FALLING FUZZY GÖDEL IDEALS OF BL-ALGEBRAS

Biao Long Meng[1]

Department of Mathematics
Northwest University
Xi'an, 710127
P.R. China
College of Science
Xi'an University of Science and Technology
Xi'an, 710054
P.R. China

Xiao Long Xin

Department of Mathematics
Northwest University
Xi'an, 710127
P.R. China

Abstract. The notions of falling fuzzy (resp. Gödel, Boolean, implicative) ideals of a BL-algebra are introduced based on the theory of falling shadows and fuzzy sets. Several characterizations and relations of these notions are studied. Finally we apply the concept of falling fuzzy inference relations to ideal theory of BL-algebras and obtain some related results.

Keywords: BL-algebra, falling shadow, ideal, fuzzy ideal, falling fuzzy ideal, falling fuzzy Gödel(resp. Boolean, implicative) ideal, falling fuzzy relation.

1. Introduction

Goodman [2] pointed out the equivalence of a fuzzy set and a class of random sets by means of combining probability and fuzzy set theory. Also, Wang and Sanchez [18] introduced the theory of falling shadows which directly relates probability concepts with the membership function of fuzzy sets. Falling shadow representation theory shows us the way of selection related on the joint degrees distributions. It is a reasonable and convenient approach for the theoretical development and the practical applications of fuzzy sets and fuzzy logics. The mathematical structure of the theory of falling shadows is formulated. In particular, Tan et al. ([12], [13]) established a theoretic approach to define a fuzzy inference

[1]Corresponding author. E-mail address: mengbl_100@139.com

relation and fuzzy set operation based on the theory of falling shadows. Yuan et al. [19] considered a fuzzy subgroup (resp. subring, ideal) as the falling shadow of a cloud of the subgroups (resp. subring, ideal). Recently, Jun et al. ([5], [6]) considered the notions of falling fuzzy (resp. positive implicative, commutative, implicative) ideals of BCK-algebras. Zhan et al. [21] applied the falling shadow theory to filter theory of BL-algebras.

The notion of BL-algebra was initiated by Hájek [3] in order to provide an algebraic proof of the completeness theorem of Basic Logic (BL, in short). Soon after Cignoli et al. [1] proved that Hájek's logic really is the logic of continuous t-norms as conjectured by Hájek. One important aspect of BL-algebras is its filter theory. Researchers started a systematic study of BL-algebras with filter theory ([4], [10], [14], [15]). Another important aspect of BL-algebras is ideal theory, which was introduced by Hájek [3]. Some properties of ideals were investigated by ([8],[11]). Fuzzy ideal theory in BL-algebras were studied by Zhang et al. [22] and Meng et al. [9]. The notions of fuzzy prime ideals, fuzzy irreducible ideals, fuzzy Gödel ideals and fuzzy Boolean ideals were introduced. Several relations among these ideals were discussed. In the present paper we introduce the notions of falling fuzzy (resp. Gödel, Boolean, implicative) ideals of a BL-algebra based on the theory of falling shadows and fuzzy sets, and study several characterizations and relations of these notions. Finally we apply the concept of falling fuzzy inference relations to BL-algebras and obtain some related results.

2. Preliminaries

Let us recall some definitions and results on BL-algebras.

Definition 2.1. [3] An algebra $(A;\ \wedge, \vee, *, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is called a BL-algebra if it satisfies the following conditions:

(BL1) $(A;\ \wedge, \vee, 0, 1)$ is a bounded lattice,
(BL2) $(A;\ *, 1)$ is a commutative monoid,
(BL3) $x * z \leq y$ if and only if $z \leq x \rightarrow y$ (residuation),
(BL4) $x \wedge y = x * (x \rightarrow y)$, thus $x * (x \rightarrow y) = y * (y \rightarrow x)$ (divisibility),
(BL5) $(x \rightarrow y) \vee (y \rightarrow x) = 1$ (prelinearity).

Throughout this paper, A means a BL-algebra without mentioned otherwise. In the sequel, we agree the operations $*$, $\vee$, $\wedge$ have priority towards the operation $\rightarrow$.

Proposition 2.2. ([3],[8],[10],[14]) *For all $x, y, z \in A$, the following are valid:*

(1) $x * (x \rightarrow y) \leq y$,
(2) $x \leq y \rightarrow (x * y)$,
(3) $x \leq y$ if and only if $x \rightarrow y = 1$,
(4) $x \rightarrow (y \rightarrow z) = (x * y) \rightarrow z = y \rightarrow (x \rightarrow z)$,
(5) $x \leq y$ implies $z \rightarrow x \leq z \rightarrow y$, $y \rightarrow z \leq x \rightarrow z$,

(6) $y \leq (y \to x) \to x$,
(7) $(x \to y) * (y \to z) \leq x \to z$,
(8) $y \to x \leq (z \to y) \to (z \to x)$,
(9) $x \to y \leq (y \to z) \to (x \to z)$,
(10) $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$,
(11) $x \leq y$ implies $y^- \leq x^-$,
(12) $1 \to x = x,\ x \to x = 1,\ x \to 1 = 1$,
(13) $x \leq y \to x$, *or equivalently,* $x \to (y \to x) = 1$,
(14) $((x \to y) \to y) \to y = x \to y$,
(15) $1^- = 0,\ 0^- = 1, 1^{--} = 1, 0^{--} = 0$,
(16) $x \to (y \vee z) = (x \to y) \vee (x \to z), x \to (y \wedge z) = (x \to y) \wedge (x \to z)$,
(17) $(x \vee y) \to z = (x \to z) \wedge (y \to z), (x \wedge y) \to z = (x \to z) \vee (y \to z)$,
(18) $(x \vee y)^- = x^- \wedge y^-,\quad (x \wedge y)^- = x^- \vee y^-$,
(19) $x \to y \leq x * z \to y * z$,
(20) $x \to y \leq x \wedge z \to y \wedge z, x \to y \leq x \vee z \to y \vee z$,
(21) $(x \to y^-)^{--} = x \to y^-$,
(22) $x * x^- = 0$,
(23) $x * (y \vee z) = (x * y) \vee (x * z), x * (y \wedge z) = (x * y) \wedge (x * z)$.

where $x^- = x \to 0$.

Definition 2.3.[3] A nonempty subset I of A is said to be an ideal of A if it satisfies:

(I1) $0 \in I$,
(I2) $x \in I$ and $(x^- \to y^-)^- \in I$ implies $y \in I$ for all $x, y \in A$.

Proposition 2.4. [8] *Let I be an ideal of A. Then for any $x \in A$, $x \in I$ if and only if $x^{--} \in I$.*

Definition 2.5. [20] A fuzzy set in X is a mapping $\mu : X \longrightarrow [0, 1]$.

Let μ be a fuzzy set in X, $t \in [0, 1]$, the set $\mu_t = \{x \in X \mid \mu(x) \geq t\}$ is called a level subset of μ.

By $\mu \leq \nu$ we mean that $\mu(x) \leq \nu(x)$ for all $x \in X$.

Definition 2.6. [22] A fuzzy set μ in A is called a fuzzy ideal of A, if for all $x, y \in A$,

(FI1) $\mu(0) \geq \mu(x)$,
(FI2) $\mu(y) \geq \mu(x) \wedge \mu((x^- \to y^-)^-)$.

Proposition 2.7. [22] *Let μ be a fuzzy set in A. Then μ is a fuzzy ideal of A if and only if, for any $t \in [0, 1]$, μ_t is either empty or an ideal of A.*

In what follows we display the basic theory of falling shadows([12],[13], [17]).

Given a universe of discourse U and $\mathscr{P}(U)$ be the power set of U. For each $u \in U$ and $E \in \mathscr{P}(U)$, denote

(FS1) $\dot{u} := \{E \in \mathscr{P}(U) \mid u \in E\}$;
(FS2) $\dot{E} := \{\dot{u} \mid u \in E\}$.

An ordered pair $(\mathscr{P}(U), \mathscr{B})$ is called a hyper measurable structure if $\mathscr{B}$ is a σ-field in $\mathscr{P}(U)$ and $\dot{U} \subseteq \mathscr{B}$.

Let $(\Omega, \mathscr{A}, P)$ be a probability space and $(\mathscr{P}(U), \mathscr{B})$ be a hyper measurable structure. Suppose that a mapping $\xi : \Omega \to \mathscr{P}(U)$ satisfies, for any $C \in \mathscr{B}$, the set $\xi^{-1}(C) = \{\omega \in \Omega \mid \xi(\omega) \in C\} \in \mathscr{A}$. Then ξ is called a random set on U.

Suppose that ξ is a random set on U. Let

$$\tilde{H}(u) := P(\{\omega \in \Omega \mid u \in \xi(\omega)\}) \text{ for each } \ u \in U.$$

Then $\tilde{H}$ is a kind of fuzzy set in U. We call $\tilde{H}$ a falling shadow of the random set ξ, and ξ is a cloud of $\tilde{H}$.

