SUBSPACE MIXING AND UNIVERSALITY CRITERION FOR A SEQUENCE OF OPERATORS

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Abstract. Let $B(X)$ denote the algebra of all bounded linear operators on an infinite-dimensional separable complex Banach space X and M be a nonzero subspace of X. We will characterize properties of being $d-M$ mixing for a $N \geq 2$ sequence $T_{1,j}, T_{2,j}, ..., T_{N,j}$ of operators in $B(X)$. Also, we will give necessary and sufficient conditions for a $N \geq 2$ sequence $T_{1,j}, T_{2,j}, ..., T_{N,j}$ of operators in $B(X)$ to satisfy $d - M$ universality criterion in terms of d-M topologically transitivity of this sequence.

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1. Introduction

Let $B(X)$ denote the algebra of all bounded linear operators on a infinite-dimensional separable complex Banach space X.

For $x \in X$, the orbit of x under T is the set $Orb(T,x) = \{T^n x : n \in \mathbb{N}\}$. A vector x is called hypercyclic for T if $Orb(T,x)$ is dense in X and the operator T is said to be hypercyclic if there is some vector $x \in X$ which is hypercyclic. More general, a sequence $(T_n)_{n\geq 0}$ of operators in $B(X)$ is called hypercyclic or universal if $\{T_n(x), n \geq 0\}$ is dense in X for some $x \in X$, in this case x is called universal for the family $(T_n)_{n\geq 0}$, see [9].

In 2007, L. Bernal-González in [3] and J.P. Bès and A. Peris in [4] introduced independently the definition of disjoint hypercyclic for tuple of linear operators. They introduced the concept of diagonally-universality for a tuple of sequences in $B(X)$. They also gave the definition of diagonally universal for a tuple of sequences in $B(X)$.

Recall that the family $(T(t))_{t\geq 0}$ of operators on X is called a strongly continuous semigroup(C_0-semigroup) of operators if:

1. $T(0) = I$;
2. $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$;
3. $\lim\limits_{t\downarrow 0} T(t)x := x$ for every $x \in X$.

The linear operator A defined in

$$D(A) = \{x \in X : \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \text{ exist } \}$$

by

$$Ax = \lim_{t\downarrow 0} \frac{T(t)x - x}{t} = \frac{d^+T(t)x}{dt} \mid_{t=0} \text{ for } x \in D(A)$$

is the infinitesimal generator of the semigroup $T(t)$ and $D(A)$ is the domain of A, see[10]. A C_0- semigroup $\tau = (T_t)_{t\geq 0}$ of operators in $B(X)$ is called hypercyclic if there exists a vector $x \in X$ such that the orbit of τ, $Orb(\tau, x) = \{T(t)x : t \geq 0\}$ is dense in X. In this case x is called the hypercyclic vector of τ [9].

Definition 1.1 Let $(T_{1,j})_{j=1}^{\infty}, (T_{2,j})_{j=1}^{\infty}, ..., (T_{N,j})_{j=1}^{\infty}$ be an $N \geq 2$ sequences in $B(X)$ and let M be a nonzero subspace of X. We say that the N sequences of operators $(T_{1,j})_{j=1}^{\infty}, (T_{2,j})_{j=1}^{\infty}, ..., (T_{N,j})_{j=1}^{\infty}$ are disjoint or diagonally subspace universal respect to M (in short $d-M$ universal), if there exists a vector $(x, x, ..., x)$ in the diagonal of X^N, such that $\{(T_{1,j}x, T_{2,j}x, ..., T_{N,j}x,), j \in \mathbb{N}\} \cap M^N$ is dense in M^N. We call x a $d-M$ universal vector. We denote by

$$dU((T_{1,j})_{j=1}^{\infty}, (T_{2,j})_{j=1}^{\infty}, ..., (T_{N,j})_{j=1}^{\infty}, M)$$

the set of all $d-M$ universal vectors of the sequences $(T_{1,j})_{j=1}^{\infty}, (T_{2,j})_{j=1}^{\infty}, ..., (T_{N,j})_{j=1}^{\infty}$.

