_{q=1}^{\infty}\Big(\bigvee_{n=1}^{\infty} m_0(E_n \cap F_q)\Big) \\ &= \bigvee_{q=1}^{\infty} m_0\Big(\bigcup_{n=1}^{\infty}(E_n \cap F_q)\Big) = \bigvee_{q=1}^{\infty} m_0(E \cap F_q) = \bigvee_{q=1}^{\infty} m_0(F_q),\end{aligned}$$

and so m_0^+ is well-defined.

We now prove that m_0^+ is monotone. Let A, $B \in \mathcal{W}^+$, $A \subset B$, $A = \bigcup_{n=1}^{\infty} A_n$, $B = \bigcup_{n=1}^{\infty} B_n$, where $(A_n)_n$ and $(B_n)_n$ are two increasing sequences in $\mathcal{W}$. For each $n \in \mathbb{N}$, set $C_n := A_n \cap B_n$. Note that $(C_n)_n$ is an increasing sequence in $\mathcal{W}$ and $\bigcup_{n=1}^{\infty} C_n = A$. By monotonicity of m_0, for any $n \in \mathbb{N}$ we get

$$m_0(C_n) \le m_0(B_n) \le \bigvee_{n=1}^{\infty} m_0(B_n) = m_0^+(B). \tag{3.2}$$

Taking in (3.2) the supremum as n varies in $\mathbb{N}$, we obtain

$$m_0^+(A) = \bigvee_{n=1}^{\infty} m_0(C_n) \le m_0^+(B).$$

From this and arbitrariness of A and B we get monotonicity of m_0^+.

We now prove that m_0^+ is k-subadditive. To this aim, choose arbitrarily A, $B \in \mathcal{W}^+$, $A = \bigcup_{n=1}^{\infty} A_n$, $B = \bigcup_{n=1}^{\infty} B_n$, where $(A_n)_n$ and $(B_n)_n$ are two increasing sequences in $\mathcal{W}$. For every $n \in \mathbb{N}$, set $D_n := A_n \cup B_n$. Note that $(D_n)_n$ is an increasing sequence in $\mathcal{W}$ and $\bigcup_{n=1}^{\infty} D_n = A \cup B$. By monotonicity and k-subadditivity of m_0 on $\mathcal{W}$ we have

$$\begin{aligned} m_0(D_n) &\le m_0(A_n) + k\, m_0(B_n) \quad \text{for each } n \in \mathbb{N}, \\ m_0^+(A \cup B) &= \bigvee_{n=1}^{\infty} m_0(D_n) \le \bigvee_{n=1}^{\infty} m_0(A_n) + k \bigvee_{n=1}^{\infty} m_0(B_n) \\ &= m_0^+(A) + k\, m_0^+(B).\end{aligned} \tag{3.3}$$

The k-subadditivity of m_0^+ follows from (3.3) and arbitrariness of A and B.

Let $(a_{t,r})_{t,r}$ be a regulator, related to global (s)-boundedness of m_0 on $\mathcal{W}$. We claim that, if $E \in \mathcal{W}^+$ with $E = \bigcup_{n=1}^{\infty} E_n$, where $(E_n)_n$ is any increasing sequence in $\mathcal{W}$, then

$$(D) \lim_n m_0^+(E \setminus E_n) = 0 \tag{3.4}$$

with respect to $(a_{t,r})_{t,r}$. From this it will follow that for each $E \in \mathcal{W}^+$ and $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is a set $E^- \in \mathcal{W}$ with

$$E^- \subset E \text{ and } m_0^+(E \setminus E^-) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}. \tag{3.5}$$

Indeed, if $(E_n)_n$ is any increasing sequence in $\mathcal{W}$, then for each $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $\overline{n} \in \mathbb{N}$ with

$$m_0(E_{n+p} \setminus E_n) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \tag{3.6}$$

whenever $n \geq \overline{n}$ and $p \in \mathbb{N}$. Otherwise there exist an element $\varphi \in \mathbb{N}^{\mathbb{N}}$ and two sequences $(n_i)_i$, $(p_i)_i$ in $\mathbb{N}$, with $n_{i+1} > n_i + p_i$ and

$$m_0(E_{n_i+p_i} \setminus E_{n_i}) \not\leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$$

for every $i \in \mathbb{N}$, getting a contradiction with global (s)-boundedness of m_0 on $\mathcal{W}$ with respect to the (D)-sequence $(a_{t,r})_{t,r}$. Moreover note that, since

$$\bigcup_{p=1}^{\infty}(E_{n+p} \setminus E_n) = E \setminus E_n \in \mathcal{W}^+$$

for every $n \in \mathbb{N}$, then

$$m_0^+(E \setminus E_n) = \bigvee_{p=1}^{\infty} m_0(E_{n+p} \setminus E_n) \text{ for each } n. \tag{3.7}$$

Taking in (3.6) the supremum as p tends to $+\infty$ and keeping fixed n, from (3.6) and (3.7) we obtain

$$m_0^+(E \setminus E_n) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \tag{3.8}$$

for each $n \geq \overline{n}$, that is the claim.