For example, $(\Omega, \mathscr{A}, P) = ([0,1], \mathscr{A}, m)$, where $\mathscr{A}$ is a Borel field on $[0,1]$ and m the usual Lebesgue measure. Let $\tilde{H}$ be a fuzzy set in U and $\tilde{H}_t := \{u \in U \mid \tilde{H}(u) \geq t\}$ be a t-cut of $\tilde{H}$. Then

$$\xi : [0,1] \to \mathscr{P}(U), t \mapsto \tilde{H}_t.$$

is a random set and ξ is a cloud of $\tilde{H}$. We shall call ξ defined above as the cut-cloud of $\tilde{H}$([2]).

3. Falling fuzzy ideals

Definition 3.1. Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\xi : \Omega \to \mathscr{P}(A)$ be a random set. If $\xi(\omega)$ is an ideal of A for any $\omega \in \Omega$, then the falling shadow $\tilde{H}$ of the random set ξ where

$$\tilde{H}(x) = P(\{\omega \in \Omega \mid x \in \xi(\omega)\}) \text{ for each } x \in A.$$

is called a falling fuzzy ideal of A.

Let $(\Omega, \mathscr{A}, P)$ be a probability space and $F(A) = \{f \mid f : \Omega \to A\}$. Define two binary operations $\otimes$ and $\rightharpoonup$ on $F(A)$ by $(f \otimes g)(\omega) = f(\omega) * g(\omega)$ and $(f \rightharpoonup g)(\omega) = f(\omega) \to g(\omega)$ for all $\omega \in \Omega$ and for all $f, g \in F(A)$.

Let $e, \theta \in F(A)$ be defined by $e(\omega) = 1, \theta(\omega) = 0$ for all $\omega \in \Omega$. For short, denotes $f^- := f \rightharpoonup \theta$. Then it is easy to check that $(F(A), \otimes, \rightharpoonup, \theta, e)$ is a BL-algebra.

For any subset B of A and $f \in F(A)$, let $B_f := \{\omega \in \Omega \mid f(\omega) \in B\}$,

$$\begin{aligned} \xi : \ & \Omega \to \mathscr{P}(F(A)), \\ & \omega \mapsto \{f \in F(A) \mid f(\omega) \in B\}. \end{aligned}$$

then $B_f \in \mathscr{A}$([21]).

Theorem 3.2. *If B is an ideal of A, then $\xi(\omega) = \{f \in F(A) \mid f(\omega) \in B\}$ is an ideal of $F(A)$.*

Proof. Let B be an ideal of A and $\omega \in \Omega$. By $\theta(\omega) = 0 \in B$ it follows that $\theta \in \xi(\omega)$. If $(f^- \rightharpoonup g^-)^- \in \xi(\omega)$ and $f \in \xi(\omega)$ for any $f, g \in F(A)$, then $((f(\omega))^- \to (g(\omega))^-)^- = (f^- \rightharpoonup g^-)^-(\omega) \in B$ and $f(\omega) \in B$. Since B is an ideal of A, we have $g(\omega) \in B$, and so $g \in \xi(\omega)$. Therefore $\xi(\omega)$ is an ideal of A. ∎

Example 3.3. [8] Let $A = \{0, a, b, 1\}$. Define $*$, $\to$, $\vee$ and $\wedge$ as follows:

$*$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	1	1	1

$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

Then $(A; \vee, \wedge, *, \to, 0, 1)$ is a BL-algebra. It is easy to check that $\{0\}, \{0, a\}$, $\{0, b\}, A$ are ideals of A. But $\{0, a, b\}$ is not an ideal of A, because $(a^- \to 1^-)^- = (b \to 0)^- = a^- = b \in \{0, a, b\}$ and $1 \notin \{0, a, b\}$. Let $(\Omega, \mathscr{A}, P) = ([0, 1], \mathscr{A}, m)$, where $\mathscr{A}$ is a Borel field on $[0, 1]$ and m the usual Lebesgue measure. Let $\xi : [0, 1] \to \mathscr{P}(A)$ be defined by

$$\xi(t) := \begin{cases} \{0, a\} & \text{if } t \in [0, 0.4), \\ \{0, b\} & \text{if } t \in [0.4, 0.9), \\ \{0, a, b, 1\} & \text{if } t \in [0.9, 1]. \end{cases}$$

Thus $\xi(t)$ is an ideal of A for all $t \in [0, 1]$. Hence $\tilde{H}$ is a falling fuzzy ideal of A where $\tilde{H}(x) = P(\{t \in [0, 1] \mid x \in \xi(t)\})$ and

$$\tilde{H}(x) := \begin{cases} 1 & \text{if } x = 0, \\ 0.5 & \text{if } x = a, \\ 0.6 & \text{if } x = b, \\ 0.1 & \text{if } x = 1. \end{cases}$$

Note 3.4. Observe that in the above example, $\tilde{H}_{0.5} = \{0, a, b\}$ is not an ideal of A, hence $\tilde{H}$ is not a fuzzy ideal in A.

Example 3.5. [22] Let $A = \{0, a, b, c, d, 1\}$. Define $\vee$, $\wedge$, $*$ and $\to$ as follows:

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	d	c	0	d	a
b	0	c	b	c	0	b
c	0	0	c	0	0	c
d	0	d	0	0	d	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	b	a	1
b	d	a	1	a	d	1
c	a	1	1	1	a	1
d	b	1	b	b	1	1
1	0	a	b	c	d	1

$\vee$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	1	a	a	1
b	b	1	b	b	1	1
c	c	a	b	c	a	1
d	d	a	1	a	d	1
1	1	1	1	1	1	1

$\wedge$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	c	c	d	a
b	0	c	b	c	0	b
c	0	c	c	c	0	c
d	0	d	0	0	d	d
1	0	a	b	c	d	1

Then $(A; \vee, \wedge, *, \to, 0, 1)$ is a BL-algebra. It can check that $\{0\}, \{0, b, c\}$ and A are ideals of A. Let $\xi : [0, 1] \to \mathscr{P}(A)$ be defined by

$$\xi(t) := \begin{cases} \{0, b, c\} & \text{if} \quad t \in [0, 0.5), \\ A & \text{if} \quad t \in [0.5, 1]. \end{cases}$$

Thus $\xi(t)$ is an ideal of A for all $t \in [0, 1]$. Hence $\tilde{H}$ is a falling fuzzy ideal of A where $\tilde{H}(x) = P(\{t \in [0, 1] \mid x \in \xi(t)\})$ and

$$\tilde{H}(x) := \begin{cases} 1 & \text{if} \quad x = 0, b, c \\ 0.5 & \text{if} \quad x = a, d, 1. \end{cases}$$

Note 3.6. Observe that in the Example 3.5, $\tilde{H}$ is also a fuzzy ideal in A.

Theorem 3.7. *Each fuzzy ideal μ in A is a falling fuzzy ideal.*

Proof. Suppose that $\mu : A \to [0, 1]$ is a fuzzy ideal in A, then for all $t \in [0, 1]$, μ_t is an ideal of A where $\mu_t \neq \emptyset$. Let $\xi : [0, 1] \to \mathscr{P}(A)$ is a random set such that $\xi(t) = \mu_t$, then μ is a falling fuzzy ideal of A. ∎

Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\tilde{H}$ be a falling shadow of a random set $\xi : \Omega \to \mathscr{P}(A)$. Denote

$$\Omega(x; \xi) := \{\omega \in \Omega \mid x \in \xi(\omega)\}.$$

Then $\Omega(x; \xi) \in \mathscr{A}$. Hence $\tilde{H}(x) = P(\Omega(x; \xi))$.

Proposition 3.8. *Let $\tilde{H}$ be the falling shadow of a random set $\xi : \Omega \to \mathscr{P}(A)$. If $\tilde{H}$ is a falling fuzzy ideal of A, then $x^- \leq y^- \Rightarrow \Omega(x; \xi) \subseteq \Omega(y; \xi)$ for any $x, y \in A$.*

Proof. If $x^- \leq y^-$ then $(x^- \to y^-)^- = 0$. Let $\omega \in \Omega(x;\xi)$, hence $x \in \xi(\omega)$ and $(x^- \to y^-)^- \in \xi(\omega)$. Since $\xi(\omega)$ is an ideal of A, we have $y \in \xi(\omega)$, and so $\omega \in \Omega(y;\xi)$. Thus $\Omega(x;\xi) \subseteq \Omega(y;\xi)$. ■

Corollary 3.9. *Let $\tilde{H}$ be the falling shadow of a random set $\xi : \Omega \to \mathscr{P}(A)$. If $\tilde{H}$ is a falling fuzzy ideal of A, then $x \leq y \Rightarrow \Omega(y;\xi) \subseteq \Omega(x;\xi)$ for any $x, y \in A$.*

Proof. Trivial. ■

Theorem 3.10. *Let $\tilde{H}$ be the falling shadow of a random set $\xi : \Omega \to \mathscr{P}(A)$. Then $\tilde{H}$ is a falling fuzzy ideal of A if and only if the following hold:*

(i) $\Omega(x;\xi) \subseteq \Omega(0;\xi)$ *for any* $x \in A$,
(ii) $\Omega((x^- \to y^-)^-;\xi) \cap \Omega(x;\xi) \subseteq \Omega(y;\xi)$ *for any* $x, y \in A$.

