RESULTS ON PRIME IDEALS IN PMV-ALGEBRAS AND MV-MODULES

S. Saidi Goraghani[1]

Department of Mathematics
Farhangian University
Iran
e-mail: SiminSaidi@yahoo.com

R.A. Borzooei

Department of Mathematics
Shahid Beheshti University
Tehran
Iran
e-mail: borzooei@sbu.ac.ir

Abstract. In this paper, by considering the notions of MV-modules and PMV-algebras, we study $\cdot$-prime ideals in PMV-algebras, prime A-ideals in MV-modules and investigate some properties on them. Also, we present the definitions of radical of a $\cdot$-ideal in PMV-algebras, radical of an A-ideal in MV-modules and verify some properties on them. Finally, we state a method to obtain the radical of a $\cdot$-ideal in PMV-algebras.

Keywords: (MV, PMV)-algebra, MV-module, $\cdot$-prime ideal, prime A-ideal, radical.

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1. Introduction

MV-algebras were defined by C.C. Chang [1], [2] as algebras corresponding to the Lukasiewicz infinite valued propositional calculus. These algebras have appeared in the literature under different names and polynomially equivalent presentation: CN-algebras, Wajsberg algebras, bounded commutative BCK-algebras and bricks. It is discovered that MV-algebras are naturally related to the Murray-von Neumann order of projections in operator algebras on Hilbert spaces and that they play an interesting role as invariants of approximately finite-dimensional C^*-algebras. They are also naturally related to Ulam's searching games with lies. MV-algebras admit a natural lattice reduct and hence a natural order structure. Many important properties can be derived from the fact, established by Chang

[1]Corresponding author.

that nontrivial MV-algebras are subdirect products of MV-chains, that is, totally ordered MV-algebras. To prove this fundamental result, Chang introduced the notion of prime ideal in an MV-algebra. The categorical equivalence between MV-algebras and lu-groups leads to the problem of defining a product operation on MV-algebras, in order to obtain structures corresponding to l-rings. A *product MV-algebra* (or PMV-algebra, for short) is an MV-algebra which has an associative binary operation ".". It satisfies an extra property which will be explained in Preliminaries. During the last years, PMV-algebras were considered and their equivalence with a certain class of l-rings with strong unit was proved. It seems quite natural to introduce modules over such algebras, generalizing the divisible MV-algebras and the MV-algebras obtained from Riesz spaces and to prove natural equivalence theorems. Hence, the notion of MV-modules was introduced as an action of a PMV-algebra over an MV-algebra by A. Di Nola [5]. In 2014, F. Forouzesh, E. Eslami and A. Borumand Saeid defined prime A-ideals in MV-modules and $\cdot$-prime ideals in PMV-algebras [7]. Also, they defined radical of A-ideals by maximal A-ideals. Since MV-modules are in their infancy, stating and opening of any subject in this field can be useful. Hence, in this paper, we investigate prime A-ideals in MV-modules and verify some properties on them. For example, we state some conditions for obtaining a prime A-ideal. Also, we present the definition of radical of an A-ideal by prime A-ideals in MV-modules, radical of a $\cdot$-ideal in PMV-algebras and verify some properties on them. Then we state a method to obtain a radical of a $\cdot$-ideal in PMV-algebras. In fact, we open new fields to anyone that is interested to studying and development of MV-modules.

2. Preliminaries

In this section, we review related lemmas and theorems that we use in the next sections.

Definition 2.1 [3] An *MV-algebra* is a structure $M = (M, \oplus, ', 0)$ of type $(2, 1, 0)$ such that:
$(MV1)$ $(M, \oplus, 0)$ is an Abelian monoid,
$(MV2)$ $(a')' = a$,
$(MV3)$ $0' \oplus a = 0'$,
$(MV4)$ $(a' \oplus b)' \oplus b = (b' \oplus a)' \oplus a$,
If we define the constant $1 = 0'$ and operations $\odot$ and $\ominus$ by $a \odot b = (a' \oplus b')'$, $a \ominus b = a \odot b'$, then
$(MV5)$ $(a \oplus b) = (a' \odot b')'$,
$(MV6)$ $x \oplus 1 = 1$,
$(MV7)$ $(a \ominus b) \oplus b = (b \ominus a) \oplus a$,
$(MV8)$ $a \oplus a' = 1$,
for every $a, b \in M$. It is clear that $(M, \odot, 1)$ is an Abelian monoid. Now, if we define auxiliary operations $\vee$ and $\wedge$ on M by $a \vee b = (a \odot b') \oplus b$ and $a \wedge b = a \odot (a' \oplus b)$, for every $a, b \in M$, then $(M, \vee, \wedge, 0)$ is a *bounded distributive lattice.*