Definition 1.2 Let $(T_{1,j})_{j=1}^{\infty}, (T_{2,j})_{j=1}^{\infty}, ..., (T_{N,j})_{j=1}^{\infty}$ be a $N \geq 2$ sequences in $B(X)$ and let M be a nonzero subspace of X. We say that the N sequences of operators $(T_{1,j})_{j=1}^{\infty}, (T_{2,j})_{j=1}^{\infty}, ..., (T_{N,j})_{j=1}^{\infty}$ are d-M topologically transitive if for any non-empty open V_0, V_1, ...V_N in M there exists $j \geq 0$ so that

$$V_0 \cap T_{1,j}^{-1}(V_1) \cap T_{2,j}^{-1}(V_2) \cap ... \cap T_{N,j}^{-1}(V_N)$$

contains a non-empty open set of M.

Let M a nonzero subspace of X. The notion diagonally subspace universal respect to M (in short $d{-}M$ universal) and the notion of d-M topologically transitive for the sequence $(T_{1,t})_{t\geq 0}, (T_{2,t})_{t\geq 0}, ..., (T_{N,t})_{t\geq 0}, (N \geq 2)$ of a C_0-semigroups of operators on X is studied in [11]. We proved that, if $(T_{1,t})_{t\geq 0}, (T_{2,t})_{t\geq 0}, ..., (T_{N,t})_{t\geq 0}$ is a sequence of C_0-semigroup with generators $A_1, A_2, ..., A_N$ and if there exists $t_0 > 0$ such that $T_{1,t_0}, T_{2,t_0}, ..., T_{N,t_0}$ are surjective and d-universal, then

$$(T_{1,t})_{t\geq 0}, (T_{2,t})_{t\geq 0}, ..., (T_{N,t})_{t\geq 0} \text{ are } d - D(A_j) \text{ universal for all } j = 1, 2, ..., N.$$

Also, we give necessary and sufficient condition for which a sequence

$$(T_{1,t})_{t\geq 0}, (T_{2,t})_{t\geq 0}, ..., (T_{N,t})_{t\geq 0} \text{ with } (N \geq 2)$$

of C_0-semigroup to be d-M topologically transitive.

Definition 1.3 We say that the $N{\geq}2$ sequences of operators $(T_{1,j}), (T_{2,j}),..., (T_{N,j})$ are $d - M$ mixing respect to nonempty subset M of X if for any non-empty open subsets $V_0, V_1, ..., V_N$ in M, there exists $n \geq 0$ such that

$$V_0 \cap T_{1,m}^{-1}(V_1) \cap T_{2,m}^{-1}(V_2) \cap ... \cap T_{N,m}^{-1}(V_N)$$

contains a non-empty open set of M for each $m \geq n$.

Definition 1.4 Let M be a nonzero subspace of X and $(T_{1,j})_{j\geq 0}, (T_{2,j})_{j\geq 0}, ...,$ $(T_{N,j})_{j\geq 0}$, $N \geq 2$ sequences of operator in $B(X)$. We say that the sequences $(T_{1,j})_{j\geq 0}, (T_{2,j})_{j\geq 0}, ..., (T_{N,j})_{j\geq 0}$ satisfy the $d{-}M$ universality criterion with respect to some (n_k), if there exist dense subsets $M_0, M_1, ..., M_N$ of M, a strictly increasing sequence of positive integers (n_k), and mapping $S_{l,k} : M_l \to M$, $(1 \leq l \leq N, k \in \mathbb{N})$ such that for each $1 \leq l \leq N$ we have:

1. $T_{l,n_k} \to_{k\to\infty} 0$ pointwise on M_0;
2. $S_{l,k} \to 0$ pointwise on M_l;
3. $(T_{l,n_k} S_{i,k} y_i - \delta_{i,l} y_i) \to_{k\to\infty} 0$ pointwise on M_l;
4. $T_{l,n_k}(M) \subset M$ $(1 \leq l \leq N)$.

