ON REVERSE AM-GM INEQUALITIES FOR n OPERATORS

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Abstract. In this paper, we generalize some operator inequalities due to Fu and He [Linear Multilinear Algebra, 63 (2015), 571-577] as follows: Let $A_i\,(i = 1, ..., n)$ be positive operators on a Hilbert space with $0 < m \leq A_i \leq M$. Then for every positive unital linear map Φ,

$$\Phi^{2p}\left(\frac{A_1 + \cdots + A_n}{n}\right) \leq \left[\frac{(M+m)^{2p}}{4M^p m^p}\right]^2 \Phi^{2p}[G(A_1, \cdots, A_n)], \quad 1 \leq p < \infty,$$

and

$$\Phi^{2p}\left(\frac{A_1 + \cdots + A_n}{n}\right) \leq \left[\frac{(M+m)^{2p}}{4M^p m^p}\right]^2 G^{2p}[\Phi(A_1), \cdots, \Phi(A_n)], \quad 1 \leq p < \infty,$$

where $G(A_1, ..., A_n)$ is Ando-Li-Mathias geometric mean.

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1. Introduction

Throughout this paper, let M, m be scalars, I be the identity operator and $\mathcal{B}(\mathcal{H})$ be the set of all bounded linear operators on a Hilbert space $(\mathcal{H}, <\cdot,\cdot>)$. The operator norm is defined by $\|\cdot\|$. We write $A \geq 0$ if the operator A is positive. If $A - B \geq 0$, then we say that $A \geq B$. Besides, for $A, B > 0$, the geometric mean $A\sharp B$ is defined by

$$A\sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}.$$

A linear map Φ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is said to be unital if $\Phi(I) = I$.

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We denote the Ando-Li-Mathias geometric mean [1] for $A_1, ..., A_n > 0$ by $G(A_1, ..., A_n)$. When $n = 2$, $G(A_1, A_2) = A_1 \sharp A_2$, but, there is no explicit formula for $G(A_1, ..., A_n)$ in terms of $A_1, ..., A_n$ when $n \geq 3$. However, we only need two basic properties of such a mean:

$$G(A_1^{-1}, ..., A_n^{-1}) = G^{-1}(A_1, ..., A_n), \tag{1.1}$$

$$G(A_1, ..., A_n) \leq \frac{A_1 + \cdots + A_n}{n}. \tag{1.2}$$

Let A and B be positive operators on a Hilbert space with $0 < m \leq A$, $B \leq M$ and Φ be positive unital linear map. Fu and He [2, Theorem 4] showed that the following reverse AM-GM inequalities for two operators:

$$\Phi^p\left(\frac{A_1 + A_2}{2}\right) \leq \left[\frac{(M+m)^2}{4^{\frac{2}{p}} Mm}\right]^p \Phi^p(A_1 \sharp A_2), \quad 2 \leq p < \infty, \tag{1.3}$$

$$\Phi^p\left(\frac{A_1 + A_2}{2}\right) \leq \left[\frac{(M+m)^2}{4^{\frac{2}{p}} Mm}\right]^p [\Phi(A_1) \sharp \Phi(A_2)]^p, \quad 2 \leq p < \infty. \tag{1.4}$$

Replacing p by $2p$ in (1.3) and (1.4), we have

$$\Phi^{2p}\left(\frac{A_1 + A_2}{2}\right) \leq \left[\frac{(M+m)^{2p}}{4M^p m^p}\right]^2 \Phi^{2p}(A_1 \sharp A_2), \quad 1 \leq p < \infty, \tag{1.5}$$

$$\Phi^{2p}\left(\frac{A_1 + A_2}{2}\right) \leq \left[\frac{(M+m)^{2p}}{4M^p m^p}\right]^2 [\Phi(A_1) \sharp \Phi(A_2)]^{2p}, \quad 1 \leq p < \infty. \tag{1.6}$$