Proof. Suppose that the falling shadow $\tilde{H}$ of a random set $\xi : \Omega \to \mathscr{P}(A)$ is a falling fuzzy ideal of A. Since $0 \leq x$ for all $x \in A$, by Corollary 3.9 we have $\Omega(x;\xi) \subseteq \Omega(0;\xi)$ for any $x \in A$. (i) holds. Now let $\omega \in \Omega((x^- \to y^-)^-;\xi) \cap \Omega(x;\xi)$, then $\omega \in \Omega((x^- \to y^-)^-;\xi)$ and $\omega \in \Omega(x;\xi)$, i.e., $(x^- \to y^-)^- \in \xi(\omega)$ and $x \in \xi(\omega)$. Thus $y \in \xi(\omega)$ since $\xi(\omega)$ is an ideal of A. This shows that $\omega \in \Omega(y;\xi)$. (ii) holds.

Conversely, suppose that $\tilde{H}$ satisfies (i) and (ii). For any $\omega \in \Omega$ and $x \in \xi(\omega)$, i.e., $\omega \in \Omega(x;\xi)$, then it follows from (i) that $\omega \in \Omega(0;\xi)$, hence $0 \in \xi(\omega)$, thus $\xi(\omega)$ satisfies (I1). Now let $x \in \xi(\omega)$ and $(x^- \to y^-)^- \in \xi(\omega)$, then $\omega \in \Omega(x;\xi)$ and $\omega \in \Omega((x^- \to y^-)^-;\xi)$. By (ii) we have

$$\omega \in \Omega((x^- \to y^-)^-;\xi) \cap \Omega(x;\xi) \subseteq \Omega(y;\xi).$$

And so $y \in \xi(\omega)$. Therefore $\xi(\omega)$ satisfies (I2). This proves that $\tilde{H}$ is a falling fuzzy ideal of A. ■

Theorem 3.11. *Let $\tilde{H}$ be the falling shadow of a random set $\xi : \Omega \to \mathscr{P}(A)$. Then $\tilde{H}$ is a falling fuzzy ideal of A if and only if , for any $x, y, z \in A$,*

$$(*) \quad z^- \to (y^- \to x^-) = 1 \Rightarrow \Omega(z;\xi) \cap \Omega(y;\xi) \subseteq \Omega(x;\xi)).$$

Proof. Suppose that $\tilde{H}$ is a falling fuzzy ideal of A. Let $z^- \to (y^- \to x^-) = 1$. If $\omega \in \Omega(z;\xi) \cap \Omega(y;\xi)$, then $z, y \in \xi(\omega)$. Since $\xi(\omega)$ is an ideal of A, it follows from $z^- \to (y^- \to x^-) = 1$ that $x \in \xi(\omega)$, i.e., $\omega \in \Omega(x;\xi)$, $(*)$ holds.

Conversely, suppose that $(*)$ is true. For any $\omega \in \Omega$, let $z^- \to (y^- \to x^-) = 1$. If $z, y \in \xi(\omega)$, then $\omega \in \Omega(z;\xi) \cap \Omega(y;\xi)$. By $(*)$ we have $\omega \in \Omega(x;\xi)$, thus $x \in \xi(\omega)$. This shows that $\xi(\omega)$ is an ideal of A. Therefore $\tilde{H}$ is a falling fuzzy ideal of A. ■

By induction we have the following conclusion.

Corollary 3.12. *Let $\tilde{H}$ be the falling shadow of a random set $\xi : \Omega \to \mathscr{P}(A)$. Then $\tilde{H}$ is a falling fuzzy ideal of A if and only if , for any $x_1, \cdots, x_n, z \in A$, $(x_n^- \to (\cdots \to (x_1^- \to z^-) \cdots) = 1$ implies $\Omega(x_1;\xi) \cap \cdots \cap \Omega(x_n;\xi) \subseteq \Omega(z;\xi)$.*

Theorem 3.13. *If $\tilde{H}$ is a falling fuzzy ideal of A, then for any $x, y \in A$,*

(i) $\tilde{H}(0) \geq \tilde{H}(x)$,
(ii) $\tilde{H}(y) \geq T(\tilde{H}((x^- \to y^-)^-), \tilde{H}(x))$,

where $T(s,t) = \max\{s+t-1, 0\}$ *for any* $s, t \in [0,1]$.

Proof. Since $\xi(\omega)$ is an ideal of A for any $\omega \in \Omega$, by Theorem 3.10(i) it follows that for any $x \in A$, $\tilde{H}(x) = P(\Omega(x;\xi)) \leq P(\Omega(0;\xi)) = \tilde{H}(0)$, (i) holds. Furthermore, by Theorem 3.10(ii) we have for any $x, y \in A$,

$$\begin{array}{rcl} \tilde{H}(y) & = & P(\Omega(y;\xi)) \\ & \geq & P(\Omega((x^- \to y^-)^-;\xi) \cap \Omega(x;\xi)) \\ & = & P(\Omega((x^- \to y^-)^-;\xi)) + P(\Omega(x;\xi)) - P(\Omega((x^- \to y^-)^-;\xi) \cup \Omega(x;\xi)) \\ & \geq & \tilde{H}((x^- \to y^-)^-) + \tilde{H}(x) - 1. \end{array}$$

Therefore

$$\tilde{H}(y) \geq \max\{\tilde{H}((x^- \to y^-)^-) + \tilde{H}(x) - 1, 0\} = T(\tilde{H}((x^- \to y^-)^-), \tilde{H}(x)).$$

(ii) holds, ending the proof. ∎

Note 3.14. Theorem 3.13 shows that every falling fuzzy ideal of A is a T-fuzzy ideal of A.

4. Falling fuzzy Gödel ideals

In this section, we introduce the notion of falling fuzzy Gödel ideals of BL-algebras and investigate some of its basic properties.

Definition 4.1.[8] Let I be an ideal of A. I is said to be a Gödel ideal if it satisfies: $(x^- \to (x^-)^2)^- \in I$ for any $x \in A$.

Definition 4.2. [9] Let μ be a fuzzy ideal of A. Then μ is said to be a fuzzy Gödel ideal if it satisfies: $\mu((x^- \to (x^-)^2)^-) = \mu(0)$ for any $x \in A$.

Lemma 4.3. [9] *Let μ be a fuzzy set in A. Then μ is a fuzzy Gödel ideal if and only if , for each $t \in [0,1]$, μ_t is a Gödel ideal of A where $\mu_t \neq \emptyset$.*

Definition 4.4. Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\xi : \Omega \to \mathscr{P}(A)$ be a random set. If $\xi(\omega)$ is a Gödel ideal of A for any $\omega \in \Omega$, then the falling shadow $\tilde{H}$ of the random set ξ is called a falling fuzzy Gödel ideal of A.

It is obvious that if A is a Gödel algebra then every falling fuzzy ideal $\tilde{H}$ of A is a falling fuzzy Gödel ideal of A.

Example 4.5. [22] Let A be the BL-algebra in Example 3.5. It can check that $\{0, b, c\}$ and A are Gödel ideals of A. Let ξ and $\tilde{H}$ be defined as in Example 3.5.

Then $\xi(t)$ is a Gödel ideal of A for all $t \in [0,1]$. Hence $\tilde{H}$ is a falling fuzzy Gödel ideal of A.

Example 4.6. In the above example, let $\xi : [0,1] \to \mathscr{P}(A)$ be defined by

$$\xi(t) := \begin{cases} \{0\} & \text{if} \quad t = [0, 0.2) \\ \{0, d\} & \text{if} \quad t \in [0.2, 0.5), \\ A & \text{if} \quad t \in [0.5, 1]. \end{cases}$$

Thus $\xi(t)$ is an ideal of A for all $t \in [0,1]$. Hence $\tilde{H}$ is a falling fuzzy ideal of A where $\tilde{H}(x) = P(\{t \in [0,1] \mid x \in \xi(t)\})$ and

$$\tilde{H}(x) := \begin{cases} 1 & \text{if} \quad x = 0, \\ 0.8 & \text{if} \quad x = d, \\ 0.5 & \text{if} \quad x = a, b, c, 1. \end{cases}$$

Because $(a^- \to (a^-)^2)^- = c \notin \{0\}$, hence $\xi(0) = \{0\}$ is an ideal of A but it is not a Gödel ideal of A. Thus $\tilde{H}$ is not a falling fuzzy Gödel ideal of A.

Theorem 4.7. *Each fuzzy Gödel ideal in A is a falling fuzzy Gödel ideal of A.*

Proof. Suppose that $\mu : A \to [0,1]$ is a fuzzy Gödel ideal in A, then by Lemma 4.3, for any $t \in [0,1]$, μ_t is a Gödel ideal of A where $\mu_t \neq \emptyset$. Let $\xi : [0,1] \to \mathscr{P}(A)$ is a random set such that $\xi(t) = \mu_t$, then μ is a falling fuzzy Gödel ideal of A. ∎

Theorem 4.8. *Let A be a Gödel algebra. Then every falling fuzzy ideal of A is a falling fuzzy Gödel ideal of A.*

Proof. Trivial. ∎

Theorem 4.9. *Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\xi : \Omega \to \mathscr{P}(A)$ be a random set. If $\tilde{H}$ is the falling fuzzy ideal of A, then the following are equivalent: for for any $x, y, z \in A$,*

(i) *$\tilde{H}$ is a falling fuzzy Gödel ideal of A;*
(ii) $\Omega(((x^-)^2 \to y^-)^-; \xi) \subseteq \Omega((x^- \to y^-)^-; \xi)$;
(iii) $\Omega(((x^- * y^-) \to z^-)^-; \xi) \subseteq \Omega(((x^- \to y^-) \to (x^- \to z^-))^-; \xi)$.

