FUZZY p-IDEALS IN MV-ALGEBRAS

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Abstract. In this paper, we introduce the notion of p-ideals of MV-algebras and characterization of p-ideals is given. Also, we show that p-ideals equivalent to Boolean ideals in MV-algebras. In addition, we introduce the notion of fuzzy p-ideals of an MV-algebra and show that in any MV-algebra, the concept of fuzzy p-ideals is equivalent to fuzzy Boolean ideals and fuzzy implicative ideals. Also, several characterizations of these fuzzy ideals are given and prove that extension theorem of fuzzy p-ideals. Furthermore, we describe the transfer principle for fuzzy p-ideals in terms of level subsets. Finally, by using the notions of maximal and normal fuzzy p-ideals, we show that under certain conditions a fuzzy p-ideal is two valued and takes the values 0 and 1.

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1. Introduction and preliminaries

C. Chang introduced the notion of MV-algebras to provide a proof for the completeness of the Łukasiewicz axioms for infinite valued propositional logic [1]. In fact MV-algebras are now algebraic counterparts of Łukasiewicz many valued logics. Also, D. Mundici [10] extended such a correspondence to a functor Γ from lattice ordered abelian groups with strong unit to MV-algebras.

In [8], Iseki proposed the notion of implicative ideals in BCK-algebras and obtained some results. Subsequently, Hoo and Sessa [6] proposed the notion of Boolean ideals in MV-algebras and proved that implicative ideals and Boolean ideals are equivalent in MV-algebras.

The concept of fuzzy set was introduced by Zadeh (1965) [13]. This idea has been applied to other algebraic structures such as semi-group, group, ideals, modules and topologies.

In 1991, Xi [12] applied the concept of fuzzy sets to BCK-algebras and proposed the notion of fuzzy implicative ideals. Afterwards, Hoo [5] proved that fuzzy implicative and fuzzy Boolean ideals are equivalent in MV-algebras.

In this paper, we introduce the notion of fuzzy p-ideals. We obtain some equivalent definitions of fuzzy p-ideals, and establish the extension theorem of fuzzy p-ideals in MV-algebras.

Furthermore, we prove that fuzzy Boolean ideals and fuzzy p-ideals are equivalent and using a level set of fuzzy set in an MV-algebra, we give a characterization of fuzzy p-ideals.

We define the concept of p-ideals and we give characterization of them. We prove that p-ideals are equivalent to Boolean ideals in MV-algebras.

Finally, we introduce normal fuzzy p-ideals and we show that the maximal fuzzy p-ideals of an MV-algebra A are normal and take only the values 0 and 1.

We recollect some definitions and results which will be used in the following:

Definition 1.1. [1], [2], [11] An MV-algebra is a structure $(A, \oplus, {}^*, 0)$ where $\oplus$ is a binary operation, * is a unary operation, and 0 is a constant such that the following axioms are satisfied for any $a, b \in A$:

$(MV1)$ $(A, \oplus, 0)$ is an abelian monoid,

$(MV2)$ $(a^*)^* = a$,

$(MV3)$ $0^* \oplus a = 0^*$,

$(MV4)$ $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

Note that $1 = 0^*$ and the auxiliary operation $\odot$ as follows: $x \odot y = (x^* \oplus y^*)^*$.

We recall that the natural order determines a bounded distributive lattice structure such that

$$x \vee y = x \oplus (x^* \odot y) = y \oplus (x \odot y^*) \quad \text{and} \quad x \wedge y = x \odot (x^* \oplus y) = y \odot (y^* \oplus x).$$

Lemma 1.2. [2], [11] *In each MV-algebra, the following relations hold for all $x, y, z \in A$:*

(1) $x \leq y$ *if and only if* $y^* \leq x^*$,

(2) *If* $x \leq y$, *then* $x \oplus z \leq y \oplus z$ *and* $x \odot z \leq y \odot z$,

(3) $x \leq y$ *if and only if* $x^* \oplus y = 1$ *if and if* $x \odot y^* = 0$,

(4) $x, y \leq x \oplus y$ *and* $x \odot y \leq x, y$, $x \leq nx = x \oplus x \oplus \cdots \oplus x$ *and* $x^n = x \odot x \odot \cdots \odot x \leq x$,

(5) $x \oplus x^* = 1$ *and* $x \odot x^* = 0$,

(6) *If* $x \leq y$ *and* $z \leq t$, *then* $x \oplus z \leq y \oplus t$,

(7) $x \odot (y \vee z) = (x \odot y) \vee (x \odot z)$.