Let Let M be a nonzero subspace of X. In this work, we will characterize properties of being $d-M$ mixing for a $N \geq 2$ sequence $T_{1,j}, T_{2,j}, ..., T_{N,j}$ of operators in $B(X)$. Also, we will give necessary and sufficient conditions for a $N \geq 2$ sequence $T_{1,j}, T_{2,j}, ..., T_{N,j}$ of operators in $B(X)$ to satisfies $d-M$ universality criterion in terms of d-M topologically transitivity of this sequence.

2. Main results

We begin with the following result.

Theorem 2.1 *Let $T_{1,j}, T_{2,j}, ..., T_{N,j}$, $N \geq 2$ sequences of operators in $B(X)$ and M a non-empty subspace of X. The following statement are equivalent:*

1. *$(T_{1,j}), (T_{2,j}), ..., (T_{N,j})$ are d-subspace mixing.*
2. *For any nonempty open subsets $V_0, V_1, ..., V_N$ in M, there exists $n \in \mathbb{N} \setminus \{0\}$ such that $V_0 \bigcap \bigcap_{i=1}^{N} T_{i,j}^{-1}(V_i)$ is a relatively nonempty open subset of M for all $j \geq n$.*
3. *For any nonempty open subsets $V_0, V_1, ..., V_N$ in M, there exists $n \in \mathbb{N} \setminus \{0\}$ such that $V_0 \bigcap \bigcap_{i=1}^{N} T_{i,j}^{-1}(V_i) \neq \emptyset$ and $T_{i,j}(M) \subset M$, for all $j \geq n$.*

Proof. (2) $\Rightarrow$ (1) is clear.
(3) $\Rightarrow$ (2) Suppose that $V_0, V_1, ..., V_N$ are $N \geq 2$ nonempty open subset of M, hence by (3) we conclude that there exists $n \in \mathbb{N} \setminus \{0\}$ such that $V_0 \bigcap \bigcap_{i=1}^{N} T_{i,j}^{-1}(V_i) \neq \emptyset$ and $T_{i,j}(M) \subset M$. Since the restricted operator $T_{i,j|M}$ is continuous, then $T_{i,j}^{-1}(V_i)$ is open $\forall \ \ j \geq n \ \ , i = 1, 2, ..., N$. Hence $V_0 \bigcap \bigcap_{i=1}^{N} T_{i,j}^{-1}(V_i)$ is a relatively open nonempty subset.
(1) $\Rightarrow$ (3) Assume that there exist $n \geq 0$such that $V_0 \bigcap \bigcap_{i=1}^{N} T_{i,j}^{-1}(V_i)$ contains a nonempty opens subset of M, then there exists $W \neq \emptyset$ an open subset of M such that $W \subset V_0 \bigcap \bigcap_{i=1}^{N} T_{i,j}^{-1}(V_i)$, hence $W \subset V_0$ and $W \subset \bigcap_{i=1}^{N} T_{i,j}^{-1}(V_i)$ this implies that $Ti,j(W) \subset V_i \ \ \forall j \geq n, \ \ i = 1, 2, ..., N$. Let $x \in M$ and $x_0 \in W$, then there exists r small enough such that $x_0 + rx \in W$, hence $T_{i,j}(x_0 + rx) \in T_{i,j}(W) \subset V_i \subset M$, $\forall i = 1, 2, ..., N$; $\forall j \geq n$, $T_{i,j}x := \frac{1}{r} T_{i,j}(x_0 + rx) - T_{i,j}(x_0) \in M$, therefore

$$T_{i,j}(x_M) \subset M \quad \text{for all} \ \ j \geq n, i = 1, 2, ..., N. \qquad \blacksquare$$

The following lemma will be used in the sequel.

Lemma 2.1 *Let M be a nonzero subspace of X and $(T_{1,j})_{j\geq 0}$, $(T_{2,j})_{j\geq 0}$,...,$(T_{N,j})_{j\geq 0}$ a $N \geq 2$ sequences satisfying the $d-M$ universal criterion with respect to some (n_k), then $(T_{1,n_k}), (T_{2,n_k}), ..., (T_{N,n_k})$ are $d-M$ mixing. In particular,*

$$(T_{1,j})_{j\geq 0}, (T_{2,j})_{j\geq 0}, ..., (T_{N,j})_{j\geq 0} \text{ are } d-M \text{ universal.}$$