Proof. (i)⇒(ii) Suppose $\tilde{H}$ is a falling fuzzy Gödel ideal of A.

For any $\omega \in \Omega$, then $\xi(\omega)$ is a Gödel ideal of A, and so $(x^- \to (x^-)^2)^- \in \xi(\omega)$. If $\omega \in \Omega(((x^-)^2 \to y^-)^-; \xi)$, then $((x^-)^2 \to y^-)^- \in \xi(\omega)$. Since

$$(x^- \to (x^-)^2) * ((x^-)^2 \to y^-) \leq x^- \to y^-,$$

it follows that

$$(x^- \to (x^-)^2) \leq ((x^-)^2 \to y^-) \to (x^- \to y^-).$$

Hence

$$\begin{aligned}(x^- \to (x^-)^2)^{--} &\leq (((x^-)^2 \to y^-) \to (x^- \to y^-))^{--}\\ &= ((x^-)^2 \to y^-) \to (x^- \to y^-)\\ &= ((x^-)^2 \to y^-)^{--} \to (x^- \to y^-)^{--},\end{aligned}$$

So we have

$$((x^- \to (x^-)^2)^{--} \to (((x^-)^2 \to y^-)^{--} \to (x^- \to y^-)^{--})^{--})^- = 0 \in \xi(\omega).$$

By $(x^- \to (x^-)^2)^- \in \xi(\omega)$ and (I2) we get

$$((x^-)^2 \to y^-)^{--} \to (x^- \to y^-)^{--})^- \in \xi(\omega).$$

By $((x^-)^2 \to y^-)^- \in \xi(\omega)$ and (I2) we obtain $(x^- \to y^-)^- \in \xi(\omega)$. Thus, $\omega \in \Omega((x^- \to y^-)^-; \xi)$. This proves that

$$\Omega(((x^-)^2 \to y^-)^-; \xi) \subseteq \Omega((x^- \to y^-)^-; \xi),$$

(ii) holds.

(ii)⇒(iii) Suppose that (ii) is true.

For any $\omega \in \Omega$, if $\omega \in \Omega(((x^- * y^-) \to z^-)^-; \xi)$, then $((x^- * y^-) \to z^-)^- \in \xi(\omega)$. Since $y^- \to z^- \leq (x^- \to y^-) \to (x^- \to z^-)$, then

$$\begin{aligned}(x^- * y^-) \to z^- &= x^- \to (y^- \to z^-)\\ &\leq x^- \to ((x^- \to y^-) \to (x^- \to z^-))\\ &= x^- \to (x^- \to ((x^- \to y^-) \to z^-))\\ &= (x^-)^2 \to ((x^- \to y^-) \to z^-),\end{aligned}$$

we have $((x^-)^2 \to ((x^- \to y^-) \to z^-))^- \leq ((x^- * y^-) \to z^-)^-$, and so

$$((x^-)^2 \to ((x^- \to y^-) \to z^-))^- \in \xi(\omega).$$

Thus $\omega \in \Omega(((x^-)^2 \to ((x^- \to y^-) \to z^-))^-; \xi)$. By (ii) it follows that

$$\omega \in \Omega((x^- \to ((x^- \to y^-) \to z^-))^-; \xi) = \Omega(((x^- \to y^-) \to (x^- \to z^-))^-; \xi).$$

This shows that (iii) is true.

(iii)⇒(i) Suppose that (iii) is true. For any $\omega \in \Omega$ and $x \in A$, then

$$(x^- \to (x^- \to (x^-)^2))^- = (x^- * x^- \to (x^-)^2))^- = 0 \in \xi(\omega),$$

i.e., $\omega \in \Omega((x^- * x^- \to (x^-)^2))^-; \xi)$. It follows from (iii) that

$$\omega \in \Omega(((x^- \to x^-) \to (x^- \to (x^-)^2)^-); \xi).$$

Since $(x^- \to x^-) \to (x^- \to (x^-)^2)^- = (x^- \to (x^-)^2)^-$, then we have

$$\omega \in \Omega((x^- \to (x^-)^2)^-; \xi),$$

thus $(x^- \to (x^-)^2)^- \in \xi(\omega)$. This shows that for each $\omega \in \Omega$, $\xi(\omega)$ is a Gödel ideal of A. Therefore $\tilde{H}$ is a falling fuzzy Gödel ideal of A. (i) holds. ∎

Theorem 4.10. *Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\xi : \Omega \to \mathscr{P}(A)$ be a random set. If $\tilde{H}$ is a falling fuzzy ideal of A, then $\tilde{H}$ is a falling fuzzy Gödel ideal of A if and only if for any $x, y, z \in A$,*

$$(**) \qquad \Omega((x^- \to (y^- \to z^-))^-; \xi) \cap \Omega((x^- \to y^-)^-; \xi) \subseteq \Omega((x^- \to z^-)^-; \xi).$$

Proof. Suppose that $\tilde{H}$ is a falling fuzzy Gödel ideal of A. For any $\omega \in \Omega$, if

$$\omega \in \Omega((x^- \to (y^- \to z^-))^-; \xi) \cap \Omega((x^- \to y^-)^-; \xi),$$

then, by Theorem 4.9(iii), we have

$$\begin{aligned} \omega &\in \Omega(((x^- \to y^-) \to (x^- \to z^-))^-; \xi) \cap \Omega((x^- \to y^-)^-; \xi) \\ &= \Omega(((x^- \to y^-)^{--} \to (x^- \to z^-)^{--})^-; \xi) \cap \Omega((x^- \to y^-)^-; \xi). \end{aligned}$$

Since $\tilde{H}$ is a falling fuzzy ideal of A, by Theorem 3.10(ii) we have

$$\omega \in \Omega((x^- \to z^-)^-; \xi).$$

Thus $(**)$ holds.

Conversely, suppose that the falling fuzzy ideal $\tilde{H}$ of A satisfies $(**)$. For any $\omega \in \Omega$ and $x \in A$, since

$$(x^- \to (x^- \to (x^-)^2))^- = ((x^-)^2 \to (x^-)^2)^- = 1^- = 0 \in \xi(\omega),$$

and

$$(x^- \to x^-)^- = 0 \in \xi(\omega),$$

we have $\omega \in \Omega\{(x^- \to (x^- \to (x^-)^2))^-; \xi\}$ and $\omega \in \{(x^- \to x^-)^-; \xi\}$, and so

$$\omega \in \Omega\{(x^- \to (x^- \to (x^-)^2))^-; \xi\} \cap \Omega\{(x^- \to x^-)^-; \xi\}.$$

It follows from $(**)$ that $\omega \in \Omega\{(x^- \to (x^-)^2)^-; \xi\}$. Hence $(x^- \to (x^-)^2)^- \in \xi(\omega)$, which shows that $\xi(\omega)$ is a Gödel ideal of A. Therefore $\tilde{H}$ is a falling fuzzy Gödel ideal of A. ∎

Theorem 4.11. *Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\xi : \Omega \to \mathscr{P}(A)$ be a random set. If $\tilde{H}$ is a falling fuzzy ideal of A, then $\tilde{H}$ is a falling fuzzy Gödel ideal of A if and only if for any $x, y, z \in A$,*

$$(***) \qquad \Omega((x^- \to ((y^-)^2 \to z^-))^-; \xi) \cap \Omega(x; \xi) \subseteq \Omega((y^- \to z^-)^-; \xi).$$

Proof. Suppose that $\tilde{H}$ is a falling fuzzy Gödel ideal of A. If

$$\omega \in \Omega((x^- \to ((y^-)^2 \to z^-))^-; \xi) \cap \Omega(x; \xi),$$

then $\omega \in \Omega((x^- \to ((y^-)^2 \to z^-))^-;\xi)$ and $\omega \in \Omega(x;\xi)$. Thus $(x^- \to ((y^-)^2 \to z^-))^- \in \xi(\omega)$ and $x \in \xi(\omega)$, i.e., $(x^- \to ((y^-)^2 \to z^-)^{--})^- \in \xi(\omega)$ and $x \in \xi(\omega)$. Since $\xi(\omega)$ is an ideal of A, it follows that $((y^-)^2 \to z^-)^- \in \xi(\omega)$. Hence $\omega \in \Omega\{((y^-)^2 \to z^-)^-;\xi\}$, and so

$$\Omega((x^- \to ((y^-)^2 \to z^-))^-;\xi) \cap \Omega(x;\xi) \subseteq \Omega\{((y^-)^2 \to z^-)^-;\xi\}.$$

By Theorem 4.9(ii) we have

$$\Omega((x^- \to ((y^-)^2 \to z^-))^-;\xi) \cap \Omega(x;\xi) \subseteq \Omega\{(y^- \to z^-)^-;\xi\}.$$

$(***)$ holds.