We now prove that m_0^+ is continuous from below on $\mathcal{W}^+$. Let $(A_l)_l$ be an increasing sequence in $\mathcal{W}^+$, and set $A = \bigcup_{l=1}^{\infty} A_l$. For each $l \in \mathbb{N}$ there is an increasing sequence $(A_l^n)_n$ in $\mathcal{W}$ with $\bigcup_{n=1}^{\infty} A_l^n = A_l$. For each l, $n \in \mathbb{N}$, set $B_l^n := \bigcup_{s=1}^{l} A_s^n$. It is not difficult to see that the sequences $(B_l^n)_n$ and $(B_l^n)_l$ are in $\mathcal{W}$, are increasing and $A_l = \bigcup_{n=1}^{\infty} B_l^n$ for any $l \in \mathbb{N}$. For each $n \in \mathbb{N}$, set $C_n = B_n^n$. It is not difficult to see that $(C_n)_n$ is an increasing sequence in $\mathcal{W}$ and $A = \bigcup_{n=1}^{\infty} C_n$. By (3.4), for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $\bar{l} \in \mathbb{N}$ with

$$m_0^+(A) - m_0(C_l) \le k\, m_0^+(A \setminus C_l) \le k \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}. \tag{3.9}$$

Since $A_l \supset C_l$ and m_0^+ is monotone, we get

$$m_0^+(A) - m_0^+(A_l) \le m_0^+(A) - m_0^+(C_l). \tag{3.10}$$

From (3.9) and (3.10) it follows that $(D) \lim_l m_0^+(A_l) = \bigvee_{l=1}^{\infty} m_0^+(A_l) = m_0^+(A)$, namely (global) continuity from below of m_0^+ on $\mathcal{W}^+$ with respect to the regulator $(a_{t,r})_{t,r}$.

We now prove that m_0^+ is continuous from above at $\emptyset$ on $\mathcal{W}^+$. Let $(E_n)_n$ be any decreasing sequence in $\mathcal{W}^+$ with $\bigcap_{n=1}^{\infty} E_n = \emptyset$, and choose arbitrarily $\varphi \in \mathbb{N}^{\mathbb{N}}$. By (3.5), there is a sequence in $(F_n)_n$ in $\mathcal{W}$ with $F_n \subset E_n$ and

$$m_0^+(E_n \setminus F_n) \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)} \tag{3.11}$$

for every $n \in \mathbb{N}$. Set $H_n = \bigcap_{j=1}^{n} F_j$, $n \in \mathbb{N}$. Note that $H_n \in \mathcal{W}$ and $H_n \subset E_n$ for each $n \in \mathbb{N}$, and hence $\bigcap_{n=1}^{\infty} H_n = \emptyset$. By continuity from above at $\emptyset$ of m_0 on $\mathcal{W}$, we have

$$\bigwedge_{n=1}^{\infty} m_0(H_n) = (D) \lim_n m_0(H_n) = 0. \tag{3.12}$$

Taking into account monotonicity and k-subadditivity of m_0^+, we get:

$$\begin{aligned} m_0^+(E_n \setminus H_n) &= m_0^+\Big(\bigcup_{j=1}^{n} (E_n \setminus F_j)\Big) \le m_0^+\Big(\bigcup_{j=1}^{n} (E_j \setminus F_j)\Big) \\ &\le k \sum_{j=1}^{n} m_0^+(E_j \setminus F_j) \le k \sum_{j=1}^{\infty} \Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t+j)}\Big). \end{aligned}$$

Moreover, by construction, $m_0^+(E_n \setminus H_n) \leq u$, where

$$u = \bigvee_{A \in \mathcal{W}} m_0(A). \tag{3.13}$$

By Lemma 2.3, we find a regulator $(b_{t,r})_{t,r}$ with

$$u \wedge \Big(k^2 \sum_{n=1}^{\infty} \Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)}\Big)\Big) \leq \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} \text{ for each } \varphi \in \mathbb{N}^{\mathbb{N}}$$

and thus, thanks to k-subadditivity of m_0^+,

$$m_0^+(E_n) - m_0(H_n) \leq k\, m_0^+(E_n \setminus H_n) \leq \bigvee_{t=1}^{\infty} b_{t,\varphi(t)},$$

namely $m_0^+(E_n) \leq m_0(H_n) + \bigvee_{t=1}^{\infty} b_{t,\varphi(t)}$. Hence, taking into account (3.12),

$$\begin{aligned} 0 \ &\leq \ \limsup_n m_0^+(E_n) \leq \limsup_n m_0(H_n) + \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} \\ &= \ (D) \lim_n m_0(H_n) + \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} = \bigvee_{t=1}^{\infty} b_{t,\varphi(t)}. \end{aligned} \tag{3.14}$$

From (3.14), arbitrariness of $\varphi \in \mathbb{N}^{\mathbb{N}}$ and weak σ-distributivity of R we obtain

$$0 \leq \limsup_n m_0^+(E_n) \leq \bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \Big(\bigvee_{t=1}^{\infty} b_{t,\varphi(t)}\Big) = 0.$$

Hence, $(D) \lim_n m_0^+(E_n) = 0$, and so we get (global) continuity from above at $\emptyset$ of m_0^+ on $\mathcal{W}^+$ with respect to the (D)-sequence $(b_{t,r})_{t,r}$.

