FINITE GROUPS HAVING EXACTLY 42 ELEMENTS OF MAXIMAL ORDER

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Abstract. Let G be a finite group, $M(G)$ denotes the number of elements of maximal order of G. In this note a finite group G with $M(G) = 42$ is determined.

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1. Introduction

All groups considered are finite. In this paper, P_r denotes Sylow r-subgroup of G for a prime r, $A \rtimes B$ denotes the semidirect product of A and B. For some natural number m and n, C_n^m always denotes the direct product of m cyclic groups of order n. All unexplained notations are standard and can be found in [3].

For a finite group G, we denote by $M(G)$ the number of elements of maximal order of G, and the maximal element order in G by $k = k(G)$. There is a topic related to one of Thompson's Conjectures:

Thompson's Conjecture. *Let G be a finite group. For a positive integer d, define $G(d) = |\{x \in G | \text{the order of } x \text{ is } d\}|$. If S is a solvable group, $G(d) = S(d)$ for $d = 1, 2, \ldots$, then G is solvable.*

Recently, some authors have investigated this topic in several articles(see [2], [5], [6], [7]). In particular, in [1] the authors gave a complete classification of the finite group with $M(G) = 30$, and the finite group with $M(G) = 24$ are classified in [4]. In this paper, we consider a finite group G satisfying $M(G) = 42$. Our main result of this paper is:

Main Theorem. *Suppose G is a finite group having exactly 42 elements of maximal order. Then G is solvable and one of the following holds:*

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(1) *if* $k = 6$, *then* $|G| = 2^\alpha \cdot 3^\beta$, *where* $2 \leq \alpha \leq 7$ *and* $1 \leq \beta \leq 5$;

(2) *if* $k = 14$, *then one of the following holds:*

(I) $G = C_2^3 \times C_7$ *or* $G = (C_2^3 \times C_7) \rtimes C_6$;

(II) $|G| = 2^\alpha \cdot 3^\beta \cdot 7$, *where* $2 \leq \alpha \leq 4$ *and* $1 \leq \beta \leq 2$;

(3) *if* $k = 18$, $|G| = 2^\alpha \cdot 3^\beta$, *where* $1 \leq \alpha \leq 4$ *and* $2 \leq \beta \leq 4$;

(4) *if* $k \in \{43, 49, 86, 98\}$, *then* $C_G(x) = \langle x \rangle \trianglelefteq G$. *Therefore,* $G/C_G(x) \lesssim Aut(C_k)$, *where* $o(x) = k$.

By the above theorem, we have:

Corollary. *Thompson's Conjecture holds if G has exactly 42 elements of maximal order.*

2. Preliminaries

The following lemma reveals the relationship of $M(G)$ and k.

Lemma 2.1 [7, Lemma 1] *Suppose G has exactly n cyclic subgroups of order l, then the number of elements of order l (denoted by $n_l(G)$) is $n_l(G) = n\phi(l)$, where $\phi(l)$ is the Euler function of l. In particular, if n denotes the number of cyclic subgroups of G of maximal order k, then $M(G) = n\phi(k)$.*

By the above lemma, we have:

Lemma 2.2 *If $M(G) = 42$ and k is maximal element order of G, then possible values of n, k and $\phi(k)$ are given in following table:*

n	$\phi(k)$	k
42	1	2
21	2	3, 4, 6
14	3	*null*
7	6	7, 14, 18
6	7	*null*
3	14	*null*
2	21	*null*
1	42	43, 49, 86, 98

In proving our main theorem, the following two results will be frequently used.

Lemma 2.3 [1, Lemma 6] *If k is prime, and the number of elements of maximal order k is m, then k divides $m + 1$.*

Lemma 2.4 [1, Lemma 8] *If the number of elements of maximal order k is m, then there exists a positive integer α such that $|G|$ divides mk^α.*

Lemma 2.5 [6, Lemma 2.5] *Let P be a p-group with order p^t, where p is a prime, and t is a positive integer. Suppose $b \in Z(P)$, where $o(b) = p^u = k$ with u a positive integer. Then P has at least $(p-1)p^{t-1}$ elements of order k.*

3. Proof of Main Theorem

By the hypothesis $M(G) = 42$, then $k \neq 2, 3, 7, 43$ by [1, Lemma 6], and $k \neq 4$ by [1, Corollary 2]. In the following we prove our theorem case by case for the remaining possible values of k.