Conversely, suppose that $(***)$ is true and $\omega \in \Omega((x^- \to (y^- \to z^-))^-;\xi) \cap \Omega((x^- \to y^-)^-;\xi)$. Since

$$(x^-)^2 * (x^- \to (y^- \to z^-)) * (x^- \to y^-) \leq z^-,$$

we have

$$\begin{aligned} &(x^- \to (y^- \to z^-)) * (x^- \to y^-) \leq (x^-)^2 \to z^-,\\ &x^- \to (y^- \to z^-) \leq (x^- \to y^-) \to ((x^-)^2 \to z^-),\\ &((x^- \to y^-) \to ((x^-)^2 \to z^-))^- \leq (x^- \to (y^- \to z^-))^-. \end{aligned}$$

By Corollary 3.9, we get

$$\Omega((x^- \to (y^- \to z^-))^-;\xi) \subseteq \Omega(((x^- \to y^-) \to ((x^-)^2 \to z^-))^-;\xi).$$

Therefore,

$$\begin{aligned} & \Omega((x^- \to (y^- \to z^-))^-;\xi) \cap \Omega((x^- \to y^-)^-;\xi)\\ \subseteq\ & \Omega(((x^- \to y^-) \to ((x^-)^2 \to z^-))^-;\xi) \cap \Omega((x^- \to y^-)^-;\xi)\\ =\ & \Omega(((x^- \to y^-)^{--} \to ((x^-)^2 \to z^-))^-;\xi) \cap \Omega((x^- \to y^-)^-;\xi)\\ \subseteq\ & \Omega((x^- \to z^-)^-). \end{aligned}$$

By Theorem 4.10, $\tilde{H}$ is a falling fuzzy ideal of A. ∎

Theorem 4.12. *Let $\tilde{H}$ be the falling shadow of a random set $\xi : \Omega \to \mathscr{P}(A)$. Then $\tilde{H}$ is a falling fuzzy Gödel ideal of A if and only if the following hold:*

(i) $\Omega(x;\xi) \subseteq \Omega(0;\xi)$ *for any* $x \in A$,

(ii) $\Omega(x;\xi) = \Omega(x^{--};\xi)$ *for any* $x \in A$,

(iii) $\Omega((x^- \to (y^- \to z^-))^-;\xi) \cap \Omega((x^- \to y^-)^-;\xi) \subseteq \Omega((x^- \to z^-)^-;\xi)$ *for any* $x, y, z \in A$.

Proof. Suppose that $\tilde{H}$ is a falling fuzzy Gödel ideal of A. By Theorem 3.10, $\tilde{H}$ satisfies (i). Since for any $x \in A$ we have

$$\omega \in \Omega(x;\xi) \Leftrightarrow x \in \xi(\omega) \Leftrightarrow x^{--} \in \xi(\omega) \Leftrightarrow \omega \in \Omega(x^{--};\xi),$$

it follows that $\Omega(x;\xi) = \Omega(x^{--};\xi)$, (ii) holds. By Theorem 4.10, (iii) holds.

Conversely, suppose that (i)-(iii) hold. For any $\omega \in \Omega$ and $x \in \xi(\omega)$, then we have $\omega \in \Omega(x;\xi)$. It follows from (i) that $\omega \in \Omega(0;\xi)$, hence $0 \in \xi(\omega)$, thus $\xi(\omega)$ satisfies (I1). Let $y \in \xi(\omega)$ and $(y^- \to x^-)^- \in \xi(\omega)$. Then $\omega \in \Omega(y;\xi)$ and $\omega \in \Omega((y^- \to x^-)^-;\xi)$. By (ii) we have $\omega \in \Omega(y^{--};\xi)$. Since $\Omega(y^{--};\xi) = \Omega((0^- \to y^-)^-;\xi)$ and $\Omega((y^- \to x^-)^-;\xi) = \Omega((0^- \to (y^- \to x^-))^-;\xi)$. It follows that $\omega \in \Omega((0^- \to y^-)^-;\xi)$ and $\omega \in \Omega((0^- \to (y^- \to x^-))^-;\xi)$. By (iii) and (ii) we have

$$\begin{aligned}\omega &\in \Omega((0^- \to y^-)^-) \cap \Omega((0^- \to (y^- \to x^-))^-;\xi)\\ &\subseteq \Omega((0^- \to x^-)^-;\xi)\\ &= \Omega(x^{--};\xi)\\ &= \Omega(x;\xi),\end{aligned}$$

thus $x \in \xi(\omega)$. This shows that $\xi(\omega)$ is an ideal of A for any $\omega \in \Omega$. Hence $\tilde{H}$ is a falling fuzzy ideal of A. By (iii) and Theorem 4.10, $\tilde{H}$ is a falling fuzzy Gödel ideal of A. ∎

Theorem 4.13. *Let $\tilde{H}$ be the falling shadow of a random set $\xi : \Omega \to \mathscr{P}(A)$. Then $\tilde{H}$ is a falling fuzzy Gödel ideal of A if and only if the following hold:*

(i) $\Omega(x;\xi) \subseteq \Omega(0;\xi)$ *for any* $x \in A$,

(ii) $\Omega(x;\xi) = \Omega(x^{--};\xi)$ *for any* $x \in A$,

(iii) $\Omega((z^- \to ((y^-)^2 \to x^-))^-;\xi) \cap \Omega(z;\xi) \subseteq \Omega((y^- \to x^-)^-;\xi)$ *for any* $x, y, z \in A$.

Proof. Suppose that $\tilde{H}$ is a falling fuzzy Gödel ideal of A. By the "only if" part of Theorem 4.12 we know (i) and (ii) hold. Then by Theorem 4.11, (iii) holds.

Conversely, suppose that (i)-(iii) are true. In (iii), let $y = 0$, we get $\Omega((z^- \to x^-)^-;\xi) \cap \Omega(z;\xi) \subseteq \Omega(x^{--};\xi)$ for any $x, y \in A$. From (ii) it follows that

$$\Omega((z^- \to x^-)^-;\xi) \cap \Omega(z;\xi) \subseteq \Omega(x;\xi).$$

By (i) and Theorem 3.10 we know that $\tilde{H}$ is a falling fuzzy ideal. Furthermore, we get that $\tilde{H}$ is a falling fuzzy Gödel ideal of A by (iii) and Theorem 4.11. ∎

Theorem 4.14. *If $\tilde{H}$ is a falling fuzzy Gödel ideal of A, then for any $x, y, z \in A$,*

(i) $\tilde{H}(0) \geq \tilde{H}(x)$,

(ii) $\tilde{H}(x) = \tilde{H}(x^{--})$,

(iii) $\tilde{H}((x^- \to z^-)^-) \geq T(\tilde{H}((x^- \to (y^- \to z^-))^-), \tilde{H}((x^- \to y^-)^-))$,

where $T(s,t) = \max\{s + t - 1, 0\}$ for any $s, t \in [0,1]$.

Proof. It is easy to see that (i) and (ii) are true. We just prove (iii). By Theorem 4.12(iii), then for any $x, y, z \in A$ we have

$$\begin{aligned}&\tilde{H}((x^- \to z^-)^-)\\ =\ & P(\Omega((x^- \to z^-)^-;\xi))\\ \geq\ & P(\Omega((x^- \to (y^- \to z^-))^-)^-;\xi) \cap \Omega((x^- \to y^-)^-;\xi))\\ =\ & P(\Omega((x^- \to (y^- \to z^-))^-)^-;\xi)) + P(\Omega((x^- \to y^-)^-;\xi))\\ & -P(\Omega((x^- \to (y^- \to z^-))^-;\xi) \cup \Omega((x^- \to y^-)^-;\xi))\\ \geq\ & \tilde{H}((x^- \to (y^- \to z^-)^-) + \tilde{H}((x^- \to y^-)^-) - 1.\end{aligned}$$

Therefore

$$\begin{array}{rcl}\tilde{H}(x) & \geq & \max\{\tilde{H}((x^- \to (y^- \to z^-)^-) + \tilde{H}((x^- \to y^-)^-) - 1, 0\} \\ & = & T(\tilde{H}((x^- \to (y^- \to z^-)^-), \tilde{H}((x^- \to y^-)^-)),\end{array}$$

and so (iii) holds. ∎

Note 4.15. Theorem 4.14 shows that every falling fuzzy Gödel ideal of A is a T-fuzzy Gödel ideal of A.

By similar argument we can prove the following conclusion and the details are omitted.

Theorem 4.16. *If $\tilde{H}$ is a falling fuzzy Gödel ideal of A, then for any $x, y, z \in A$*

(i) $\tilde{H}(0) \geq \tilde{H}(x)$,
(ii) $\tilde{H}(x) = \tilde{H}(x^{--})$,
(iii) $\tilde{H}((y^- \to z^-)^-) \geq T(\tilde{H}((x^- \to ((y^-)^2 \to z^-))^-), \tilde{H}(x))$,

where $T(s, t) = \max\{s + t - 1, 0\}$ for any $s, t \in [0, 1]$.

5. Falling fuzzy Boolean ideals

In this section, we introduce the notion of falling fuzzy Boolean ideals of BL-algebras and investigate some of its basic properties. We also discuss relation between falling fuzzy Boolean ideals and falling fuzzy Gödel ideals.

Definition 5.1. An ideal I of A is said to be a Boolean ideal if $x \wedge x^- \in I$ for all $x \in A$.