Now, we prove continuity from above of m_0^+ on $\mathcal{W}^+$ in the general case. Let $(E_n)_n$ be a decreasing sequence in $\mathcal{W}^+$, with $\bigcap_{n=1}^{\infty} E_n = E \in \mathcal{W}^+$. There is an increasing sequence $(V_n)_n$ in $\mathcal{W}$, with $E := \bigcup_{n=1}^{\infty} V_n \in \mathcal{W}^+$ and $(D) \lim_n m_0(V_n) = \bigvee_{n=1}^{\infty} m_0(V_n) = m_0^+(E)$, and so the sequence $(E_n \setminus V_n)_n$ is in $\mathcal{W}^+$, decreasing and $\bigcap_{n=1}^{\infty} (E_n \setminus V_n) = \emptyset$. Then, by the previous step, we get $(D) \lim_n m_0^+(E_n \setminus V_n) = 0$ with respect to the regulator $(b_{t,r})_{t,r}$. Moreover, thanks to monotonicity and k-subadditivity of m_0^+, we have

$$0 \leq m_0^+(E_n) - m_0(V_n) \leq k\, m_0^+(E_n \setminus V_n),$$

and hence $(D)\lim\limits_n (m_0^+(E_n) - m_0(V_n)) = 0$. So,

$$m_0^+(E) = (D)\lim_n m_0^+(E_n) = \bigwedge_{n=1}^{\infty} m_0^+(E_n).$$

Thus, m_0^+ is (globally) continuous from above on $\mathcal{W}^+$ with respect to the regulator $(k\, b_{t,r})_{t,r}$. ∎

Theorem 3.3 *Let $\mathcal{W}^* = \{A \subset G$: there is $D \in \mathcal{W}^+$ with $D \supset A\}$. For each $A \in \mathcal{W}^*$, set $m_0^*(A) = \bigwedge \Big\{ m_0^+(D)\colon D \supset A,\ D \in \mathcal{W}^+ \Big\}$. Then m_0^* is a k-subadditive capacity on $\mathcal{W}^*$, such that for every $A \in \mathcal{W}^*$ there exists a set $D \in \mathcal{W}^{+-}$, $D \supset A$, with $m_0^*(A) = m_0^*(D)$. Moreover, if $(A_n)_n$ is any decreasing sequence in $\mathcal{W}^+$ with $A = \bigcap\limits_{n=1}^{\infty} A_n$, then $m_0^*(A) = \bigwedge\limits_{n=1}^{\infty} m_0^+(A_n)$.*

Proof. It is not difficult to check that m_0^* is a k-subadditive capacity on $\mathcal{W}^*$. We now claim that for each $A \in \mathcal{W}^*$ there is $D \in \mathcal{W}^{+-}$, $D \supset A$, with

$$m_0^*(A) = m_0^*(D). \tag{3.15}$$

Choose arbitrarily $A \in \mathcal{W}^*$. Since R is super Dedekind complete, there is a sequence $(C_n)_n$ in $\mathcal{W}^+$, with $C_n \supset A_n$ for every $n \in \mathbb{N}$ and $m_0^*(A) = \bigwedge\limits_{n=1}^{\infty} m_0^+(C_n)$. For each $n \in \mathbb{N}$, set

$$D_n = \bigcap_{i=1}^{n} C_i. \tag{3.16}$$

Then $D_n \in \mathcal{W}^+$, $D_n \supset A$ and the sequence $(D_n)_n$ is decreasing.

Put $D = \bigcap\limits_{n=1}^{\infty} D_n = \bigcap\limits_{n=1}^{\infty} C_n$. We get $D \supset A$. By monotonicity of m_0^*, we have

$$\begin{aligned} m_0^*(A) \le m_0^*(D) \ &\le\ \bigwedge_{n=1}^{\infty} m_0^+(D_n) = (D)\lim_n m_0^+(D_n) \\ &\le\ \bigwedge_{n=1}^{\infty} m_0^+(C_n) = m_0^*(A). \end{aligned} \tag{3.17}$$

Thus, all inequalities in (3.17) are equalities, and so we get (3.15). Furthermore, note that, by (3.17), we find a (D)-sequence $(w_{t,r})_{t,r}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $\overline{n} \in \mathbb{N}$ with

$$m_0^+(D_n) \le m_0^+(A) + \bigvee_{t=1}^{\infty} w_{t,\varphi(t)} \tag{3.18}$$

whenever $n \geq \overline{n}$. Taking in (3.18) the set $A^+ = D_{\overline{n}}$, we get that for every $A \in \mathcal{W}^*$ and $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $A^+ \in \mathcal{W}^+$, $A^+ \supset A$, with

$$m_0^+(A^+) \leq m_0^+(A) + \bigvee_{t=1}^{\infty} w_{t,\varphi(t)}. \tag{3.19}$$

We now prove that, if $(A_n)_n$ is any decreasing sequence in $\mathcal{W}^+$ and $A = \bigcap_{n=1}^{\infty} A_n$, then

$$\bigwedge_{n=1}^{\infty} m_0^+(A_n) = (D) \lim_n m_0^+(A_n) = m_0^*(A). \tag{3.20}$$

Indeed, if $(D_q)_q$ is a decreasing sequence in $\mathcal{W}^+$ associated with A as in (3.16), then, by (3.17), monotonicity of m_0^+ and continuity from above of m_0^+ on $\mathcal{W}^+$, we get

$$\begin{aligned} \bigwedge_{n=1}^{\infty} m_0^+(A_n) &\leq \bigwedge_{n=1}^{\infty}\Big(\bigwedge_{q=1}^{\infty} m_0^+(A_n \cup D_q)\Big) = \bigwedge_{q=1}^{\infty}\Big(\bigwedge_{n=1}^{\infty} m_0^+(A_n \cup D_q)\Big) \\ &= \bigwedge_{q=1}^{\infty} m_0^+(D_q) = m_0^*(A) \leq \bigwedge_{n=1}^{\infty} m_0^+(A_n), \end{aligned}$$

and so we obtain (3.20). This ends the proof. ■

We now prove the following result about the existence of extensions of continuous k-subadditive capacities.