Definition 1.3. [1], [11] An ideal of an MV-algebra A is a nonempty subset I of A satisfying the following conditions:

$(I1)$ If $x \in I$, $y \in A$ and $y \leq x$ then $y \in I$,

$(I2)$ If $x, y \in I$, then $x \oplus y \in I$.

We denote by $Id(A)$ the set of ideals of an MV-algebra A.

Definition 1.4. [2] Let I be an ideal of an MV-algebra A. Then I is a proper ideal of A if $I \neq A$.

- [6] An ideal I of an MV-algebra A is called Boolean ideal if $x \wedge x^* \in I$, for all $x \in A$.
- [6] An ideal I of an MV-algebra A is called an implicative ideal of A if for any $x, y, z \in A$ such that $z \odot (y^* \odot x^*) \in I$ and $y \odot x^* \in I$, then $z \odot x^* \in I$.

Definition 1.5. Let X and Y be two MV-algebras. A function $f : X \to Y$ is called homomorphism of MV-algebras if and only if

(1) $f(0) = 0$,

(2) $f(x \oplus y) = f(x) \oplus f(y)$,

(3) $f(x^*) = (f(x))^*$.

Remark 1.6. [2] In an MV-algebra A, the distance function is

$$d: \quad A \times A \longrightarrow A, \qquad d(x, y) := (x \odot y^*) \oplus (y \odot x^*).$$

Suppose that I is an ideal of an MV-algebra A. Define $x \sim_I y$ if and only if $d(x, y) \in I$ if and only if $x \odot y^* \in I$ and $y \odot x^* \in I$. Then $\sim_I$ is a congruence relation on A. The set of all congruence classes is denoted by A/I then $A/I = \{[x] : x \in A\}$, where $[x] = \{y \in A : x \sim_I y\}$. We can easily to see that $x \in I$ if and only if $x/I = 0/I$. The MV-algebra operations on A/I given by $x/I \oplus y/I = (x \oplus y)/I$ and $(x/I)^* = x^*/I$, are well defined. Hence $(A/I, \oplus, *, [0])$ becomes an MV-algebra [2], [11].

Definition 1.7. [4] A fuzzy set in A is a mapping $\mu : A \to [0, 1]$. Let A be an MV-algebra and μ be a fuzzy set on A. Then μ is a fuzzy ideal of A, if it satisfies

$(MV1)$ $\mu(0) \geq \mu(x)$, for all $x \in A$,

$(MV2)$ $\mu(y) \geq \mu(x) \wedge \mu(y \odot x^*)$, for all $x, y \in A$.

Lemma 1.8. [4] *Let A be an MV-algebra and μ be a fuzzy set on A. Then μ is called a fuzzy ideal on A, if and only if*

(1) *$\mu(x) \leq \mu(0)$, for all $x \in A$ and*

(2) *$\mu(x \oplus y) \geq \mu(x) \wedge \mu(y)$, for all $x, y \in A$,*

(3) *If $x \leq y$, then $\mu(x) \geq \mu(y)$.*

Definition 1.9. [13] Let X, Y be two MV-algebras. μ is a fuzzy subset of X, μ' is a fuzzy subset of Y and $f : X \to Y$ is a homomorphism. The image of μ under f denoted by $f(\mu)$ is a fuzzy set of Y defined by:

For all $y \in Y$,

$$\begin{aligned} f(\mu)(y) &= \sup_{x \in f^{-1}(y)} \mu(x), \quad \text{if } f^{-1}(y) \neq \emptyset, \text{ and} \\ f(\mu)(y) &= 0 \qquad\qquad\qquad \text{if } f^{-1}(y) = \emptyset. \end{aligned}$$

The preimage of μ' under f denoted by $f^{-1}(\mu')$ is a fuzzy set of X defined by:

For all $x \in X$,

$$f^{-1}(\mu')(x) = \mu'(f(x)).$$

Definition 1.10. [13] A fuzzy subset μ of X has sup-property if for any nonempty subset Y of X, there exists $y_0 \in Y$ such that $\mu(y_0) = \sup_{y \in Y} \mu(y)$.

Definition 1.11. [13] Let μ be a fuzzy set in A, $t \in [0, 1]$. The set $\mu_t = \{x \in A : \mu(x) \geq t\}$ is called a level subset of μ. For any fuzzy sets μ, ν in A, we define

$$\mu \subseteq \nu \text{ if and only if } \mu(x) \leq \nu(x) \text{ for all } x \in A.$$

Theorem 1.12. [4] *Let μ be a fuzzy ideal in A. For any $x, y, z \in A$, the following hold:*

(1) $\mu(x \oplus y) = \mu(x) \wedge \mu(y)$,

(2) $\mu(x \odot y) \geq \mu(x) \wedge \mu(y)$,

(3) $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$,

(4) $\mu(nx) = \mu(x)$,

(5) $\mu(x \vee y) = \mu(x) \wedge \mu(y)$.