Let $0 < m \leq A_i \leq M$ $(i = 1, ..., n)$. J.I. Fujii et al. [3, (12)] showed a reverse AM-GM inequality for n operators:

$$\frac{A_1 + \cdots + A_n}{n} \leq \frac{(M+m)^2}{4Mm} G(A_1, ..., A_n). \tag{1.7}$$

Recently, Lin [4, Theorem 3.2] showed that the reverse AM-GM inequality (1.7) can be squared:

$$\left(\frac{A_1 + \cdots + A_n}{n}\right)^2 \leq \left[\frac{(M+m)^2}{4Mm}\right]^2 G^2(A_1, ..., A_n). \tag{1.8}$$

The inequality (1.7) can be regarded as a counterpart to (1.2). By (1.7), it is easy to obtain the following reverse operator AM–GM inequality:

$$\Phi\left(\frac{A_1 + \cdots + A_n}{n}\right) \leq \frac{(M+m)^2}{4Mm} \Phi[G(A_1, ..., A_n)].$$

Are the following inequalities true for $p \geq 1$?

$$\Phi^{2p}\left(\frac{A_1+\cdots+A_n}{n}\right) \le \left[\frac{(M+m)^{2p}}{4M^pm^p}\right]^2 \Phi^{2p}[G(A_1,...,A_n)],$$

$$\Phi^{2p}\left(\frac{A_1+\cdots+A_n}{n}\right) \le \left[\frac{(M+m)^{2p}}{4M^pm^p}\right]^2 G^{2p}[\Phi(A_1),...,\Phi(A_n)].$$

We will answer this question in the next section.

In this paper, we will present some operator inequalities for n operators which are generalizations of (1.5) and (1.6).

2. Main results

We begin this section with the following lemmas.

Lemma 1. [5] *Let $A, B > 0$. Then the following norm inequality holds:*

$$\|AB\| \le \frac{1}{4}\|A+B\|^2. \tag{2.1}$$

Lemma 2. [6] *Let A and B be positive operators. Then for $1 \le r < \infty$,*

$$\|A^r + B^r\| \le \|(A+B)^r\|. \tag{2.2}$$

Lemma 3. [7] *Let A be a positive operator on a Hilbert space. Then for every positive unital linear map Φ,*

$$\Phi(A^{-1}) \ge \Phi^{-1}(A). \tag{2.3}$$

Theorem 1. *Let $0 < m \le A_i \le M (i = 1,...,n)$. Then for every positive unital linear map Φ,*

$$\Phi^{2p}\left(\frac{A_1+\cdots+A_n}{n}\right) \le \left[\frac{(M+m)^{2p}}{4M^pm^p}\right]^2 \Phi^{2p}[G(A_1,...,A_n)],\ 1 \le p < \infty. \tag{2.4}$$

Proof. Inequality (2.4) is equivalent to

$$\left\|\Phi^{p}\left(\frac{A_1+\cdots+A_n}{n}\right)\Phi^{-p}[G(A_1,...,A_n)]\right\| \le \frac{(M+m)^{2p}}{4M^pm^p},\ 1 \le p < \infty. \tag{2.5}$$

Compute

$$\begin{aligned}
&\left\|\Phi^p\left(\frac{A_1+\cdots+A_n}{n}\right)M^pm^p\Phi^{-p}[G(A_1,...,A_n)]\right\| \\
&\le \frac{1}{4}\left\|\Phi^p(\frac{A_1+\cdots+A_n}{n})+M^pm^p\Phi^{-p}[G(A_1,...,A_n)]\right\|^2 \qquad (\text{by}\,(2.1)) \\
&\le \frac{1}{4}\left\|\Phi\left(\frac{A_1+\cdots+A_n}{n}\right)+Mm\Phi^{-1}[G(A_1,...,A_n)]\right\|^{2p} \qquad (\text{by}\,(2.2)) \\
&\le \frac{1}{4}\left\|\Phi\left(\frac{A_1+\cdots+A_n}{n}\right)+Mm\Phi[G^{-1}(A_1,...,A_n)]\right\|^{2p} \qquad (\text{by}\,(2.3)) \\
&= \frac{1}{4}\left\|\Phi\left[\frac{A_1+\cdots+A_n}{n}+MmG^{-1}(A_1,...,A_n)\right]\right\|^{2p} \\
&\le \frac{1}{4}(M+m)^{2p}. \qquad (\text{by}\,(1.8))
\end{aligned}$$