Theorem 3.4 *Let $m_0 : \mathcal{W} \to R$ be a globally (s)-bounded k-subadditive capacity, continuous from above at $\emptyset$. Then there exists a (unique) k-subadditive capacity $m : \sigma(\mathcal{W}) \to R$, continuous from above and from below, with $m(A) = m_0(A)$ for every $A \in \mathcal{W}$.*

Proof. Let $\mathcal{S} := \{A \in \mathcal{W}^*$ such that there are $E \in \mathcal{W}^{+-}$, $F \in \mathcal{W}^{-+}$ with $F \subset A \subset E$ and $m_0^*(E \setminus F) = 0\}$. We begin with proving that for every $A \in \mathcal{S}$ there is a regulator $(\gamma_{t,r})_{t,r}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there are $D \in \mathcal{W}^+$ and $H \in \mathcal{W}^-$ with $H \subset A \subset D$ and

$$m_0^+(D \setminus H) \leq \bigvee_{t=1}^{\infty} \gamma_{t,\varphi(t)}. \tag{3.21}$$

Choose arbitrarily $A \in \mathcal{S}$ and, in correspondence with A, let E and F be two sets satisfying the conditions in the definition of $\mathcal{S}$. There are a decreasing sequence $(E_n)_n$ in $\mathcal{W}^+$ and an increasing sequence $(F_n)_n$ in $\mathcal{W}^-$ with $E = \bigcap_{n=1}^{\infty} E_n$, $F = \bigcup_{n=1}^{\infty} F_n$.

Note that $E_n \setminus F_n \in \mathcal{W}^+$ and $E \setminus F = \bigcap_{n=1}^{\infty}(E_n \setminus F_n)$. Then, by (3.20), we get

$$\bigwedge_{n=1}^{\infty} m_0^+(E_n \setminus F_n) = (D)\lim_n m_0^+(E_n \setminus F_n) = m_0^*(E \setminus F) = 0, \tag{3.22}$$

that is there exists a (D)-sequence $(\gamma_{t,r})_{t,r}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $\overline{n} \in \mathbb{N}$ with

$$m_0^+(E_n \setminus F_n) \le \bigvee_{t=1}^{\infty} \gamma_{t,\varphi(t)} \tag{3.23}$$

for each $n \geq \overline{n}$. Taking in (3.23) $D = E_{\overline{n}}$ and $H = F_{\overline{n}}$, we obtain (3.21).

We now prove that $\mathcal{S}$ is a σ-ring containing $\mathcal{W}$. First of all, it is readily seen that $\mathcal{W} \subset \mathcal{S}$. Now, choose arbitrarily A_1, $A_2 \in \mathcal{S}$, and let E_1, $E_2 \in \mathcal{W}^{+-}$, F_1, $F_2 \in \mathcal{W}^{-+}$ be with $F_i \subset A_i \subset E_i$ and $m_0^*(E_i \setminus F_i) = 0$, $i = 1, 2$. It is not difficult to check that $E_1 \cup E_2$, $E_1 \setminus F_2 \in \mathcal{W}^{+-}$, $F_1 \cup F_2$, $F_1 \setminus E_2 \in \mathcal{W}^{-+}$. Taking into account also monotonicity and k-subadditivity of m_0^*, we get

$$\begin{array}{rcl} & & F_1 \cup F_2 \subset A_1 \cup A_2 \subset E_1 \cup E_2, \\ & & F_1 \setminus E_2 \subset A_1 \setminus A_2 \subset E_1 \setminus F_2, \\ 0 & \le & m_0^*((E_1 \cup E_2) \setminus (F_1 \cup F_2)) \le m_0^*((E_1 \setminus F_1) \cup (E_2 \setminus F_2)) \\ & \le & m_0^*(E_1 \setminus F_1) + k\, m_0^*(E_2 \setminus F_2) = 0, \\ 0 & \le & m_0^*((E_1 \setminus F_2) \setminus (F_1 \setminus E_2)) \le m_0^*((E_1 \setminus F_1) \cup (E_2 \setminus F_2)) \\ & \le & m_0^*(E_1 \setminus F_1) + k\, m_0^*(E_2 \setminus F_2) = 0. \end{array} \tag{3.24}$$