Case 1. $k = 6$. In this case $|G| = 2^\alpha 3^\beta$, where $\alpha > 0$ and $\beta > 0$ by Lemma 2.4. Let x be an element of order 6. Then $|C_G(\langle x\rangle)| = 2^u \cdot 3^v$. Since there exists no element of order 9 or 4 in $C_G(x)$, we have $v \leq 3$ and $u \leq 4$ by $M(G) = 42$. Since G has exactly 21 cyclic subgroups of order 6, we have $|G : N_G(\langle x\rangle)| = 1, 2, 3, 4, 6, 8, 9, 12, 16$ or 18. If there is an element y of order 6 in G such that $|G : N_G(\langle y\rangle)| = 12, 16$ or 18, then there exists another element z of order 6 in G such that $|G : N_G(\langle z\rangle)| = 1, 2, 3, 4, 8$ or 9. That is to say, G always has an element x of order 6 such that $|G : N_G(\langle x\rangle)| = 1, 2, 3, 4, 8$ or 9. Therefore $|G|\Big|2^7 \cdot 3^5$ since $|G| = |G : N_G(\langle x\rangle)| \cdot |N_G(\langle x\rangle) : C_G(\langle x\rangle)| \cdot |C_G(\langle x\rangle)|$. Thus (1) follows.

Case 2. $k = 14$. By Lemma 2.4, we may assume that $|G| = 2^\alpha \cdot 3^\beta \cdot 7^\gamma$, where $\alpha, \gamma > 0$ and $\beta = 0$ or 1. Let x be an element of order 14. Then $|C_G(\langle x\rangle)| = 2^u \cdot 7^v$. Since there exists no element of order 49 or 4 in $C_G(x)$, we have $v = 1$ and $u \leq 3$ by Lemma 2.5. Since G has exactly 7 cyclic subgroups of order 14, we have $|G : N_G(\langle x\rangle)| = 1, 2, 3, 4, 6$ or 7. If there is an element y of order 14 in G such that $|G : N_G(\langle y\rangle)| = 2, 4$ or 6, then there exists another element z of order 14 in G such that $|G : N_G(\langle z\rangle)| = 1$ or 3. That is to say, G always has an element x of order 6 such that $|G : N_G(\langle x\rangle)| = 1, 3$ or 7.If there exists an element a of order 14 in G satisfies $|G : N_G(\langle a\rangle)| = 1$, then $\langle a\rangle$ is normal in G, and hence $\langle a^7\rangle$ lies in $Z(G)$, the center of G and $\langle a^2\rangle$ is normal in G, which implies that P_7, the Sylow 7-subgroup of G is normal in G and is of order 7. Thus all elements of order 14 lie in $C_G(a)$ and hence $C_G(a) = C_2^3 \times C_7$. Since $N_G(\langle a\rangle)/C_G(\langle a\rangle) \leq C_6$, we have $G = C_2^3 \times C_7$ or $G = (C_2^3 \times C_7) \rtimes C_6$. Thus (2) follows.If there exists an element a of order 14 in G satisfies $|G : N_G(\langle a\rangle)| = 3$, then we get $|G| = 2^\alpha \cdot 3^\beta \cdot 7$, where $2 \leq \alpha \leq 4$ and $1 \leq \beta \leq 2$ since $|G| = |G : N_G(\langle a\rangle)| \cdot |N_G(\langle a\rangle) : C_G(\langle a\rangle)| \cdot |C_G(\langle a\rangle)|$. Thus (2) follows.If there exists an element a of order 14 in G satisfies $|G : N_G(\langle a\rangle)| = 7$, then $|P_7| = 7^2$, a contradiction to [1, Lemma 7].

Case 3. $k = 18$. By Lemma 2.4, we may assume that $|G| = 2^\alpha \cdot 3^\beta \cdot 7^\gamma$, where $\alpha, \beta > 0$ and $\gamma = 0$ or 1. If $\gamma = 0$, then G is a $\{2, 3\}$-group and $|G| = 2^\alpha \cdot 3^\beta$. Since the number of cyclic subgroups of order 18 in G is 7, it follows that $|G : N_G(\langle x\rangle)| = 1, 2, 3, 4$ or 6 for some element x of order 18. If $|G : N_G(\langle x\rangle)| = 4$ or 6, then there must be another element y of order 18 such that $|G : N_G(\langle y\rangle)| = 1, 2$ or 3. Hence there is always an element x of order 18 such that $|G : N_G(\langle x\rangle)| = 1, 2$ or 3. Let $|C_G(\langle x\rangle)| = 2^u \cdot 3^v$. It is easy to see the Sylow 2-subgroups of $C_G(x)$ are abelian and $u = 1$, $2 \leq v \leq 3$ or $u = 2$, $v = 2$. So we get $|G|\Big|2^4 \cdot 3^4$ since $|G| = |G : N_G(\langle x\rangle)| \cdot |N_G(\langle x\rangle) : C_G(\langle x\rangle)| \cdot |C_G(\langle x\rangle)|$. Thus (3) follows.If $\gamma = 1$, there exists an element x of order 18 such that $|G : N_G(\langle x\rangle)| = 7$ and hence all the cyclic subgroups of order 18 are conjugate in G and also the centralizers. Let $|C_G(x)| = 2^u \cdot 3^v$. By Lemma2.5 and our assumption, we have $u = 1$, $2 \leq v \leq 3$ or $u = 2$, $v = 2$.If $u = 1$, $v = 3$ or $u = 2$, $v = 2$, then $C_G(x)$ contains 18 elements of order 18. Choose $y \in G\backslash C_G(x)$ be an element of order 18, then $C_G(y)$ also contains