Theorem 1.13. [3] *Let μ be a fuzzy set in A. μ is a fuzzy ideal if and only if for all $t \in [0, 1]$, μ_t is either empty or an ideal of A.*

Corollary 1.14. *Let I be a nonempty subset of A. I is an ideal if and only if χ_I is a fuzzy ideal of A, where χ_I is characteristic function of I.*

2. P-ideals of MV-algebras

Definition 2.1. I is a p-ideal if it satisfies the following conditions:

(i) $0 \in I$,

(ii) For all $x, y, z \in A$, if $y \odot (z^* \oplus y) \odot x^* \in I$ and $x \in I$, then $y \in I$.

Example 2.2. Let $A = \{0, a, b, 1\}$, where $0 < a, b < 1$. Define $\odot$, $\oplus$ and $*$ as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

$*$	0	a	b	1
	1	b	a	0

Then $(A, \oplus, \odot, *, 0, 1)$ is an MV-algebra [7]. Simple computations prove that $I_1 = \{0, a\}$ and $I_2 = \{0, b\}$ are p-ideals of A.

The following example shows that an ideal may not be a p-ideal.

Example 2.3. Let $A = \{0, a, b, c, d, 1\}$, where $0 < a, c < d < 1$ and $0 < a < b < 1$. Define $\odot$, $\oplus$ and $*$ as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	a	0	0	a
b	0	a	b	0	a	b
c	0	0	0	c	c	c
d	0	0	a	c	c	d
1	0	a	b	c	d	1

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	b	b	d	1	1
b	b	b	b	1	1	1
c	c	d	1	c	d	1
d	d	1	1	d	1	1
1	1	1	1	1	1	1

$*$	0	a	b	c	d	1
	1	d	c	b	a	0

Then $(A, \oplus, \odot, *, 0, 1)$ is an MV-algebra [7]. It is clear that $I = \{0, c\}$ is an ideal but I is not a p-ideal, since $a \odot (b^* \oplus a) \odot c^* = 0 \in I$ and $c \in I$ while $a \notin I$.

Proposition 2.4. *Every p-ideal is an ideal.*

Proof. Let I be a p-ideal of A, it is clear that $0 \in I$. Suppose that $y \leq x$ and $x \in I$, for all $x, y \in A$. By Lemma 1.2, we imply that $y \odot x^* = 0$, by setting $z = 0$ in the definition of p-ideal, we obtain $y \odot (0^* \oplus y) \odot x^* = y \odot x^* = 0 \in I$ and $x \in I$, it follows that $y \in I$.

Now, let $x, y \in I$. We prove that $x \oplus y \in I$. Since $x^* \wedge y \leq y \in I$, hence $x^* \wedge y \in I$. So $x^* \wedge y = x^* \odot (x \oplus y) \geq (x \oplus y) \odot (y^* \oplus (x \oplus y)) \odot x^*$ and $(x \oplus y) \odot (y^* \oplus (x \oplus y)) \odot x^* \in I$. Since I is a p-ideal of A and $x \in I$, it follows that $x \oplus y \in I$. ∎

The following proposition gives a characterization of p-ideals of A.

Proposition 2.5. *The following conditions are equivalent for any ideal I:*

(i) *I is a p-ideal,*

(ii) *For all $x, y \in A$, $x \odot (y^* \oplus x) \in I$ implies $x \in I$,*

(iii) *If $x^2 \in I$, then $x \in I$.*

Proof. (i) → (ii) Suppose that I is a p-ideal of A and $x \odot (y^* \oplus x) \in I$, since $x \odot (y^* \oplus x) \odot 0^* \in I$ and $0 \in I$, we apply the fact that I is p-ideal of A and obtain the result.

(ii) → (iii) We obtain the result by setting $y = 1$ in the equation (ii).

(iii) → (i) Suppose that $y \odot (z^* \oplus y) \odot x^* \in I$ and $x \in I$. By ideal property, we conclude that $y \odot (z^* \oplus y) \leq x \vee (y \odot (z^* \oplus y)) = x \oplus (x^* \odot y \odot (z^* \oplus y)) \in I$. Hence $y \odot (z^* \oplus y) \in I$. On the other hand, $y \odot y = y \odot (1^* \oplus y) \leq y \odot (z^* \oplus y) \in I$. Since I is an ideal, we obtain $y^2 = y \odot y \in I$ and we apply the hypothesis and obtain $y \in I$. ∎

Theorem 2.6. *A proper ideal I is a p-ideal if and only if I is a Boolean ideal of A.*

Proof. Let I be a Boolean ideal of A. By Proposition 2.5, it sufficient to show that if $x^2 \in I$, then $x \in I$. On the other hand, we have for all $x \in A$, $x^2 = x \odot x \in I$ and $x \wedge x^* \in I$. Since I is an ideal, $(x \wedge x^*) \oplus x^2 = x \odot (x^* \oplus x^*) \oplus x^2 = (x \odot (x^2)^*) \oplus x^2 \in I$. On the other hand, $x \leq x^2 \vee x \in I$, thus $x \in I$.