An MV-algebra M is a *Boolean* algebra if and only if the operation " $\oplus$ " is idempotent, that is, $x \oplus x = x$, for every $x \in M$. In an MV-algebra M, the following conditions are equivalent: (i) $a' \oplus b = 1$, (ii) $a \odot b' = 0$, (iii) $b = a \oplus (b \ominus a)$, (iv) $\exists c \in M$ such that $a \oplus c = b$, for every $a, b, c \in M$. For any two elements a, b of the MV-algebra M, $a \leq b$ if and only if a, b satisfy the above equivalent conditions $(i) - (iv)$. An ideal of MV-algebra M is a subset I of M, satisfying the following conditions: $(I1)$: $0 \in I$, $(I2)$: $x \leq y$ and $y \in I$ imply $x \in I$, $(I3)$: $x \oplus y \in I$, for every $x, y \in I$. A proper ideal I of M is a prime ideal of M if and only if $x \ominus y \in I$ or $y \ominus x \in I$ (or $x \wedge y \in I$ implies that $x \in I$ or $y \in I$), for every $x, y \in M$. In an MV-algebra M, the *distance function* $d : M \times M \to M$ is defined by $d(x, y) = (x \ominus y) \oplus (y \ominus x)$ which satisfies (i): $d(x, y) = 0$ if and only if $x = y$, (ii): $d(x, y) = d(y, x)$, (iii): $d(x, z) \leq d(x, y) \oplus d(y, z)$, (iv): $d(x, y) = d(x', y')$, (v): $d(x \oplus z, y \oplus t) \leq d(x, y) \oplus d(z, t)$, for every $x, y, z, t \in M$. Let I be an ideal of an MV-algebra M. We denote $x \sim y$ ($x \equiv_I y$) if and only if $d(x, y) \in I$, for every $x, y \in M$. So $\sim$ is a congruence relation on M. Denote the equivalence class containing x by $\frac{x}{I}$ and $\frac{M}{I} = \{\frac{x}{I} : x \in X\}$. Then $(\frac{M}{I}, \oplus, ', \frac{0}{I})$ is an MV-algebra, where $(\frac{x}{I})' = \frac{x'}{I}$ and $\frac{x}{I} \oplus \frac{y}{I} = \frac{x \oplus y}{I}$, for all $x, y \in M$. Let M and K be two MV-algebras. A mapping $f : M \to K$ is called an MV-*homomorphism* if $(H1)$: $f(0) = 0$, $(H2)$: $f(x \oplus y) = f(x) \oplus f(y)$ and $(H3)$: $f(x') = (f(x))'$, for every $x, y \in M$. If f is one to one (onto), then f is called an MV-monomorphism (epimorphism) and if f is onto and one to one, then f is called an MV-isomorphism (see [4, 10])

Proposition 2.2 [3] *Let M be an MV-algebra and $z \in M$. Then the principal ideal generated by z is denoted by $\prec z \succ$ and*

$$\prec z \succ = \{x \in M : nz = \underbrace{z \oplus \cdots \oplus z}_{n\ times} \geq x,\ for\ some\ n \geq 0\}.$$

Proposition 2.3 [3] *Let I be an ideal of A. Then*

$$\prec I \cup \{z\} \succ = \{x \in A : x \leq nz \oplus a,\ for\ some\ n \in \mathbb{N}\ and\ a \in I\}.$$

Proposition 2.4 [3] *In every MV-algebra A, the natural order " $\leq$ " has the following properties:*
(i) $x \leq y$ if and only if $y' \leq x'$,
(ii) if $x \leq y$, then $x \oplus z \leq y \oplus z$ and $x \odot z \leq y \odot z$, for every $x, y, z \in A$.

Definition 2.5 [5], [6] (i) An l-group is an algebra $(G, +, -, 0, \vee, \wedge)$, where the following properties hold:
(a) $(G, +, -, 0)$ is a group,
(b) $(G, \vee, \wedge)$ is a lattice,
(c) $x \leq y$ implies that $x + a \leq y + a$, for any $x, y, a, b \in G$.
A strong unit $u > 0$ is a positive element with property that for any $g \in G$ there exits $n \in \omega$ such that $g \leq nu$. The Abelian l-groups with strong unit will be simply called lu-groups.