Thus, all inequalities in (3.24) are equalities, and hence $A_1 \cup A_2$, $A_1 \setminus A_2 \in \mathcal{S}$. Therefore, $\mathcal{S}$ is a ring. So, in order to prove that $\mathcal{S}$ is a σ-ring it will be enough to show that, if $(A_n)_n$ is an increasing sequence in $\mathcal{S}$ and $A = \bigcup_{n=1}^{\infty} A_n$, then $A \in \mathcal{S}$. To this aim, for technical reasons it is more advisable to proceed dealing with (O)-convergence rather than (D)-convergence. By (3.22), to every $n \in \mathbb{N}$ it is possible to associate two sequences $(E_{h,n})_h$ and $(F_{h,n})_h$ in $\mathcal{W}^+$ and $\mathcal{W}^-$, respectively, with $F_{h,n} \subset A_n \subset E_{h,n}$ for every h and n, and

$$\begin{array}{rcl} \bigwedge_{h=1}^{\infty} m_0^+(E_{h,n} \setminus F_{h,n}) & = & (O)\lim_h m_0^+(E_{h,n} \setminus F_{h,n}) \\ & = & (D)\lim_h m_0^+(E_{h,n} \setminus F_{h,n}) = 0. \end{array} \tag{3.25}$$

Hence, taking into account monotonicity and k-subadditivity of m_0^+, for every $n \in \mathbb{N}$ there is an (O)-sequence $(\sigma_p^{(n)})_p$ such that for every $p \in \mathbb{N}$ there exists $\overline{h} \in \mathbb{N}$ with

$$m_0^+\Big(\bigcup_{i=1}^{n}(E_{h,i} \setminus F_{h,i})\Big) \le k \sum_{i=1}^{n} m_0^+(E_{h,i} \setminus F_{h,i}) \le \sigma_p^{(n)} \tag{3.26}$$

for every $h \geq \overline{h}$. Let u be as in (3.13). Without loss of generality, we can assume that $\sigma_p^{(n)} \leq u$ for each n, $p \in \mathbb{N}$ since, by construction, $m_0^+(A) \leq u$ for every $A \in \mathcal{W}^+$. Thus, taking into account that R is super Dedekind complete and weakly σ-distributive, by Lemma 2.4 there is an (O)-sequence $(b_j)_j$ such that for every j and $n \in \mathbb{N}$ there is $p \in \mathbb{N}$ with $\sigma_p^{(n)} \leq b_j$. From this and (3.26) it follows that for every n and $j \in \mathbb{N}$ there is $h' \in \mathbb{N}$ with

$$m_0^+\Big(\bigcup_{i=1}^{n}(E_{h,i} \setminus F_{h,i})\Big) \leq k \sum_{i=1}^{n} m_0^+(E_{h,i} \setminus F_{h,i}) \leq b_j \tag{3.27}$$

for every $h \geq h'$. Passing to the supremum as n varies in $\mathbb{N}$ in (3.26), taking into account continuity from below of m_0^+, from (3.27) we obtain

$$m_0^+\Big(\bigcup_{i=1}^{\infty}(E_{h,i} \setminus F_{h,i})\Big) \leq b_j \tag{3.28}$$

whenever $h \geq h'$. Let now $E_h := \bigcup_{n=1}^{\infty} E_{h,n}$, $E := \bigcap_{h=1}^{\infty} E_h$, $F_h := \bigcap_{n=1}^{\infty} F_{h,n}$, $F := \bigcup_{h=1}^{\infty} F_h$. It is not difficult to check that $E_h \in \mathcal{W}^+$, $F_h \in \mathcal{W}^-$ for all $h \in \mathbb{N}$, $E \in \mathcal{W}^{+-}$, $F \in \mathcal{W}^{-+}$, $F \subset A \subset E$,

$$E \setminus F \subset E_h \setminus F_h \subset \bigcup_{n=1}^{\infty}(E_{h,n} \setminus F_{h,n}) \text{ for every } h \in \mathbb{N}. \tag{3.29}$$

From (3.27), (3.28), (3.29), positivity and monotonicity of m_0^* we obtain

$$0 \leq m_0^*(E \setminus F) \leq m_0^*(E_h \setminus F_h) \leq m_0^+\Big(\bigcup_{n=1}^{\infty}(E_{h,n} \setminus F_{h,n})\Big) \leq b_j. \tag{3.30}$$

By arbitrariness of j, we get $m_0^*(E \setminus F) = 0$. Thus, E and F are the required sets associated with A and satisfying the conditions in the definition of $\mathcal{S}$. Thus, $A \in \mathcal{S}$, and so we deduce that $\mathcal{S}$ is a σ-ring. Since $\mathcal{S} \supset \mathcal{W}$, then $\mathcal{S} \supset \sigma(\mathcal{W})$.

Now we prove that m_0^* is continuous from above at $\emptyset$ on $\mathcal{S}$. Let $(A_n)_n$ be a decreasing sequence in $\mathcal{S}$, with $\bigcap_{n=1}^{\infty} A_n = \emptyset$. By (3.21), taking into account monotonicity and k-subadditivity of m_0^*, for every $n \in \mathbb{N}$ there exists a (D)-sequence $(v_{t,r})_{t,r}$ such that for each $\varphi \in \mathbb{N}^{\mathbb{N}}$, in correspondence with A_n, there is $C_n \in \mathcal{W}^+$ with $C_n \supset A_n$ and

$$m_0^*\Big(\bigcup_{i=1}^{n}(C_i \setminus A_i)\Big) \leq k \sum_{i=1}^{n} m_0^*(C_i \setminus A_i) \leq \bigvee_{t=1}^{\infty} v_{t,\varphi(t+n)}^{(n)}. \tag{3.31}$$

For each $n \in \mathbb{N}$, set $D_n := \bigcap_{i=1}^{n} C_i$. Then $(D_n)_n$ is a decreasing sequence in $\mathcal{W}^+$ and $D_n \supset A_n$ for each $n \in \mathbb{N}$.