Lemma 5.2. *Let I be a Boolean ideal of A. Then for any $x \in A$, $(x \to x^-)^- \in I$ implies $x \in I$.*

Proof. Since

$$\begin{array}{rl} & ((x \to x^-)^{--} \to x^-)^- \\ = & ((x \to x^-) \to x^-)^- \\ = & ((x \to x^-) \to (x \to 0))^- \\ = & (((x \to x^-) * x) \to 0)^- \\ = & ((x \wedge x^-) \to 0)^- \\ = & (x^- \vee x^{--})^- \\ = & x^{--} \wedge x^- \in I,\end{array}$$

by $(x \to x^-)^- \in I$ and (I2) we get $x \in I$. ∎

Definition 5.3. [22] A fuzzy ideal μ in A is said to be a fuzzy Boolean ideal if $\mu(x \wedge x^-) = \mu(0)$ for all $x \in A$. In this case we also say that the fuzzy ideal μ in A is Boolean.

Proposition 5.4. *A fuzzy ideal μ in A is Boolean if, for any $t \in [0, 1]$, μ_t is a Boolean ideal of A where $\mu_t \neq \emptyset$.*

Definition 5.5. Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\xi : \Omega \to \mathscr{P}(A)$ be a random set. If $\xi(\omega)$ is a Boolean ideal of A for any $\omega \in \Omega$, then the falling shadow $\tilde{H}$ of the random set ξ is called a falling fuzzy Boolean ideal of A.

Example 5.6. Let A be the BL-algebra in Example 3.3. It is easy to see that $\{0\}, \{0, a\}, \{0, b\}, A$ are ideals of A. Since

$$\begin{array}{r|l}
1^- = 0 & 0 \wedge 0^- = 0 \\
0^- = 1 & 1 \wedge 1^- = 0 \\
a^- = b & a \wedge a^- = a \wedge b = 0 \\
b^- = a & b \wedge b^- = b \wedge a = 0,
\end{array}$$

It follows that $\{0\}, \{0, a\}, \{0, b\}, A$ are Boolean ideals of A. Let $(\Omega, \mathscr{A}, P) = ([0, 1], \mathscr{A}, m)$, where $\mathscr{A}$ is a Borel field on $[0, 1]$ and m the usual Lebesgue measure. Let $\xi : [0, 1] \to \mathscr{P}(A)$ be defined by

$$\xi(t) := \begin{cases} \{0\} & \text{if} \quad t \in [0, 0.2), \\ \{0, a\} & \text{if} \quad t \in [0.2, 0.4), \\ \{0, b\} & \text{if} \quad t \in [0.4, 0.9), \\ \{0, a, b, 1\} & \text{if} \quad t \in [0.9, 1]. \end{cases}$$

Thus $\xi(t)$ is a Boolean ideal of A for all $t \in [0, 1]$. Hence $\tilde{H}(x)$ is a falling fuzzy Boolean ideal of A where $\tilde{H}(x) = P(\{t \in [0, 1] \mid x \in \xi(t)\})$ and

$$\tilde{H}(x) := \begin{cases} 1 & \text{if} \quad x = 0, \\ 0.3 & \text{if} \quad x = a, \\ 0.6 & \text{if} \quad x = b, \\ 0.1 & \text{if} \quad x = 1. \end{cases}$$

If $\xi : [0, 1] \to \mathscr{P}(A)$ be defined by

$$\xi(t) := \begin{cases} \{0\} & \text{if} \quad t \in [0, 0.2), \\ \{0, a, b\} & \text{if} \quad t \in [0.4, 0.9), \\ \{0, a, b, 1\} & \text{if} \quad t \in [0.9, 1]. \end{cases}$$

then $\tilde{H}$ is not a falling fuzzy Boolean ideal of A, because $\{0, a, b\}$ is not an ideal of A.

Theorem 5.7. *Each fuzzy Boolean ideal in A is a falling fuzzy Boolean ideal of A.*

Proof. It is similar to Theorem 4.7. ∎

Theorem 5.8. *Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\xi : \Omega \to \mathscr{P}(A)$ be a random set. If $\tilde{H}$ is the falling fuzzy Boolean ideal of A, then $\tilde{H}$ is a falling fuzzy Gödel ideal of A.*

Proof. Let $\tilde{H}$ is the falling fuzzy Boolean ideal of A. For any $\omega \in \Omega$, suppose $\omega \in \Omega(((x^-)^2 \to y^-)^-;\xi)$, then $((x^-)^2 \to y^-)^- \in \xi(\omega)$. Since

$$\begin{array}{rl} & ((x \wedge x^-)^- \to (x^- \to y^-)^{--})^- \\ = & ((x \wedge x^-)^- \to (x^- \to y^-))^- \\ = & ((x^- \vee x^{--}) \to (x^- \to y^-))^- \\ = & (((x^- \vee x^{--}) * x^-) \to y^-)^- \\ = & (((x^-)^2 \vee (x^- * x^{--})) \to y^-)^- \\ = & (((x^-)^2 \vee 0) \to y^-)^- \\ = & (((x^-)^2 \to y^-)^- \\ \in & \xi(\omega), \end{array}$$

from $x \wedge x^- \in \xi(\omega)$ it follows that $(x^- \to y^-)^- \in \xi(\omega)$, hence $\omega \in \Omega((x^- \to y^-)^-;\xi)$. This shows that $\Omega(((x^-)^2 \to y^-)^-;\xi) \subseteq \Omega((x^- \to y^-)^-;\xi)$. By Theorem 4.9, $\tilde{H}$ is a falling fuzzy Gödel ideal of A. ■

Note 5.9. From above we know that a falling fuzzy Boolean ideal is a falling fuzzy Gödel ideal, but the converse is whether or not true?

Theorem 5.10. *Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\xi : \Omega \to \mathscr{P}(A)$ be a random set. If $\tilde{H}$ is a falling fuzzy ideal of A and satisfies: for any $x, y \in A$,*

$$(*_1) \quad \Omega((x \to (y \to x)^-)^-;\xi) \subseteq \Omega(x;\xi),$$

then $\tilde{H}$ is a falling fuzzy Boolean ideal of A.

Proof. For any $\omega \in \Omega$ and $(x \to (y \to x)^-)^- \in \xi(\omega)$, then $\omega \in \Omega((x \to (y \to x)^-)^-;\xi)$. By $(*_1)$ we get $\omega \in \Omega(x;\xi)$, that is, $x \in \xi(\omega)$. Thus, we have proved that

$$(*') \qquad (x \to (y \to x)^-)^- \in \xi(\omega) \text{ implies } x \in \xi(\omega) \text{ for any } x, y \in A.$$

Since, for any $x \in A$, we have

$$\begin{array}{rcl} ((x \wedge x^-) \to (1 \to (x \wedge x^-))^-)^- & = & ((x \wedge x^-) \to (x \wedge x^-)^-)^- \\ & = & ((x \wedge x^-) \to (x^- \vee x^{--}))^- \\ & = & 1^- = 0 \in \xi(\omega), \end{array}$$

it follows from $(*')$ that $x \wedge x^- \in \xi(\omega)$. Hence $\xi(\omega)$ is a Boolean ideal of A, and so $\tilde{H}$ is a falling fuzzy Boolean ideal of A. ■

Definition 5.11. An ideal I of A is said to be an *implicative ideal* if it satisfies: for any $x, y \in A$,

$$(*_2) \qquad (x \to (y \to x)^-)^- \in I \text{ implies } x \in I.$$

Definition 5.12. [22] A fuzzy ideal μ of A is said to be an *fuzzy implicative ideal* if it satisfies: for any $x, y \in A$,

$$(*_3) \qquad \mu((x \to (y \to x)^-)^-) = \mu(0) \text{ implies } \mu(x) = \mu(0).$$

Definition 5.13. Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\xi : \Omega \to \mathscr{P}(A)$ be a random set. If $\xi(\omega)$ is an implicative ideal of A for any $\omega \in \Omega$, then the falling shadow $\tilde{H}$ of the random set ξ is called a falling fuzzy implicative ideal of A.

Theorem 5.14. *Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\xi : \Omega \to \mathscr{P}(A)$ be a random set. If $\tilde{H}$ is the falling fuzzy ideal of A, then $\tilde{H}$ is a falling fuzzy implicative ideal of A if and only if it satisfies: for any $x, y \in A$,*

$$(*_4) \qquad \Omega((x \to (y \to x)^-)^-; \xi) \subseteq \Omega(x; \xi).$$

Proof. It is easy and omitted. ∎

Theorem 5.15. *Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\xi : \Omega \to \mathscr{P}(A)$ be a random set. If $\tilde{H}$ is a falling fuzzy ideal of A, then $\tilde{H}$ is a falling implicative ideal of A if and only if $\tilde{H}$ is a falling fuzzy Boolean ideal of A.*

Proof. $(\Rightarrow)$ It is immediate by Theorem 5.9.