Proof. Let $V_0, V_1, ..., V_N$ be nonempty open subsets of M, let $y_l \in V_l \bigcap M_l$ and $\varepsilon \geq 0$, so that

$$B(y_l, (N+1)\varepsilon) \subset V_l, \quad (0 \leq l \leq N).$$

Since $(T_{1,j})_{j\geq 0}, (T_{2,j})_{j\geq 0}, ..., (T_{N,j})_{j\geq 0}$ is a $N \geq 2$ sequences satisfying the $d-M$ universal criterion with respect to some (n_k), then by Definition 1.4, there exists $n_0 \in \mathbb{N}$ so that $T_{l,n_k}y_0, S_{l,k}y_l$ and $(T_{l,n_k}y_0 S_{i,k}y_i - \delta_{i,l}y_i)$ belong to $B(0,\varepsilon)$ for $k \geq k_0$ and $1 \leq i \leq N$. For each $k \geq k_0$, set $z_k = y_0 + \sum_{i=1}^{N} S_{i,k}y_i$, we have $S_{i,k}y_i \in B(0,\varepsilon)$, this implies that $\sum_{i=1}^{N} S_{i,k}y_i \in B(0, N\varepsilon)$, hence $z_k \in B(y_0,\varepsilon) \subset B(y_0,(N+1)\varepsilon) \subset V_0$ and $T_{l,n_k}z_k = T_{l,n_k}y_0 + \sum_{i=1}^{N} T_{l,n_k}S_{i,k}y_i$. Since $(T_{l,n_k}S_{i,k}y_i - \delta_{i,l}y_i) \in B(0,\varepsilon)$, then there exists $r \in B(0,\varepsilon)$ such that $\sum_{i=1}^{N} T_{l,n_k}\delta_{i,k}y_i = \sum_{i=1}^{N}(r + \delta i, l y_i = \sum_{i=1}^{N} r + y_l$, hence

$$T_{l,n_k}z_k = T_{l,n_k}y_0 + \sum_{i=1}^{N} r + y_l.$$

We have $T_{l,n_k}y_0 \in B(0,\varepsilon)$ and $r \in B(0,\varepsilon)$, then $T_{l,n_k}z_k \in B(y_l,(N+1)\varepsilon) \subset V_l$, this implies implies that $z_k \in T_{l,n_k}^{-1}(V_l)$ for each $1 \leq l \leq N$, so

$$V_0 \cap T_{1,n_k}^{-1}(V_1) \cap T_{2,n_k}^{-1}(V_2) \cap ... \cap T_{N,n_k}^{-1}(V_N) \neq \emptyset \text{ for } k \geq k_0. \qquad \blacksquare$$

In the following theorem, we will give necessary and sufficient conditions for a $N \geq 2$ sequence $T_{1,j}, T_{2,j}, ..., T_{N,j}$ of operators in $B(X)$ to satisfy $d-M$ universality criterion in terms of d-M topologically transitivity of this sequence. Note that that the proof of the following theorem is inspired by Bès and Peris [4, Theorem 2. 7].

Theorem 2.2 *Let M be a nonzero subspace of X, and $(T_{1,j})_{j\geq 0}, (T_{2,j})_{j\geq 0}, ...,$ $(T_{N,j})_{j\geq 0}$, $N \geq 2$, sequences of operator in $B(X)$. The following are equivalent :*

1. *$(T_{1,j})_{j\geq 0}, (T_{2,j})_{j\geq 0}, ..., (T_{N,j})_{j\geq 0}$ satisfy the $d-M$ universality criterion.*
2. *There exists a strictly increasing sequence of positive integers (n_k) such that for each subsequence (n_{k_j}) of (n_k), there exists a dense set of vectors $z \in X$ for which $\{(T_{1,n_{k_j}}z, T_{2,n_{k_j}}z, ..., T_{N,n_{k_j}}z), j \in \mathbb{N}\} \bigcap M^N$ is dense in M^N*
3. *for each $r \in \mathbb{N}$,*

   $$T_{1,j} \bigoplus T_{1,j} \bigoplus ... \bigoplus T_{1,j}(r\ time), ..., T_{N,j} \bigoplus T_{N,j} \bigoplus ... \bigoplus T_{N,j}(r\ time)$$

   are d-M topologically transitive.