Indeed, by induction, if $D_n \supset A_n$, then $D_{n+1} = D_n \cap C_{n+1} \supset A_n \cap A_{n+1} = A_{n+1}$, since $(A_n)_n$ is decreasing. By monotonicity of m_0^*, from (3.31) it follows that

$$m_0^*\Big(\bigcup_{i=1}^{n}(D_i \setminus A_i)\Big) \leq \bigvee_{t=1}^{\infty} v^{(n)}_{t,\varphi(t+n)}. \tag{3.32}$$

By (3.19), in correspondence with $D_n \setminus A_n$ there are a regulator $(w^{(n)}_{t,r})_{t,r}$ and a set $I_n \in \mathcal{W}^+$, $I_n \supset D_n \setminus A_n$, with

$$k\sum_{i=1}^{n} m_0^+(I_i) \leq k\sum_{i=1}^{n} m_0^+(D_i \setminus A_i) + \bigvee_{t=1}^{\infty} w^{(n)}_{t,\varphi(t+n)},$$

and hence, taking into account monotonicity and k-subadditivity of m_0^+,

$$m_0^+\Big(\bigcup_{i=1}^{n} I_i\Big) \leq k\sum_{i=1}^{n} m_0^+(I_i) \leq \bigvee_{t=1}^{\infty} v^{(n)}_{t,\varphi(t+n)} + \bigvee_{t=1}^{\infty} w^{(n)}_{t,\varphi(t+n)}. \tag{3.33}$$

Let $x_{t,r} := 2(v_{t,r} + w_{t,r})$, t, $r \in \mathbb{N}$, and u be as in (3.13). By virtue of Lemma 2.3, we find a (D)-sequence $(\tau_{t,r})_{t,r}$ with

$$u \wedge \Big(\sum_{n=1}^{\infty}\Big(\bigvee_{t=1}^{\infty} x^{(n)}_{t,\varphi(t+n)}\Big)\Big) \leq \bigvee_{t=1}^{\infty} \tau_{t,\varphi(t)} \text{ for every } \varphi \in \mathbb{N}^{\mathbb{N}}.$$

Since, by construction, $m_0^+\Big(\bigcup_{i=1}^{n} I_i\Big) \leq u$, then $m_0^+\Big(\bigcup_{i=1}^{n} I_i\Big) \leq \bigvee_{t=1}^{\infty} \tau_{t,\varphi(t)}$ for each $n \in \mathbb{N}$.

If $I = \bigcup_{n=1}^{\infty} I_n$, then $I \in \mathcal{W}^+$, $I \supset \bigcup_{n=1}^{\infty}(D_n \setminus A_n)$ and, by continuity from below of m_0^+, we get

$$m_0^+(I) = \bigvee_{n=1}^{\infty} m_0^+\Big(\bigcup_{i=1}^{n} I_i\Big) \leq \bigvee_{t=1}^{\infty} \tau_{t,\varphi(t)}. \tag{3.34}$$

As $\bigcap_{n=1}^{\infty} A_n = \emptyset$, then

$$\bigcap_{n=1}^{\infty} D_n = \Big(\bigcap_{n=1}^{\infty} D_n\Big) \setminus \Big(\bigcap_{n=1}^{\infty} A_n\Big) \subset \bigcup_{n=1}^{\infty}(D_n \setminus A_n) \subset I$$

and so, from (3.20), (3.34) and monotonicity of m_0^* we obtain

$$0 \leq \bigwedge_{n=1}^{\infty} m_0^*(A_n) \leq \bigwedge_{n=1}^{\infty} m_0^+(D_n) = m_0^*\Big(\bigcap_{n=1}^{\infty} D_n\Big) \leq m_0^+(I) \leq \bigvee_{t=1}^{\infty} \tau_{t,\varphi(t)}. \tag{3.35}$$

By arbitrariness of $\varphi \in \mathbb{N}^{\mathbb{N}}$ and weak σ-distributivity of R, we obtain

$$0 \leq \bigwedge_{n=1}^{\infty} m_0^*(A_n) \leq \bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \Big(\bigvee_{t=1}^{\infty} \tau_{t,\varphi(t)} \Big) = 0,$$

that is $\bigwedge_{n=1}^{\infty} m_0^*(A_n) = 0$. So we get that m_0^* is continuous from above at $\emptyset$ on $\mathcal{S}$. Continuity from above and from below of m_0^* on $\mathcal{S}$ follows from continuity from above at $\emptyset$ of m_0^* on $\mathcal{S}$ and Proposition 3.1. Thus, the restriction $m : \sigma(\mathcal{W}) \to R$ of m_0^* satisfies the thesis of the theorem. ∎

Now, we are in position to prove our main result on extensions of lattice group-valued k-subadditive capacities.