Conversely, suppose that I is a p-ideal. Let $x \in A$. Setting $t = x \wedge x^*$, we show that $t \in I$. Since $t \leq x$, we have $x^* \wedge x \leq x^* \leq t^*$ and then $t \leq t^*$ or $t^2 = t \odot t = 0 \in I$. So since I is a p-ideal, by Proposition 2.5, we imply that $t \in I$. Thus I is Boolean ideal of A. ∎

By the above theorem, the extension theorem of p-ideals is obtained from the following result:

Theorem 2.7. *Let I_1 and I_2 two ideals of A such that $I_1 \subseteq I_2$. If I_1 is a p-ideal, then so is I_2.*

3. Fuzzy p-ideals in MV-algebras

Definition 3.1. Let μ be a fuzzy set in A. μ is called a fuzzy p-ideal if it satisfies

(i) $\mu(0) \geq \mu(x)$,

(ii) For all $x, y, z \in A$, $\mu(y \odot (z^* \oplus y) \odot x^*) \wedge \mu(x) \leq \mu(y)$.

The following example shows that fuzzy p-ideals exist.

Example 3.2. Consider Example 2.2. Define a fuzzy set μ in A by

$$\mu(1) = \mu(b) = \mu(a) = t_1,$$
$$\mu(0) = t_2 \quad (0 \leq t_1 < t_2 \leq 1).$$

We can easily verify that μ is a fuzzy p-ideal of A.

Theorem 3.3. *Every fuzzy p-ideal is a fuzzy ideal.*

Proof. Let μ be a fuzzy p-ideal in A. Taking $z = 0$ in Definition 3.1 (ii), we get

$$\mu(y \odot x^*) \wedge \mu(x) \leq \mu(y), \text{ for all } x, y \in A.$$

Hence μ is a fuzzy ideal. ■

Remark 3.4. The converse of the above theorem may not be true.

Let $A = \{0, a, b, c, d, 1\}$, where $0 < a, b < c < 1$ and $0 < b < d < 1$.

Define $\oplus$, $\odot$ and $*$ as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	c	c	1	1
b	b	c	d	1	d	1
c	c	c	1	1	1	1
d	d	1	d	1	d	1
1	1	1	1	1	1	1

$*$	0	a	b	c	d	1
	1	d	c	b	a	0

Then $(A, \oplus, \odot, *, 0, 1)$ is an MV-algebra [7].

Define a fuzzy set μ in A by

$$\mu(1) = \mu(a) = \mu(b) = \mu(c) = \mu(d) = t_2,\ \mu(0) = t_1\ (0 \leq t_2 < t_1 \leq 1).$$

Routine calculation shows that μ is a fuzzy ideal. But μ is not a fuzzy p-ideal, because

$$\mu(b \odot (d^* \oplus b) \odot 0^*) \wedge \mu(0) = t_1 \nleq \mu(b) = t_2.$$

The following Theorem gives a characterization of fuzzy p-ideals.

Theorem 3.5. *Let μ be a fuzzy ideal in A. The following are equivalent:*

(i) *μ is a fuzzy p-ideal,*

(ii) *For all $x, y \in A$, $\mu(x \odot (y^* \oplus x)) \leq \mu(x)$,*

(iii) *For all $x, y \in A$, $\mu(x^2) = \mu(x)$.*

Proof. (i) $\rightarrow$ (ii) Suppose that μ is a fuzzy p-ideal of A and we have

$$\mu(x \odot (y^* \oplus x)) = \mu(x \odot (y^* \oplus x) \odot 0^*) \wedge \mu(0) \leq \mu(x).$$

(ii) $\rightarrow$ (iii) By setting $y = 1$ in equation (ii), we obtain $\mu(x^2) \leq \mu(x)$. Also, since $x^2 \leq x$, then $\mu(x) \leq \mu(x^2)$. Thus, $\mu(x) = \mu(x^2)$.

(iii) $\rightarrow$ (i) We show that $\mu(y \odot (z^* \oplus y) \odot x^*) \wedge \mu(x) \leq \mu(y)$.