Proof. (1) $\Rightarrow$ (2) Suppose that the sequences $(T_{1,j})_{j\geq 0}, (T_{2,j})_{j\geq 0}, ..., (T_{N,j})_{j\geq 0}$, $N \geq 2$ satisfy the $d-M$ universality criterion, then there exists a increasing sequence of positive integers (n_k) which satisfies conditions of Definition 1.4. If (n_{k_j}) is any subsequence of (n_k), then $(T_{1,j})_{j\geq 0}, (T_{2,j})_{j\geq 0}, ..., (T_{N,j})_{j\geq 0}$ satisfies the $d-M$ universality criterion with respect to it. Hence by lemma 2.1, $T_{1,n_{k_j}}, T_{2,n_{k_j}}, ..., T_{N,n_{k_j}}$, are $d-M$ mixing, so $T_{1,n_{k_j}}, T_{2,n_{k_j}}, ..., T_{N,n_{k_j}}$, are $d-M$ topologically transitive. Thus, $\{(T_{1,n_{k_j}}z, T_{2,n_{k_j}}z, ..., T_{N,n_{k_j}}z), j \in \mathbb{N}\} \bigcap M^N$ is dense in M^N.

(2) $\Rightarrow$ (3) Suppose that there exists a strictly increasing sequence of positive integers (n_k) such that, for any subsequence (n_{k_j}) of (n_k), there exists a dense set of vectors z for which $\{(T_{1,n_{k_j}}z, T_{2,n_{k_j}}z, ..., T_{N,n_{k_j}}z), j \in \mathbb{N}\} \bigcap M^N$ is dense in M^N. Let $r \in \mathbb{N}$ be fixed and for each $l = 0, 1, ..., N$ and $k = 1, 2, ..., r$, let $V_{l,k} \subset M$ be open and nonempty, we have to show that there exist $m \in \mathbb{N}$ so that:

$$\emptyset \neq V_{0,k} \bigcap_{l=1}^{N} T_{l,m}^{-1}(V_{l,k}) \quad (1 \leq k \leq r).$$

Let $(n_{1,k})$ be a subsequence of (n_k), since $dU(T_{1,n_{1,k}}, T_{2,n_{1,k}}, ..., T_{N,n_{1,k}}, M) \bigcap M^N$ is dense in M^N. Then

$$\emptyset \neq V_{0,1} \bigcap_{l=1}^{N} T_{l,n_{1,k}}^{-1}(V_{l,1}) \quad (k \in \mathbb{N}).$$

Next, since $dU(T_{1,n_{1,k}}, T_{2,n_{1,k}}, ..., T_{N,n_{1,k}}, M) \bigcap M^N$ is dense in M^N, then there exist a subsequence $(n_{2,k})$ of $(n_{1,k})$ so that $\emptyset \neq V_{0,2} \bigcap_{l=1}^{N} T_{l,n_{2},k}^{-1}(V_{l,2})$. By the same way and after r steps, we obtain a chain of subsequences $(n_{r,k}) \subset ... \subset (n_{1,k}) \subset (n_k)$, so that

$$\emptyset \neq V_{0,j} \bigcap_{l=1}^{N} T_{l,n_{r,k}}^{-1}(V_{l,j}) \qquad (1 \leq j \leq r) \text{ for all } k \in \mathbb{N},$$

hence we can pick $m := n_{r,1}$.

(3) $\Rightarrow$ (1): Suppose that $(T_{1,j}), (T_{2,j}), ..., (T_{N,j})$ satisfy: for each $r \in \mathbb{N}$ and nonempty open $V_{l,k} (0 \leq l \leq N \quad , \quad 1 \leq k \leq r)$ of M there exists $m \in \mathbb{N}$ arbitrarily large with

$$V_{0,k} \bigcap_{l=1}^{N} T_{l,m}^{-1}(V_{l,k}) \neq \emptyset \quad (1 \leq k \leq r). \tag{$*$}$$

Let $(A_{0,n})_{n \geq 0}$ be a basis for the topology of M and $\{(A_{1,n}), (A_{2,n}), ..., (A_{N,n})\}$ be a basis of nonempty set for the product topology of M^N. For each $n \in \mathbb{N}$ and $l = 0, 1, 2, ..., N$, let $A_{l,n,0} := A_{l,n}$ and $W_n = B(0, \frac{1}{n})$.