Theorem 3.5 *Let $\mathcal{W}$, $\sigma(\mathcal{W})$ and m be as in Theorem 3.4. Then m is globally (s)-bounded, and there exists a regulator $(c_{t,r})_{t,r}$ such that for each $A \in \sigma(\mathcal{W})$ and $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $F \in \mathcal{W}$ with*

$$m(A \triangle F) \leq \bigvee_{t=1}^{\infty} c_{t,\varphi(t)}. \tag{3.36}$$

Proof. Let $m_0 : \mathcal{W} \to R$ and $m : \sigma(\mathcal{W}) \to R$ be as in Theorem 3.4, and $(a_{t,r})_{t,r}$ be a regulator, related with global (s)-boundedness of m_0 on $\mathcal{W}$. We begin with proving (3.36). Let $u = \bigvee_{A \in \sigma(\mathcal{W})} m(A)$. By Lemma 2.3, there is a regulator $(A_{t,r})_{t,r}$ with

$$u \wedge \Big(\sum_{n=1}^{\infty} \Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)} \Big) \Big) \leq \bigvee_{t=1}^{\infty} A_{t,\varphi(t)}.$$

Set $b_{t,r} = 2\,(a_{t,r} + A_{t,r})$ and $c_{t,r} = 2\,k\,b_{t,r}$, $t, r \in \mathbb{N}$. We prove that $(c_{t,r})_{t,r}$ satisfies (3.36). To this aim, we first recall that by (3.4), if $H \in \mathcal{W}^+$, $H = \bigcup_{n=1}^{\infty} H_n$, where $(H_n)_n$ is an increasing sequence in $\mathcal{W}$, then for each $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $\overline{n} \in \mathbb{N}$ with

$$m(H \setminus H_n) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} \quad \text{whenever } n \geq \overline{n}. \tag{3.37}$$

Choose arbitrarily an element $B \in \mathcal{W}^{+-}$ and let $(V_n)_n$ be any decreasing sequence in $\mathcal{W}^+$ with $B = \bigcap_{n=1}^{\infty} V_n$. Pick any element $\varphi \in \mathbb{N}^{\mathbb{N}}$. By (3.37), for each $n \in \mathbb{N}$ there exists $E_n \in \mathcal{W}$ such that $E_n \subset V_n$ and

$$m(V_n \setminus E_n) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t+n)}.$$

Set now $F_n := \bigcap_{i=1}^{n} E_i$, $n \in \mathbb{N}$. Then $(F_n)_n$ is a decreasing sequence in $\mathcal{W}$.

Proceeding analogously as in (3.6), we find a positive integer n_0 with $m(F_n \setminus F_{n+p}) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}$ for every $n \geq n_0$ and $p \in \mathbb{N}$. Since m is k-subadditive, we get

$$m(V_n \setminus B) \leq m(V_n \setminus V_{n+p}) + k\, m(V_{n+p} \setminus B)$$

for every n, $p \in \mathbb{N}$. It is possible to check that

$$\begin{aligned} V_n \setminus F_n &\subset (V_1 \setminus E_1) \cup \ldots \cup (V_n \setminus E_n), \\ (V_n \setminus V_{n+p}) \triangle (F_n \setminus F_{n+p}) &\subset (V_1 \setminus E_1) \cup \ldots \cup (V_{n+p} \setminus E_{n+p}) \end{aligned}$$

for every n, $p \in \mathbb{N}$, and hence

$$m(V_n \setminus V_{n+p}) \leq m(F_n \setminus F_{n+p}) + u \wedge \Big(k \sum_{h=1}^{n+p} m(V_h \setminus E_h)\Big) \leq k \Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t)} + \bigvee_{t=1}^{\infty} A_{t,\varphi(t)}\Big),$$

for each $n \geq n_0$ and $p \in \mathbb{N}$. Thus, we get

$$m(V_n \setminus B) \leq k \Big(\bigvee_{t=1}^{\infty} a_{t,\varphi(t)} + \bigvee_{t=1}^{\infty} A_{t,\varphi(t)}\Big) + k\, m(V_{n+p} \setminus B)$$

for every $n \geq n_0$ and $p \in \mathbb{N}$. Letting p tend to $+\infty$ and taking into account continuity from above of m, we obtain $m(V_n \setminus B) \leq k \bigvee_{t=1}^{\infty} b_{t,\varphi(t)}$ for any $n \geq n_0$. Taking $H = V_{n_0}$, we get that for every $B \in \mathcal{W}^{+-}$ and $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists a set $H \in \mathcal{W}^{+}$ with $H \supset B$ and

$$m(H \setminus B) \leq k \bigvee_{t=1}^{\infty} b_{t,\varphi(t)}. \tag{3.38}$$

Moreover by (3.37), in correspondence with H and φ, there is a set $A \in \mathcal{W}$, $A \subset H$, with

$$m(H \setminus A) \leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}. \tag{3.39}$$

From (3.38) and (3.39), monotonicity and k-subadditivity of m, it follows that

$$\begin{aligned} m(A \triangle B) &\leq m((H \setminus A) \cup (H \setminus B)) \leq m(H \setminus A) + k\, m(H \setminus B) \leq \\ &\leq \bigvee_{t=1}^{\infty} a_{t,\varphi(t)} + k \bigvee_{t=1}^{\infty} b_{t,\varphi(t)} \leq \bigvee_{t=1}^{\infty} c_{t,\varphi(t)}. \end{aligned}$$

By (3.15), since $\sigma(\mathcal{W}) \subset \mathcal{W}^{*}$, we get that for every $E \in \sigma(\mathcal{W})$ there is $B \in \mathcal{W}^{+-}$ with $B \supset E$ and $m(E) = m(B)$. Thus, if E is any element of $\sigma(\mathcal{W})$ and A is as in (3.39), then, using monotonicity and k-subadditivity of m, we obtain

$m(A\triangle E) \le m(A\triangle B) + k\, m(B\triangle E) = m(A\triangle B) \le \bigvee_{t=1}^{\infty} c_{t,\varphi(t)}$, and hence (3.36) is proved.