By ideal property, we get

$$y \odot (z^* \oplus y) \leq x \vee (y \odot (z^* \oplus y)) = x \oplus (x^* \odot y \odot (z^* \oplus y)).$$

Hence we have

$$\begin{aligned} \mu(y) &= \mu(y^2) = \mu(y \odot (1^* \oplus y)), \\ &\geq \mu(y \odot (z^* \oplus y)), \\ &\geq \mu(x \oplus (x^* \odot y \odot (z^* \oplus y)), \\ &= \mu(x) \wedge \mu(x^* \odot y \odot (z^* \oplus y)). \end{aligned}$$

Thus μ is a fuzzy p-ideal of A. ∎

Now, we describe the transfer principle [9] for fuzzy p-ideals in terms of level subsets:

Theorem 3.6. *Let μ be a fuzzy set in A. μ is a fuzzy p-ideal if and only if for each $t \in [0, 1]$, μ_t is either empty or a p-ideal of A.*

Proof. Let μ be a fuzzy p-ideal and for each $t \in [0, 1]$, $\mu_t \neq \emptyset$. We assume that $x_0 \in \mu_t$, i.e., $\mu(x_0) \geq t$. Since μ is a fuzzy p-ideal, $\mu(0) \geq \mu(x_0) \geq t$. On the other hand $0 \in \mu_t$. Next, if $y \odot (z^* \oplus y) \odot x^* \in \mu_t$ and $x \in \mu_t$, then $\mu(y \odot (z^* \oplus y) \odot x^*) \geq t$ and $\mu(x) \geq t$. Hence $\mu(y) \geq \mu(y \odot (z^* \oplus y) \odot x^*) \wedge \mu(x) \geq t$. Thus $\mu(y) \geq t$. Therefore, $y \in \mu_t$. This results μ_t is a p-ideal of A.

Conversely, let for each $t \in [0, 1]$, μ_t is either empty or a p-ideal in A. Since $x \in \mu_{\mu(x)}$, for any $x \in A$, we have $\mu_{\mu(x)}$ is a p-ideal of A. Thus $0 \in \mu_{\mu(x)}$. Hence $\mu(0) \geq \mu(x)$, for all $x \in A$.

Now, let $t = \mu(y \odot (z^* \oplus y) \odot x^*) \wedge \mu(x)$. Then $y \odot (z^* \oplus y) \odot x^* \in \mu_t$ and $x \in \mu_t$. Since μ_t is a p-ideal, $y \in \mu_t$. It follows that $\mu(y) \geq t = \mu(y \odot (z^* \oplus y) \odot x^*) \wedge \mu(x)$. Thus μ is a fuzzy p-ideal of A. ∎

Corollary 3.7. *Let I be a nonempty set of A. I is a p-ideal of A if and only if χ_I is a fuzzy p-ideal of A.*

Proof. Let I be a p-ideal of A. We will prove that χ_I is a fuzzy p-ideal. By Theorem 3.5, we show that $\chi_I(x^2) = \chi_I(x)$, for all $x \in A$.

If $x \in I$, then $x^2 = x \odot x \leq x$. Hence $x^2 \in I$. Thus $\chi_I(x^2) = \chi_I(x) = 1$.

If $x \notin I$, then since I is a p-ideal of A, by Proposition 2.5, we obtain $x^2 \notin I$. Hence $\chi_I(x^2) = \chi_I(x) = 0$.

Conversely, let χ_I be a fuzzy p-ideal. It is sufficient to show that by Proposition 2.5, if $x^2 \in I$, then $x \in I$.

If $x^2 \in I$, then $1 = \chi_I(x^2) = \chi_I(x)$. Hence $\chi_I(x) = 1$. It follows that $x \in I$. Thus by Proposition 2.5, I is a p-ideal of A. ■

The following theorem, shows the relation between fuzzy p-ideals and fuzzy Boolean ideals in A.

Theorem 3.8. *Every fuzzy ideal μ is fuzzy Boolean ideal if and only if μ is fuzzy p-ideal in A.*

Proof. Let μ be a fuzzy Boolean ideal of A. Then $\mu(x \wedge x^*) = \mu(0)$. By Theorem 1.12 (1) and Lemma 1.8 (3), we have:

$$\begin{aligned} \mu(x^2) &= \mu(0) \wedge \mu(x^2) \\ &= \mu(x \wedge x^*) \wedge \mu(x^2) \\ &= \mu((x \wedge x^*) \oplus x^2) \\ &= \mu((x \odot (x^* \oplus x^*)) \oplus (x \odot x)) \\ &= \mu((x \odot (x^2)^*) \oplus x^2) \\ &= \mu(x^2 \vee x) \leq \mu(x). \end{aligned}$$

Hence $\mu(x^2) \leq \mu(x)$. Also, since $\mu(x) \leq \mu(x^2)$, we get $\mu(x^2) = \mu(x)$. Thus by Theorem 3.5, we obtain μ is a fuzzy p-ideal of A.