$(\Leftarrow)$ Suppose that $\tilde{H}$ is a falling fuzzy Boolean ideal of A and $\omega \in \Omega((x \to (y \to x)^-)^-; \xi)$. Hence $(x \to (y \to x)^-)^- \in \xi(\omega)$. Since

$$\begin{array}{rl} & ((x \to (y \to x)^-)^{--} \to (x \to x^-)^{--})^- \\ = & ((x \to x^-)^- \to (x \to (y \to x)^-)^-))^- \\ = & ((x \to (y \to x)^-) \to (x \to x^-)^{--})^- \\ = & ((x \to (y \to x)^-) \to (x \to x^-))^- \\ \leq & ((y \to x)^- \to x^-)^- \\ \leq & (x \to (y \to x))^- = 1^- = 0 \in \xi(\omega), \end{array}$$

we have $((x \to (y \to x)^-)^{--} \to (x \to x^-)^{--})^- \in \xi(\omega)$.

From $(x \to (y \to x)^-)^- \in \xi(\omega)$ it follows that $(x \to x^-)^- \in \xi(\omega)$. Since $\xi(\omega)$ is a Boolean ideal of A, by lemma 5.2 we get $x \in \xi(\omega)$, i.e., $\omega \in \Omega(x; \xi)$. This shows that $\Omega((x \to (y \to x)^-)^-; \xi) \subseteq \Omega(x; \xi)$. Therefore $\tilde{H}$ is a falling implicative ideal of A. ∎

6. Falling fuzzy inference relations

Based on the theory of falling shadows, Tan et al.[12] establish a theoretical approach to define a fuzzy inference relation. Let B and C be fuzzy sets in the universes U and V, respectively, ξ and η be cut-clouds of B and C, respectively. Then fuzzy inference relation $T_{B \to C}$ of the implication $B \to C$ be defined to be

$$\begin{array}{rl} I_{B \to C}(u, v) = & P(\{(s, t) \mid (u, v) \in I_{B_s \to C_t}\}) \\ = & P(\{(s, t) \mid (u, v) \in (B_s \times C_t) \cup (B_s^c \times V)\}) \end{array}$$

where P is a joint probability on $[0, 1]^2$. So different probability distribution P will generate different formulas for the fuzzy inference relations. The following three basic cases are considered.

Theorem 6.1. [12]

(1) *If the whole probability P of (s,t) on $[0,1]^2$ is concentrated and uniformly distributed on the main diagonal $\{(s,s) \mid s \in [0,1]\}$ of the unit square $[0,1]^2$, then P is the diagonal distribution and $I_{B\to C}(s,t) = \min\{1 - B(s) + C(t), 1\}$.*

(2) *If the whole probability P of (s,t) on $[0,1]^2$ is concentrated and uniformly distributed on the anti-diagonal $\{(s,1-s) \mid s \in [0,1]\}$ of the unit square $[0,1]^2$, then P is the anti-diagonal distribution and $I_{B\to C}(s,t) = \max\{1 - B(s), C(t)\}$.*

(3) *If the whole probability P of (s,t) on $[0,1]^2$ is uniformly distributed on the unit square $[0,1]^2$, then P is the independence distribution and $I_{B\to C}(s,t) = 1 - B(s) + B(s)C(t)$.*

We call the three fuzzy inference relations falling implication operators on $[0,1]$. In what follows we consider the concept of I-fuzzy ideals of BL-algebras.

Definition 6.2. Let μ be a fuzzy set of A, I be a falling implication operator over $[0,1]$ and $t \in (0,1]$. Then μ is called an I-fuzzy ideal of A if, for all $x, y \in A$, the following conditions are satisfied:

(FFI1) $I(\mu(x), \mu(0)) \geq t$;
(FFI2) $I(\min\{\mu((x^- \to y^-)^-), \mu(x)\}, \mu(y)) \geq t$.

Obviously, if $t = 1$ and P is the diagonal distribution, then Definition 6.2 is equivalent to Definition 2.6.

Theorem 6.3. *Let μ be a fuzzy set of A and $t = 0.5$. Then*

(1) *if P is the diagonal distribution then μ is an I-fuzzy ideal of A if and only if it satisfies* (a1) *and* (b1), *where*

(a1) $\mu(x) \leq \mu(0)$ *or* $0 < \mu(x) - \mu(0) \leq 0.5$ *for all* $x, y \in A$,

(b1) $\min\{\mu((x^- \to y^-)^-), \mu(x)\} \leq \mu(y)$ *or* $0 < \min\{\mu((x^- \to y^-)^-), \mu(x)\} - \mu(y) \leq 0.5$ *for all* $x, y \in A$.

(2) *if P is the anti-diagonal distribution then μ is an I-fuzzy ideal of A if and only if it satisfies* (a2) *and* (b2), *where*

(a2) $\mu(x) \leq \max\{\mu(0), 0.5\}$ *or* $\min\{\mu(x), 0.5\} \leq \mu(0)$ *for all* $x, y \in A$,

(b2) $\min\{\mu((x^- \to y^-)^-), \mu(x)\} \leq \max\{\mu(y), 0.5\}$ *or* $\min\{\mu((x^- \to y^-)^-), \mu(x), 0.5\} \leq \mu(y)$ *for all* $x, y \in A$.

(3) *if P is the independent distribution then μ is an I-fuzzy ideal of A if and only if it satisfies* (a3) *and* (b3), *where*

(a3) $\mu(x)(1 - \mu(0)) \leq 0.5$ *for all* $x, y \in A$,

(b3) $\min\{\mu((x^- \to y^-)^-), \mu(x)\}(1 - \mu(y)) \leq 0.5$ *for all* $x, y \in A$.

Proof. (1) Suppose P is the diagonal distribution. Then

$$I(\mu(x), \mu(0)) = \min\{1 - \mu(x) + \mu(0), 1\}.$$

If μ is an I-fuzzy ideal of A, then $\min\{1 - \mu(x) + \mu(0), 1\} \geq 0.5$ by (FFI1). When $\mu(x) > \mu(0)$, we have $\mu(x) - \mu(0) > 0$, and so $1 - \mu(x) + \mu(0) < 1$. Thus $\min\{1 - \mu(x) + \mu(0), 1\} = 1 - \mu(x) + \mu(0) \geq 0.5$, and $0 < \mu(x) - \mu(0) \leq 0.5$. Hence (a1) holds. By (FFI2) we have $I(\min\{\mu((x^- \to y^-)^-), \mu(x)\}, \mu(y)) \geq 0.5$, that is,

$$\min\{\min\{\mu((x^- \to y^-)^-), \mu(x)\} - \mu(y), 1\} > 0.5.$$

If $\min\{\mu((x^- \to y^-)^-), \mu(x)\} > \mu(y)$, then $0 < \min\{\mu((x^- \to y^-)^-), \mu(x)\} - \mu(y) \leq 0.5$. Thus (b1) holds.

Conversely, suppose that (a1) and (b1) hold. Let P be the diagonal distribution. By Theorem 6.1 we have $I(\mu(x), \mu(0)) = \min\{1 - \mu(x) + \mu(0), 1\}$. When $\mu(x) \leq \mu(0)$, we have $1 - \mu(x) + \mu(0) = 1 + [\mu(0) - \mu(x)] \geq 1$, and so $I(\min\{1 - \mu(x) + \mu(0), 1\}) = 1 > 0.5$. When $0 < \mu(x) - \mu(0) \leq 0.5$, we have

$$\begin{aligned} I(\mu(x), \mu(0)) &= \min\{1 - \mu(x) + \mu(0), 1\} \\ &= 1 - [\mu(x) - \mu(0)] \\ &\geq 0.5. \end{aligned}$$

Hence (FFI1) is true. Also by Theorem 6.1,

$$\begin{aligned} & I(\min\{\mu((x^- \to y^-)^-), \mu(x)\}, \mu(y)) \\ = \; & \min\{1 - \min\{\mu((x^- \to y^-)^-), \mu(x)\} + \mu(y), 1\}. \end{aligned}$$

When $\min\{\mu((x^- \to y^-)^-), \mu(x)\} \leq \mu(y)$, we have

$$\begin{aligned} & 1 - \min\{\mu((x^- \to y^-)^-), \mu(x)\} + \mu(y) \\ = \; & 1 + [\mu(y) - \min\{\mu((x^- \to y^-)^-), \mu(x)\}] \\ \geq \; & 1, \end{aligned}$$

and so $I(\min\{\mu((x^- \to y^-)^-), \mu(x)\}, \mu(y)) = 1 > 0.5$.

When $0 < \min\{\mu((x^- \to y^-)^-), \mu(x)\} - \mu(y) \leq 0.5$, we have

$$\begin{aligned} & I(\min\{\mu((x^- \to y^-)^-), \mu(x)\}, \mu(y)) \\ = \; & \min\{1 - \min\{\mu((x^- \to y^-)^-), \mu(x)\} + \mu(y), 1\} \\ = \; & \min\{1 - [\min\{\mu((x^- \to y^-)^-), \mu(x)\} - \mu(y)], 1\} \\ \geq \; & 0.5, \end{aligned}$$

i.e., $I(\min\{\mu((x^- \to y^-)^-), \mu(x)\}, \mu(y)) \geq 0.5$. (FFI2) holds. This proves (1).