18 elements of order 18. We prove that for every $y \in G \backslash C_G(x)$ with $o(y) = 18$, $C_G(y) \cap C_G(x)$ contains no elements of order 18. Otherwise, there is $z \in C_G(x) \cap C_G(y)$ with $o(z) = 18$. Since $C_G(x)$ and $C_G(y)$ are abelian, we have $C_G(y) \leq C_G(z)$ and $C_G(x) \leq C_G(z)$. Noting that all centralizers of cyclic subgroups of order 8 are conjugate, we know that $C_G(x), C_G(y)$ and $C_G(z)$ are also conjugate. Hence $C_G(x) = C_G(y) = C_G(z)$, a contradiction. Thus if there exists an element a of order 18 such that $a \in G \backslash C_G(x) \cup C_G(y)$, then $C_G(x) \cup C_G(y) \cup C_G(a)$ contains 54 elements of order 18, a contradiction. If $n_{18}(G) = n_{18}(C_G(x) \cup C_G(y))$, then $M(G) = n_{18}(C_G(x) \cup C_G(6)) = 36$, a contradiction too.If $u = 1$, $v = 2$, then $|C_G(x)| = 18$ and $|G| = 2^{\alpha+1} \cdot 3^{\beta+2} \cdot 7$, where $\alpha = 0$ or 1 and $\beta = 0$ or 1. Suppose that $\alpha = 0$. Then there exists a normal subgroup M of G such that $|M| = 3^{\beta+2} \cdot 7$. Now by Sylow's Theorem we can easily know $P_7 \trianglelefteq G$, where $P_7 \in Syl_7(G)$, which implies that G has elements of order 21, a contradiction. Therefore $\alpha = 1$. If $\beta = 0$, then $|G| = 2^2 \cdot 3^2 \cdot 7$. If that $P_7 \trianglelefteq G$, then G has elements of order 21, a contradiction. Suppose that P_7 is not normal in G. Then $|G : N_G(P_7)| = 2^2 \cdot 3^2$ by Sylow's Theorem and hence $N_G(P_7) = C_G(P_7)$. Now by the Burnside Theorem we know that there exists a normal subgroup N of G such that $|N| = 2^2 \cdot 3^2$. It is easy to see that either Sylow 2-subgroup of N or Sylow 3-subgroup of N is normal in N and hence normal in G. Thus $|N_G(P_7)| > 7$, a contradiction. Suppose that P_7 is normal in G. Then G has elements of order 21, a contradiction too. Thus $\beta \neq 0$. If $\beta = 1$, then $|G| = 2^2 \cdot 3^3 \cdot 7$. By the same argument as above we can get that $|G : N_G(P_7)| = 2^2 \cdot 3^2$ and hence $|N_G(P_7)| = 21$. Obviously, G is not a simple group, so $F(G)$, the Fitting subgroup of G is not trivial. If $|F(G)|_2 \neq 1$, then $|N_G(P_7)|_2 \neq 1$, a contradiction. It implies that $|F(G)| = 3^\gamma$, where $\gamma > 0$. Now P_7 acts on $F(G)$. Then we have $7 \; \big| \; |Aut(F(G))| \; \big| (3^3 - 1)(3^3 - 3)(3^3 - 9)$, a contradiction.

Case 4. $k \in \{43, 49, 86, 98\}$. Let x be an element of order k. Then $C_G(x) = \langle x \rangle \trianglelefteq G$. Therefore, $G/C_G(x) \lesssim Aut(C_k)$ and $C_G(x) \cong C_k$. Thus (4) follows.

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