Conversely, let μ be a fuzzy p-ideal and $x \in A$. Setting $t = x \wedge x^*$, we show that $\mu(t) = \mu(0)$. Since $t \leq x$, $t = x^* \wedge x \leq x^* \leq t^*$ and then $t \leq t^*$ or $t^2 = t \odot t = 0$. Hence

$$\mu(t^2) = \mu(0) \tag{1}$$

Also, since μ is a fuzzy p-ideal, $\mu(t^2) = \mu(t)$. It follows from (1) that $\mu(0) = \mu(t)$. Therefore, μ is a fuzzy Boolean ideal of A. ■

The extension theorem of fuzzy p-ideals is obtained from the following result:

Theorem 3.9. *Let μ, ν be two fuzzy ideals which satisfy $\mu \leq \nu$, $\mu(0) = \nu(0)$. If μ is a fuzzy p-ideal, so is v.*

In general, it is not difficult to see the following:

Theorem 3.10.

(i) *Let μ_i $(i \in \Gamma)$ be a fuzzy p-ideal. Then $\wedge_{i\in\Gamma}\mu_i$ is a fuzzy p-ideal of A.*

(ii) *If $\wedge_{i\in\Gamma}\mu_i$ is a fuzzy p-ideal of A, then by Theorem* 3.9 *and $\wedge_{i\in\Gamma}\mu_i \leq \mu_i$, we get that μ_i, for all $i \in \Gamma$ is a fuzzy p-ideal of A.*

(iii) *Let μ_i, for all $i \in \Gamma$ be a fuzzy p-ideal of A. Then, by Theorem* 3.9, $\vee_{i\in\Gamma}\mu_i$ *is a fuzzy p-ideal of A.*

Remark 3.11. Let $f : X \to Y$ be onto MV-homomorphism. Then prove that preimage of a fuzzy p-ideal μ under f is also a fuzzy p-ideal of A.

Proof. Suppose that μ is a fuzzy p-ideal of Y. By Theorem 3.5, we have

$$f^{-1}(\mu)(x^2) = \mu(f(x^2)) = \mu((f(x))^2) = \mu(f(x)) = f^{-1}(\mu)(x).$$

It follows from Theorem 3.5 that $f^{-1}(\mu)$ is a fuzzy p-ideal of X. ■

Proposition 3.12. *Let $f : X \to Y$ be an onto MV-homomorphism. The image $f(\mu)$ of a fuzzy p-ideal μ with a sup-property is also a fuzzy p-ideal of A.*

Proof. By Theorem 3.5, it is sufficient to show that $f(\mu)(y^2) = f(\mu)(y)$, for all $y \in Y$.

Let $y \in Y$ and $x \in f^{-1}(y)$ such that $\mu(x) = sup_{t\in f^{-1}(y)}\mu(t)$. We have

$$f(\mu)(y) = \sup_{t\in f^{-1}(y)} \mu(t) = \mu(x) = \mu(x^2) = \sup_{t\in f^{-1}(y^2)} \mu(t) = f(\mu)(y^2).$$ ■

Theorem 3.8 and [5, Theorem 2.8] state that relations among fuzzy implicative, fuzzy Boolean and fuzzy p-ideals in A.

Corollary 3.13. *Let μ be a fuzzy ideal of A. The following are equivalent:*

(1) *μ is a fuzzy implicative ideal,*

(2) *μ is a fuzzy Boolean ideal,*

(3) *μ is a fuzzy p-ideal,*

(4) *$A/\mu_{\mu(0)}$ is a Boolean algebra.*

4. Normalizations of fuzzy p-ideals

Definition 4.1. A fuzzy p-ideal μ of A is said to be normal if there exists $x \in A$ such that $\mu(x) = 1$.

Example 4.2. Let $A = \{0, a, b, 1\}$ be an MV-algebra in Example 2.2. Then the fuzzy set μ in A defined by $\mu(0) = 1$ and $\mu(a) = \mu(b) = \mu(1) = t$, $(0 \leq t \leq 1)$ is a normal fuzzy p-ideal of A.