Now, we prove global (s)-boundedness of m. Let u, $(a_{t,r})_{t,r}$ and $(c_{t,r})_{t,r}$ be as above. By Lemma 2.3, there is a regulator $(C_{t,r})_{t,r}$ with

$$u \wedge \sum_{n=1}^{\infty}\Big(\bigvee_{t=1}^{\infty} c_{t,\varphi(t+n)}\Big) \le \bigvee_{t=1}^{\infty} C_{t,\varphi(t)}.$$

Put

$$d_{t,r} := 2k(C_{t,r} + a_{t,r}),\ \ t, r \in \mathbb{N}. \tag{3.40}$$

We prove that m is globally (s)-bounded on $\sigma(\mathcal{W})$ with respect to $(d_{t,r})_{t,r}$. Choose arbitrarily a disjoint sequence $(H_n)_n$ in $\sigma(\mathcal{W})$ and an element $\varphi \in \mathbb{N}^{\mathbb{N}}$. By (3.36), for each $n \in \mathbb{N}$ there is a set $F_n \in \mathcal{W}$ with

$$m(H_n\triangle F_n) \le \bigvee_{t=1}^{\infty} c_{t,\varphi(t+n)}.$$

Set $F_1^* := F_1,\ F_n^* := F_n \setminus \Big(\bigcup_{j=1}^{n-1} F_j\Big)$ for every $n \ge 2$. We get

$$F_n^* \in \mathcal{W} \text{ for every } n \in \mathbb{N} \text{ and } H_n\triangle F_n^* \subset \bigcup_{j=1}^{n}(H_j\triangle F_j). \tag{3.41}$$

From (3.41), monotonicity and k-subadditivity of m we obtain

$$\begin{aligned} m(H_n) &\le m(H_n\triangle F_n^*) + k\, m(F_n^*) \\ &\le u \wedge \Big(k\sum_{j=1}^{n} m(H_j\triangle F_j)\Big) + k\, m(F_n^*) \\ &\le \bigvee_{t=1}^{\infty} C_{t,\varphi(t)} + k\, m(F_n^*). \end{aligned} \tag{3.42}$$

Since the sequence $(F_n^*)_n$ is disjoint, then, by global (s)-boundedness of m on $\mathcal{W}$ with respect to $(a_{t,r})_{t,r}$, for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $\overline{n} \in \mathbb{N}$ with

$$m(F_n^*) \le \bigvee_{t=1}^{\infty} a_{t,\varphi(t)}. \tag{3.43}$$

From (3.42) and (3.43) it follows that $m(H_n) \le \bigvee_{t=1}^{\infty} d_{t,\varphi(t)}$. From this and arbitrariness of the chosen sequence $(H_n)_n$ we get global (s)-boundedness of m on $\sigma(\mathcal{W})$ with respect to the regulator $(d_{t,r})_{t,r}$.

Furthermore, by construction, taking into account that the set functions m_0^+ and m_0^* are well-defined, using weak σ-distributivity of R, it is not difficult to check that the extension $m : \sigma(\mathcal{W}) \to R$ of m_0 is unique. This ends the proof. ■

Remark 3.6 Observe that the set function m in Theorem 3.4 is also globally continuous on $\sigma(\mathcal{W})$. Indeed, let $(E_n)_n$ be a decreasing sequence in $\sigma(\mathcal{W})$ with empty intersection, and $(d_{t,r})_{t,r}$ be as in (3.40). Using global (s)-boundedness of m on $\sigma(\mathcal{W})$, analogously as in (3.6) it is possible to see that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $\overline{n} \in \mathbb{N}$ with

$$m(E_n \setminus E_{n+p}) \leq \bigvee_{t=1}^{\infty} d_{t,\varphi(t)} \tag{3.44}$$

for every $n \in \mathbb{N}$, $n \geq \overline{n}$ and $p \in \mathbb{N}$. Taking into account continuity from below of m on $\sigma(\mathcal{W})$, from (3.44), keeping fixed n and letting p tend to $+\infty$, we get

$$m(E_n) \leq \bigvee_{t=1}^{\infty} d_{t,\varphi(t)} \tag{3.45}$$

for every $n \in \mathbb{N}$, $n \geq \overline{n}$. By arbitrariness of the chosen sequence $(E_n)_n$, from (3.45) we get global continuity from above at $\emptyset$ of m on $\sigma(\mathcal{W})$. Global continuity from above and from below of m on $\sigma(\mathcal{W})$ follows from global continuity from above at $\emptyset$, by proceeding analogously as in Proposition 3.1. ∎

Open problems:

(a) Find some types of extensions for continuous set functions with values in a not necessarily super Dedekind complete or weakly σ-distributive lattice group.

(b) Is the extension found in [4] still valid for lattice group-valued measures on effect algebras or even on pseudo-effect algebras?

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