That is

$$\left\|\Phi^p\left(\frac{A_1+\cdots+A_n}{n}\right)\Phi^{-p}[G(A_1,...,A_n)]\right\| \le \frac{(M+m)^{2p}}{4M^pm^p}.$$

Thus, (2.5) holds. This completes the proof. ∎

Remark 1. When $n=2$, by (2.4) we obtain the operator inequality (1.5). Thus, (2.4) is a generalization of (1.5).

Theorem 2. *Let $0<m\le A_i\le M$ $(i=1,...,n)$. Then for every positive unital linear map Φ,*

$$\Phi^{2p}\left(\frac{A_1+\cdots+A_n}{n}\right)\le\left[\frac{(M+m)^{2p}}{4M^pm^p}\right]^2G^{2p}[\Phi(A_1),...,\Phi(A_n)],\ 1\le p<\infty. \tag{2.6}$$

Proof. Inequality (2.6) is equivalent to

$$\left\|\Phi^p\left(\frac{A_1+\cdots+A_n}{n}\right)G^{-p}[\Phi(A_1),...,\Phi(A_n)]\right\|\le\frac{(M+m)^{2p}}{4M^pm^p},\ 1\le p<\infty. \tag{2.7}$$

Compute

$$\begin{aligned}
&\left\|\Phi^p\left(\frac{A_1+\cdots+A_n}{n}\right)M^p m^p G^{-p}[\Phi(A_1),...,\Phi(A_n)]\right\| \\
&\le \frac{1}{4}\left\|\Phi^p\left(\frac{A_1+\cdots+A_n}{n}\right)+M^p m^p G^{-p}[\Phi(A_1),...,\Phi(A_n)]\right\|^2 && (\text{by } (2.1)) \\
&\le \frac{1}{4}\left\|\Phi\left(\frac{A_1+\cdots+A_n}{n}\right)+MmG^{-1}[\Phi(A_1),...,\Phi(A_n)]\right\|^{2p} && (\text{by } (2.2)) \\
&= \frac{1}{4}\left\|\frac{\Phi(A_1)+\cdots+\Phi(A_n)}{n}+MmG[\Phi^{-1}(A_1),...,\Phi^{-1}(A_n)]\right\|^{2p} && (\text{by } (1.1)) \\
&\le \frac{1}{4}\left\|\frac{\Phi(A_1)+\cdots+\Phi(A_n)}{n}+Mm\frac{\Phi^{-1}(A_1)+\cdots+\Phi^{-1}(A_n)}{n}\right\|^{2p} && (\text{by } (1.2)) \\
&\le \frac{1}{4}\left\|\frac{\Phi(A_1)+\cdots+\Phi(A_n)}{n}+Mm\frac{\Phi(A_1^{-1})+\cdots+\Phi(A_n^{-1})}{n}\right\|^{2p} && (\text{by } (2.3)) \\
&\le \frac{1}{4}(M+m)^{2p}. && (\text{by } [8, 2.3])
\end{aligned}$$

That is

$$\left\|\Phi^p\left(\frac{A_1+\cdots+A_n}{n}\right)G^{-p}[\Phi(A_1),...,\Phi(A_n)]\right\| \le \frac{(M+m)^{2p}}{4M^p m^p}.$$

Thus, (2.7) holds. This completes the proof. ■

Remark 2. When $n=2$, by (2.6) we obtain the operator inequality (1.6). Thus, (2.6) is a generalization of (1.6).

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