Proposition 4.3. *Given a fuzzy p-ideal μ of A, let μ^+ be a fuzzy set in A defined by $\mu^+(x) = \mu(x) + 1 - \mu(0)$, for all $x \in A$. Then μ^+ is a normal fuzzy p-ideal of A which contains μ*

Proof. Let μ be a fuzzy p-ideal of A. We have $\mu^+(0) = \mu(0)+1-\mu(0) = 1 \geq \mu(x)$, for all $x \in A$. For any $x, y, z \in A$, by Theorem 3.5 (ii) and Theorem 1.12 (1), we have

$$\begin{aligned} & \mu^+(y \odot (z^* \oplus y) \odot x^*) \wedge \mu^+(x) \\ = \ & [\mu(y \odot (z^* \oplus y) \odot x^*) + 1 - \mu(0)] \wedge (\mu(x) + 1 - \mu(0)) \\ = \ & [\mu(y \odot (z^* \oplus y) \odot x^*) \wedge \mu(x)] + 1 - \mu(0) \\ = \ & \mu((y \odot (z^* \oplus y) \odot x^*) \oplus x) + 1 - \mu(0) \\ = \ & [\mu(x \vee (y \odot (z^* \oplus y)] + 1 - \mu(0) \\ \leq \ & \mu(y \odot (z^* \oplus y)) + 1 - \mu(0) \\ \leq \ & \mu(y) + 1 - \mu(0). \end{aligned}$$

Hence μ^+ is a normal fuzzy p-ideal of A and obviously $\mu \subseteq \mu^+$. ■

Corollary 4.4. *If there is $x \in A$ such that $\mu^+(x) = 0$, then $\mu(x) = 0$.*

Corollary 4.5. *$(\mu^+)^+ = \mu^+$, for any fuzzy p-ideal of A. If μ is normal, then $\mu^+ = \mu$.*

Remark 4.6. Let μ be a fuzzy p-ideal of A. If there is a fuzzy p-ideal ν of A satisfying $\nu^+ \subseteq \mu$, then μ is normal.

Proof. Assume that there is a fuzzy p-ideal ν of A such that $\nu^+ \subseteq \mu$. Then $1 = \nu^+(0) \leq \mu(0)$, and so $\mu(0) = 1$. Hence μ is normal. ■

Given a fuzzy p-ideal of A, we construct a new normal fuzzy p-ideal of A.

Theorem 4.7. *Let μ be a fuzzy p-ideal of A and let $f : [0, \mu(0)] \to [0, 1]$ be an increasing function. Let $\mu_f : A \to [0, 1]$ be a fuzzy set in A define by $\mu_f = f(\mu(x))$, for all $x \in A$. Then μ_f is a fuzzy p-ideal of A. In particular, if $f(\mu(0)) = 1$, then μ_f is normal and if $f(t) \geq t$, for all $t \in [0, \mu(0)]$, then $\mu \subseteq \mu_f$.*

Proof. Since $\mu(0) \geq \mu(x)$, for all $x \in A$ and f is increasing, we have $\mu_f(0) = f(\mu(0)) \geq f(\mu(x)) = \mu_f(x)$ for all $x \in A$. For any $x, y, z \in A$, we get

$$\begin{aligned} & \mu_f(y \odot (z^* \oplus y) \odot x^*) \wedge \mu_f(x) \\ = \ & f(\mu(y \odot (z^* \oplus y) \odot x^*) \wedge f(\mu(x)) \\ = \ & f(\mu(y \odot (z^* \oplus y) \odot x^*) \wedge \mu(x)) \\ \leq \ & f(\mu(y)) \\ = \ & \mu_f(y). \end{aligned}$$

Hence μ_f is a fuzzy p-ideal of A. If $f(\mu(0)) = 1$, then clearly μ_f is normal. Assume that $f(t) \geq t$, for all $t \in [0, \mu(0)]$. Then $\mu_f(x) = f(\mu(x)) \geq \mu(x)$, for all $x \in A$, which proves $\mu \subseteq \mu_f$. ■

Let $N(A)$ denote the set of all normal fuzzy p-ideals of A.

Theorem 4.8. *Let $\mu \in N(A)$ be nonconstant such that it is a maximal element of the poset $(N(A), \subseteq)$. Then μ takes only the values 0 and 1.*

Proof. Since μ is normal, we have $\mu(0) = 1$. Let $x \in A$ such that $\mu(x) \neq 1$. It is sufficient to show that $\mu(x) = 0$. If not, then there exists $a \in A$ such that $0 < \mu(a) < 1$. Define a fuzzy set ν in A by $\nu(x) = (1/2)\{\mu(x) + \mu(a)\}$, for all $x \in A$. Clearly, ν is well defined, and we get

$$\nu(0) = 1/2\{\mu(0) + \mu(a)\} = 1/2\{1 + \mu(a)\} \geq 1/2\{\mu(x) + \mu(a)\} = \nu(x) \quad \forall x \in A.$$