First step:

Denote by $D(A)$ the diameter of nonempty set A.

Let $A_{l,1,1} \subset A_{l,1,0} \quad (1 \leq l \leq N)$ open set such that $D(A_{l,1,1}) < \frac{1}{2} D(A_{l,1,0})$, hence $\overline{A_{l,1,1}} \subset A_{l,1,0}$ by $(*)$, there exists $n_1 > 1$ so that

$$\begin{cases} \emptyset \neq A_{0,1,0} \bigcap \bigcap_{l=1}^{N} T_{l,n_1}^{-1}(W_l); \\ \emptyset \neq W_1 \bigcap T_{l,n_1}^{-1}(A_{l,1,1}) \bigcap \bigcap_{s \neq l} T_{s,n_1}^{-1}(W_l), \quad (1 \leq l \leq N). \end{cases} \tag{$**$}$$

Next, get $A_{0,1,1}$ nonempty open subset of $A_{0,1,0}$ such that $D(A_{0,1,1}) < \frac{1}{2} D(A_{0,1,0})$, then $\overline{A_{0,1,1}} \subset A_{0,1,0}$, this implies that $T_{l,n_1}(\overline{A_{0,1,1}}) \subset W_1 \qquad (1 \leq l \leq N)$ also by $(**)$, we pick $W_{s,1,1} \in W_1 \quad (1 \leq s \leq N)$ so that for each $1 \leq l \leq N$ we have

$$T_{l,n_1} W_{s,1,1} \in \begin{cases} A_{l,1,0}, & \text{if } s = l; \\ W_1, & \text{if } s \neq l. \end{cases}$$

Second step:
for $k = 1, 2$ let $A_{l,k,3-k}$ nonempty open subset of $A_{l,k,2-k}$ such that

$$D(A_{l,k,3-K}) < \frac{1}{3} D(A_{l,k,2-k}),$$

so that $\overline{A_{l,k,3-k}} \subset A_{l,k,3-k}$ and $\overline{A_{l,2,1}} \bigcap \overline{A_{l,1,2}} = \emptyset \quad (1 \leq l \leq N)$. By $(*)$ there exist $n_2 > n_1$ such that

$$\begin{cases} \emptyset \neq A_{0,k,2-k} \bigcap \bigcap_{l=1}^{N} T_{l,n_2}(W_2); \\ \emptyset \neq W_2 \bigcap T_{l,n_2}(A_{l,k,3-k}) \bigcap \bigcap_{s \neq l} T_{s,n_2}(W_2), \quad (1 \leq l \leq N) \quad (k = 1, 2). \end{cases}$$

Next, for $k = 1, 2$ and $l = 1, 2, ..., N$, we get $W_{l,k,3-k} \in W_2$ and nonempty open subset $A_{0,k,3-k}$ of $A_{0,k,2-k}$ such that

$$D(A_{0,k,3-K}) < \frac{1}{3} D(A_{0,k,2-k}),\ T_{l,n_2}(\overline{A_{0,k,3-k}}) \subset W_2$$

and for: $1 \leq s \leq N$,

$$T_{l,n_2} W_{s,k,3-k} \in \begin{cases} A_{l,k,3-k} & if s = l \\ W_2; & if s \neq l. \end{cases}$$

If we continue this process inductively by $(*)$, on each step, we obtain an increasing sequence of positives integer $1 < n_1 < n_2 < ....$ and for each $l \in \{1, 2, ..., N\}$ and each $i \in \mathbb{N}$ the nonempty open sets $A_{l,k,i+1-k} \quad (1 \leq k \leq i)$ such that $D(A_{l,k,i+1-k}) < \frac{1}{i+1} D(A_{l,k,i-k})$ and $W_{l,k,i+1-k} \in W_i$ satisfy