(2) Suppose that P is the anti-diagonal distribution and μ is an I-fuzzy ideal of A. By (FFI1), $I(\mu(x), \mu(0)) = \max\{1 - \mu(x), \mu(0)\} \geq 0.5$. If $\mu(x) > \max\{\mu(0), 0.5\}$, then $\mu(x) > 0.5$, and $1 - \mu(x) < 0.5$. Hence we have $\mu(0) = \max\{1 - \mu(x), \mu(0)\} \geq 0.5$, and so $\min\{\mu(x), 0.5\} \leq \mu(0)$. Thus (a2) holds. By the same argument we can prove that (b2) holds.

Conversely, suppose that (a2) and (b2) hold. Let P be the anti-diagonal distribution. By Theorem 6.1 we have $I(\mu(x), \mu(0)) = \max\{1 - \mu(x), \mu(0)\}$. Let $\mu(x) \leq \max\{\mu(0), 0.5\}$. If $\mu(0) \geq 0.5$, then $I(\mu(x), \mu(0)) \geq 0.5$. If $\mu(0) < 0.5$, then $\mu(x) \leq 0.5$. Thus $1 - \mu(x) \geq 0.5$, and $I(\mu(x), \mu(0)) \geq 0.5$. Let $\min\{\mu(x), 0.5\} \leq \mu(0)$. If $0.5 \leq \mu(x)$ then $0.5 \leq \mu(0)$. Thus $I(\mu(x), \mu(0)) = \max\{1 - \mu(x), \mu(0)\} \geq 0.5$. If $0.5 \geq \mu(x)$ then $1-\mu(x) \geq 0.5$. Thus $I(\mu(x), \mu(0)) = \max\{1-\mu(x), \mu(0)\} \geq 0.5$. Therefore (FFI1) holds. This proves (2).

(3) Suppose that P is the independent distribution and μ is an I-fuzzy ideal of A. By (FFI1), $I(\mu(x), \mu(0)) = 1 - \mu(x) + \mu(x)\mu(0) \geq 0.5$. Hence $0.5 \leq \mu(x) - \mu(x)\mu(0) = \mu(x)(1-\mu(0))$, (a3) holds. By the same argument we can prove that (b3) holds.

Conversely, suppose that (a3) and (b3) hold. Let P be the independent distribution. By Theorem 6.1, $I(\mu(x), \mu(0)) = 1-\mu(x)+\mu(x)\mu(0)$. By (a3) we have $0.5 \geq \mu(x)(1-\mu(0))$. Thus $1-\mu(x)+\mu(x)\mu(0) = 1-\mu(x)(1-\mu(0)) \geq 1-0.5 = 0.5$ (FFI1) is true. By the same argument we can prove that (FFI2) holds. This proves (3). ∎

Definition 6.4. Let μ be a fuzzy set of A, I be a falling implication operator over $[0, 1]$ and $t \in (0, 1]$. Then μ is called an I-fuzzy Gödel ideal of A, if it is satisfies (FFI1) and (FFI3), where

(FFI3) $I(\min\{\mu((x^- \to (y^- \to z^-))^-), \mu((x^- \to y^-)^-)\}, \mu((x^- \to z^-)^-)) \geq t$ for any $x, y, z \in A$.

If $t = 1$ and P is the diagonal distribution, then we easily prove that Definition 6.4 is equivalent to Definition 4.2.

By Theorem 6.3 and Definition 6.4 we can prove the following

Theorem 6.5. *Let μ be a fuzzy set of A and $t = 0.5$, then for all $x, y \in A$*

(1) if P is the diagonal distribution then μ is an I-fuzzy Gödel ideal of A if and only if it satisfies (a1) and (b4), where

(b4) $\min\{\mu((x^- \to (y^- \to z^-))^-), \mu((x^- \to y^-)^-)\} \leq \mu((x^- \to z^-)^-)$ *or* $0 < \min\{\mu((x^- \to (y^- \to z^-))^-), \mu((x^- \to y^-)^-)\} - \mu((x^- \to z^-)^-) \leq 0.5$.

(2) *if P is the anti-diagonal distribution then μ is an I-fuzzy Gödel ideal of A if and only if it satisfies* (a2) *and* (b5), *where*

(b5) $\min\{\mu(\mu((x^- \to (y^- \to z^-))^-), \mu((x^- \to y^-)^-)\} \leq \max\{\mu((x^- \to z^-)^-), 0.5\}$ *or* $\min\{\mu(\mu((x^- \to (y^- \to z^-))^-), \mu((x^- \to y^-)^-), 0.5\} \leq \mu((x^- \to z^-)^-)$.

(3) *if P is the independent distribution then μ is an I-fuzzy Gödel ideal of A if and only if it satisfies* (a3) *and* (b6), *where*

(b6) $\min\{\mu((x^-\to (y^- \to z^-))^-), \mu((x^-\to y^-)^-)\}(1-\mu((x^-\to z^-)^-)) \leq 0.5$.

7. Conclusion

The theory of falling shadows relates to probability concepts with the membership functions of fuzzy sets. Falling shadow representation theory is a reasonable and convenient approach for the theoretical development and the practical applications of fuzzy sets and fuzzy logics. In this paper we apply falling theory to ideal theory of BL-algebras, and obtain some results. Also we consider falling fuzzy inference relations to BL-algebras. As the continuation of these results, we will further apply falling shadow theory and falling fuzzy inference relations to information systems and computer.

Acknowledgements. The authors would like to express their sincere thanks to the referees for their valuable suggestions and comments. This research is supported by a grant of National Natural Science Foundation of China (11571281).

References

[1] Cignoli, R., Esteva, F., Godo,, L., Torrens, A., *Basic fuzzy logic is the logic of continuous T-norm and their residua*, Soft Comput., **12**(2000), 106-112.

[2] Goodman, I.R., *Fuzzy sets as equivalence classes of random sets*, in: R.Yager(ed.), Recent developments in fuzzy sets and probability theory, Pergamon, New York, NY, USA, 1982.

[3] Hájek, P., *Metamathematics of fuzzy logic*, Kluwer Academic Publishers, Dordrecht, 1998.

[4] Haveshki, M., Saeid, A.B., Eslami, E., *Some types of filters in BL-algebras*, Soft Computing, 10 (2006), 657-664.

[5] Jun, Y.B., Kang, M.S., *Fuzzy positive implicative ideals of BCK-algebras based on the theory of falling shadows*, Comput. Math. Apll., 61 (2011), 62-67.

[6] Jun, Y.B., Kang, M.S., Park, C.H., *Implicative ideals of BCK-algebras based on the fuzzy sets and the theory of falling shadows*, Int. J. Math Math Sci., Article ID 819463 (2011), 11 pages.

[7] Liu, L.Z., Li, K.T., *Fuzzy filters in BL-algebras*, Inform Sci., 173 (2005), 141-154.

[8] Meng, B.L., Xin, X.L., *Prime ideals and Gödel ideals of BL-algebras*, Journal of Advances in Mathematics, 9 (2015), 2989-3005.

[9] Meng, B.L., Xin, X.L., *On fuzzy ideal theory of BL-algebras*, Scientific World J., vol. 2014, Article ID 757382, 12 pp.

[10] Saeid, A.B., Motamed, S., *Normal filters in BL-Algebras*, World Applied Sciences Journal, 7 (2009), 70-76.

[11] Saeid, A.B., Ahadpanah, A., Torkzadeh, L., *Smarandache BL-Algebras*, Journal of Applied Logic, 8 (2010), 253-261.

[12] Tan, S.K., Wang, P.Z., Zhang, X.Z., *Fuzzy inference relation based on the theory of falling shadows*, Fuzzy Sets and Systems, 53 (1993), 179-188.

[13] Tan, S.K., Wang, P.Z., Zhang, X.Z., *Fuzzy set operations based on the theory of falling shadows*, J. Math. Anal. Appl., 174 (1993), 242-255.

[14] Turunen, E., *BL-algebras of basic fuzzy logic,* Mathware & Soft Comp., 6 (1999), 49-61.

[15] Turunen, E., *Boolean deductive systems of BL-algebras,* Arch. Math. Logic, 40 (2001), 467-473.

[16] Wang, G.J., *On the logic foundation of fuzzy reasoning*, Inform. Sci., 117 (1999), 47-88.

[17] Wang, P.Z., *Fuzzy sets and falling shadows of random sets*, Beijing Normal Univ. Press, People's Republic of China (in Chinese), 1985.

[18] Wang, P.Z., Sanchez, E., *Treating a fuzzy subsets as a projectable random set in Fuzzy Information and Decision*, M.M. Gupta and E. Sanchez, Eds., Pergamon, New York, NY, USA, 1982, 212-219.

[19] Yuan, X.H., Lee, E.S., *A fuzzy algebraic system based on the theory of falling shadows*, J. Math. Anal. Appl., 208 (1997), 243-251.

[20] Zadeh, L.A., *Fuzzy sets*, Inform. Control, 8 (1965), 338-353.

[21] Zhan, J.M., Jun, Y.B., Kim, H.S., *Some types of falling fuzzy filters in BL-algebras and its applications*, J. Intelligent & Fuzzy systems, 26 (2014), 1675-1685.

[22] Zhang, X.H., Jun, Y.B., Doh, M.I., *On fuzzy filters and fuzzy ideals of BL-algebras*, Fuzzy Systems and Mathematics, 20 (2006), 8-20.

Accepted: 12.04.2016