Let $x, y, z \in A$. Then

$$\begin{aligned} \nu(y) &= 1/2\{\mu(y) + \mu(a)\} \\ &\geq 1/2\{[\mu(y \odot (z^* \oplus y) \odot x^*) \wedge \mu(x)] + \mu(a)\} \\ &= 1/2[(\mu(y \odot (z^* \oplus y) \odot x^*) + \mu(a)] \wedge 1/2(\mu(x) + \mu(a)) \\ &= \nu(y \odot (z^* \oplus y) \odot x^*) \wedge \nu(x). \end{aligned}$$

Hence ν is a fuzzy p-ideal of A. By Proposition 4.3, ν^+ is a maximal fuzzy p-ideal of A, where ν^+ is defined by $\nu^+(x) = \nu(x) + 1 - \nu(0)$, for all $x \in A$. Note that

$$\begin{aligned} \nu^+(a) &= \nu(a) + 1 - \nu(0) \\ &= 1/2\{\mu(a) + \mu(a)\} + 1 - 1/2\{\mu(0) + \mu(a)\} \\ &= 1/2\{\mu(a) + 1\} > \mu(a) \end{aligned}$$

and $\nu^+(a) < 1 = \nu^+(0)$. It follows that ν^+ is nonconstant and μ is not a maximal element of $(N(A), \subseteq)$. This is a contradiction. ∎

Definition 4.9. A nonconstant fuzzy p-ideal μ of A is called maximal if μ^+ is a maximal element of the poset $N(A)$.

Theorem 4.10. *A maximal fuzzy p-ideal μ of A is normal and takes only the values* 0 *and* 1*.*

Proof. Let μ be a maximal fuzzy p-ideal μ of A. Then μ^+ is a nonconstant maximal element of the poset $N(A)$ and by Theorem 4.8, μ^+ takes only the values 0 and 1. Clearly $\mu^+(x) = 1$ if and only if $\mu(x) = \mu(0)$ and $\mu^+(x) = 0$ if and only if $\mu(x) = \mu(0) - 1$. But $\mu \subseteq \mu^+$ (by Theorem 4.3). So $\mu^+(x) = 0$ implies that $\mu(x) = 0$, consequently $\mu(0) = 1$. Therefore μ is normal. ∎

Theorem 4.11. *Let μ be a fuzzy p-ideal of A, $\mu(1) \neq 0$ and $\overline{\mu}$ be the fuzzy set of A defined by $\overline{\mu}(x) = \mu(x)/\mu(1)$ for all $x \in A$. Then $\overline{\mu}$ is a normal fuzzy p-ideal of A and $\mu \subseteq \overline{\mu}$.*

Proof. Let $x, y \in A$. We have

$$\overline{\mu}(0) = \mu(0)/\mu(1) \geq \mu(x)/\mu(1) = \overline{\mu}(x).$$

Also, we have

$$\begin{aligned}\overline{\mu}(y) &= \mu(y)/\mu(1)\\ &\leq [\mu(y \odot (z^* \oplus y) \odot x^*) \wedge \mu(x)]/\mu(1)\\ &= [\mu(y \odot (z^* \oplus y) \odot x^*)/\mu(1)] \wedge \mu(x)/\mu(1)\\ &= \overline{\mu}(y \odot (z^* \oplus y) \odot x^*) \wedge \overline{\mu}(x).\end{aligned}$$

Hence $\overline{\mu}$ is a fuzzy p-ideal of A. Clearly, $\overline{\mu}$ is normal and $\mu \subseteq \overline{\mu}$. ∎

5. Conclusion

MV-algebras were introduced by C. Chang [1] in 1958 in order to provide an algebraic proof for the completeness theorem of the Łukasiewicz infinite valued propositional logic.

In this paper, we defined the concept of p-ideals and given characterization of them. We proved that p-ideals equivalent to Boolean ideals in MV-algebras.

We introduced the notion of fuzzy p-ideals of MV-algebras and described the transfer principle for fuzzy p-ideals in terms of level subsets.

We have also presented several characterizations and many important properties of fuzzy p-ideals in MV-algebras. Moreover, we obtained the extension theorem of fuzzy p-ideals in MV-algebras. We showed that in any MV-algebra, the concept of fuzzy p-ideals are equivalent to fuzzy Boolean ideals and are equivalent to fuzzy implicative ideals. Finally, by using the notion of maximal and normal fuzzy p-ideals, we showed that under certain conditions a fuzzy p-ideal is two valued and takes the values 0 and 1.

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