1. $\overline{A_{l,k,i+1-k}} \subset A_{l,k,i-k} \subset A_{l,k}$.
2. Each collection $\{\overline{A_{l,k,i+1-k}} : \quad 1 \leq k \leq i\}$ is pairwise disjoint.
3. $T_{l,n_i}(A_{0,k,i+1-k}) \subset W_i$.
4. For $1 \leq s \leq N$, $T_{l,n_i} W_{s,k,i+1-k} \in \begin{cases} A_{l,k,i+1-k}, & \text{if } s = l; \\ W_i, & \text{if } s \neq l. \end{cases}$

For each fixed l, $(0 \leq l \leq N)$ and $m \in \mathbb{N}$ there exists a unique $a_{l,m} \in M$ so that $\{a_{l,m}\} = \bigcap_{j=m+1}^{\infty} \overline{A_{l,m,j-m}}$ note that $a_{l,m} \neq a_{l,n}$ by (2) if $n \neq m$, and that $M_l := \{a_{l,m} : m \in \mathbb{N}\}$ is dense in M. Consider $S_{l,m} : M_l \to M$ is defined by

$$S_{l,m} a_{l,k} := \begin{cases} W_{l,k,m+1-k}, & \text{if } m \geq k, \\ 0, & \text{if } 1 \leq m < k. \end{cases}$$

From (4), $S_{l,k} \to 0, k \to \infty$ point wise on $M_l \quad (1 \leq l \leq N)$. Also, by (4) we have,

$$T_{s,n_m} S_{l,m} a_{l,k} = T_{s,n_m} W_{l,k,m+1-k} \in \begin{cases} A_{l,k,m+1-k}, & \text{if } s = l, \\ W_m, & \text{if } s \neq l. \end{cases}$$

Hence $(T_{s,n_k} s_{l,k} - \delta_{s,l} Id M_l) \to 0,\ k \to \infty$ point wise on $M_l (1 \leq l \leq N)$. we have also $T_{l,n_k} \to 0, \quad k \to \infty$ point wise on $M_0 \quad (1 \leq l \leq N)$. It easy to see that

$T_{l,n_k}(M) \subset M$ for $1 \leq l \leq N, k \in \mathbb{N}$. Finally, $(T_{1,j}),(T_{2,j}),...,(T_{N,j})$ satisfies the $d-M$ universality criterion. ∎

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References

[1] Ansari, S.I., *Existance of hypercyclic operatos on topological vector space,* J. Funct. Anal., 148 (1997), 384-390.

[2] Bayart, F., Matheron, E., *Dynamics of Linear Operators,* Cambridge Tracts in Mathematics 179, Cambridge University Press, 2009.

[3] Bernal-González, L., *Disjoint hypercyclic operators,* Studia Math, 182 (2) (2007), 113-130.

[4] Bès, J.P., Peris, A., *Disjointness in hypercyclicity.* J. Math. Anal. Appl., 336 (2007), 297-315.

[5] Bès, J.P., Martin, Ö., Sanders, R., *Weighted Shifts And Disjoint Hypercyclicity,* J. Operator. Theory, (72) (2014), 15-40.

[6] Madore, B.F., Matínez-Avendaño, R.A., *Subspace Hypercyclicity,* J. Math. Anal. Appl., (2011), 502-511.

[7] Costakis, G., Peris, A., *Hypercyclic semigroups and somewhere dense orbits,* C.R. Math. Acad. Sci. Paris, 335 (2002), 895-898.

[8] Engel, K.-J., Nagel, R., *One-Parameter Semigroups for Linear Evolution Equations,* Springer-Verlag New York, 2000.

[9] Erdmann, K.G., Peris, A., *Linear Chaos* , Universitext. Springer, London, 2011.

[10] Pazy, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag New York, 1983.

[11] Tajmouati, A., Amouch, M., Alhomidi Zakaria, M.R.F., Abkari, M., *Universality and transitivity of Semigroups of Operators*, International Journal of Pure and Applied Mathematics, 106 (4) (2016), 1087-1094.

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