NOTE ON YOUNG AND ARITHMETIC-GEOMETRIC MEAN INEQUALITIES FOR MATRICES

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Abstract. In this short note, we prove that the conjecture of singular value Young inequality holds when $j = n$. Meanwhile, we also present a refinement of the arithmetic-geometric mean inequality for unitarily invariant norms.

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1. Introduction

Let M_n be the space of $n \times n$ complex matrices. We shall always denote the singular values of A by $s_1(A) \geq \cdots \geq s_n(A) \geq 0$. If $A \in M_n$ has real eigenvalues, we label them as $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$. Let $\|\cdot\|$ denote any unitarily invariant norm on M_n.

Let A, B be positive semidefinite. Ando proved in [1] that if $v \in [0,1]$, then

$$s_j\left(A^v B^{1-v}\right) \leq s_j\left(vA + (1-v)B\right),\ j = 1, ..., n.$$

This is Young inequality for singular values. Recently, Lin [2] posed the following.

Conjecture. *Let A, B be positive semidefinite and $v \in [0,1]$. Then*

$$s_j\left(A^v B^{1-v}\right) \leq s_j\left(vA^{1/2} + (1-v)B^{1/2}\right)^2,\ j = 1, ..., n.$$

If the conjecture holds, then it is a strength of Ando's inequality. In this short note, we prove that the conjecture holds when $j = n$.

Let A, B be positive semidefinite. Bhatia and Kittaneh [3] proved that

$$\|AB\| \leq \frac{1}{4}\left\|(A+B)^2\right\|, \tag{1.1}$$

which is a arithmetic-geometric mean inequality for unitarily invariant norms. In this short note, we also obtain a refinement of inequality (1.1).

2. Main results

In this section, we first prove that Lin's conjecture holds when $j=n$.

Theorem 2.1. *Let A, B be positive semidefinite and $v \in [0,1]$. Then*

$$s_n\left(A^v B^{1-v}\right) \leq s_n\left(vA^{1/2} + (1-v)B^{1/2}\right)^2.$$

Proof. This is obviously true if either A, or B is not invertible. So assume A and B are invertible. Then

$$\begin{aligned} s_n\left(A^v B^{1-v}\right) &= \lambda_n^{1/2}\left(B^{1-v}A^{2v}B^{1-v}\right) \\ &= \lambda_1^{-1/2}\left(B^{v-1}A^{-2v}B^{v-1}\right) \end{aligned} \tag{2.1}$$

On the other hand, we have

$$\begin{aligned} \lambda_1^{1/2}\left(B^{v-1}A^{-2v}B^{v-1}\right) &= s_1\left(A^{-v}B^{v-1}\right) \\ &\geq \lambda_1\left(A^{-v}B^{v-1}\right) \end{aligned} \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\begin{aligned} s_n\left(A^v B^{1-v}\right) &\leq \lambda_1^{-1}\left(A^{-v}B^{v-1}\right) \\ &= \lambda_1^{-1}\left(A^{-v/2}B^{v-1}A^{-v/2}\right) \\ &= \lambda_n\left(A^{v/2}B^{1-v}A^{v/2}\right) \\ &= s_n^2\left(A^{v/2}B^{(1-v)/2}\right). \end{aligned}$$

So, Lemma 2.1 and the last inequality complete the proof. ∎

Next, we give a refinement of inequality (1.1). To do this, we need the following lemma [3].

Lemma 2.2. *Let A, B be positive semidefinite. Then*

$$\left\|A^{1/2}(A+B)B^{1/2}\right\| \leq \frac{1}{2}\left\|(A+B)^2\right\|$$

Theorem 2.2. *Let A, B be positive semidefinite. Then*

$$\|AB\| + \left(\int_{1/2}^{3/2}\left\|A^v B^{2-v} + A^{2-v}B^v\right\| dv - 2\|AB\|\right) \leq \frac{1}{4}\left\|(A+B)^2\right\|.$$

Proof. It is known [4, p.265)]that the function

$$g(r) = \left\|A^r B^{1-r} + A^{1-r}B^r\right\|$$

is convex on $[0,1]$. Replacing A by A^2, B by B^2, and $2r$ by v, we know that the function

$$f(v) = \left\| A^v B^{2-v} + A^{2-v} B^v \right\|$$

is convex on $[0,2]$, it follows that this function is also convex on $\left[\frac{1}{2}, \frac{3}{2}\right]$. Therefore, if $v \in \left[\frac{1}{2}, 1\right]$, then by the convexity of the function of $f(v)$, we have

$$f\left(\lambda \times \frac{1}{2} + (1-\lambda) \times 1\right) \leq \lambda f\left(\frac{1}{2}\right) + (1-\lambda) f(1). \tag{2.3}$$

Let

$$v = \lambda \times \frac{1}{2} + (1-\lambda) \times 1.$$

Then, we know that inequality (2.3) is equivalent to

$$f(v) \leq (2-2v) f\left(\frac{1}{2}\right) + (2v-1) f(1),$$

which yields

$$\int_{1/2}^{1} f(v)\, dv \leq \frac{1}{4}\left(f\left(\frac{1}{2}\right) + f(1)\right). \tag{2.4}$$

On the other hand, if $v \in \left[1, \frac{3}{2}\right]$, then by the convexity of the function of $f(v)$, we have

$$f\left(\lambda \times 1 + (1-\lambda) \times \frac{3}{2}\right) \leq \lambda f(1) + (1-\lambda) f\left(\frac{3}{2}\right),$$

which implies

$$\int_{1}^{3/2} f(v)\, dv \leq \frac{1}{4}\left(f(1) + f\left(\frac{3}{2}\right)\right). \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$\int_{1/2}^{3/2} f(v)\, dv \leq \frac{1}{2}\left(f(1) + f\left(\frac{1}{2}\right)\right),$$

which is equivalent to

$$\|AB\| + \left(\int_{1/2}^{3/2} \left\| A^v B^{2-v} + A^{2-v} B^v \right\| dv - 2\|AB\|\right) \leq \frac{1}{2}\left\| A^{1/2}(A+B) B^{1/2} \right\|.$$

So, Lemma 2.2 and the last inequality complete the proof. ∎

Remark 2.1. By the convexity of the function of $f(v)$, we know that

$$2\|AB\| \leq \left\| A^v B^{2-v} + A^{2-v} B^v \right\|$$

and hence

$$\int_{1/2}^{3/2} \left\| A^v B^{2-v} + A^{2-v} B^v \right\| dv - 2 \left\| AB \right\| \geq 0.$$

So Theorem 2.2 is a refinement of the arithmetic-geometric mean inequality (1.1).

References

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