The category whose objects are MV-algebras and whose homomorphisms are MV-homomorphisms is denoted by MV. The category whose objects are pairs (G, u), where G is an Abelian l-group and u is a strong unit of G and whose homomorphisms are l-group homomorphisms is denoted by Ug. The functor that establishes the categorial equivalence between MV and Ug is

$$\Gamma : Ug \longrightarrow MV,$$

where $\Gamma(G, u) = [0, u]_G$, for every lu-group (G, u) and $\Gamma(h) = h|_{[0,u]}$, for every lu-group homomorphism h. The above results allows us to consider an MV-algebra, when necessary, as an interval in the positive cone of an l-group. Thus, many definitions and properties can be transferred from l-groups to MV-algebras. For example, the group addition becomes a partial operation when it is restricted to an interval, so we define a *partial addition* on an MV-algebra M as follows:
$x + y$ is defined if and only if $x \leq y'$ and in this case, $x + y = x \oplus y$, for every $x, y \in M$. Moreover, if $z + x \leq z + y$, then $x \leq y$.
(ii) An l-ring is a structure $(R, +, ., 0, \leq)$, where $(R, +, 0, \leq)$ is an L-group such that, for any $x, y \in R$,

$$x \geq 0 \textit{ and } y \geq 0 \textit{ implies } x.y \geq 0.$$

(iii) A *product MV-algebra* (or PMV-algebra, for short) is a structure $A = (A, \oplus, ., ', 0)$, where $(A, \oplus, ', 0)$ is an MV-algebra and "." is a binary associative operation on A such that the following property is satisfied: if $x+y$ is defined, then $x.z+y.z$ and $z.x+z.y$ are defined and $(x+y).z = x.z+y.z$, $z.(x+y) = z.x+z.y$, for every $x, y, z \in A$, where " $+$ " is the partial addition on A. A unity for the product is an element $e \in A$ such that $e.x = x.e = x$, for every $x \in A$. If A has a unity for product, then $e = 1$. A PMV-homomorphism is an MV-homomorphism which also commutes with the product operation. A $\cdot$-ideal of A is an ideal I of A such that if $a \in I$ and $b \in A$ entail $a.b \in I$ and $b.a \in I$. The set of $\cdot$-ideals of A is denoted by $Id(A)$.

Lemma 2.6 [4] *Let A be a PMV-algebra. Then $a \leq b$ implies that $a.c \leq b.c$ and $c.a \leq c.b$ for every $a, b, c \in A$.*

Lemma 2.7 [5] *Let M be an MV-algebra . Then for every $x, y, z \in M$,*
(b) $x + 0 = x$,
(c) $x \vee y = x + (x' \odot y)$,
(d) *if $x+y$ and $(x+y)+z$ are defined, then $y+z$ and $x+(y+z)$ are defined and* $(x + y) + z = x + (y + z)$,
(f) *if $z + x \leq z + y$, then $x \leq y$,*
(h) *if $z + x = z + y$, then $x = y$,*
where $+$ is the partial addition on M.

Definition 2.8 [5] Let $A = (A, \oplus, ., ', 0)$ be a PMV-algebra, $M = (M, \oplus, ', 0)$ be an MV-algebra and the operation $\Phi : A \times M \longrightarrow M$ is defined by $\Phi(a, m) = am$, which satisfies the following axioms:

$(AM1)$ If $x+y$ is defined in M, then $ax+ay$ is defined in M and $a(x+y)=ax+ay$,
$(AM2)$ If $a+b$ is defined in A, then $ax+bx$ is defined in M and $(a+b)x=ax+bx$,
$(AM3)$ $(a.b)x=a(bx)$, for every $a,b\in A$ and $x,y\in M$.
Then M is called a (left) *MV-module* over A or briefly an A-module. We say that M is a *unitary* MV-module if A has a unity 1_A for the product that is
$(AM4)$ $1_Ax=x$, for every $x\in M$.

Lemma 2.9 [5] *Let A be a PMV-algebra and M be an A-module. Then*
(a) $0x=0$,
(b) $a0=0$,
(c) $ax'\le(ax)'$,
(d) $a'x\le(ax)'$,
(e) $(ax)'=a'x+(1x)'$,
(f) $x\le y$ implies $ax\le ay$,
(g) $a\le b$ implies $ax\le bx$,
(h) $a(x\oplus y)\le ax\oplus ay$,
(i) $d(ax,ay)\le ad(x,y)$,
(j) if M is a unitary MV-module, then $(ax)'=a'x+x'$,
(k) $(ax)\odot(ay)'\le a(x\odot y)$, for every $a,b\in A$ and $x,y\in M$.

Definition 2.10 [5] Let A be a PMV-algebra, M_1 and M_2 be two A-modules. A map $f:M_1\to M_2$ is called an *A-module* homomorphism or (*A-homomorphism*, for short) if f is an MV-homomorphism and
$(H4)$: $f(ax)=af(x)$, for every $x\in M_1$ and $a\in A$.

Definition 2.11 [5] Let A be a PMV-algebra and M be an A-module. Then an ideal $N\subseteq M$ is called an *A-ideal* of M if $(I4)$: $ax\in N$, for every $a\in A$ and $x\in N$.

Definition 2.12 [7] Let M be an A-module and N be a proper A-ideal of M. Then N is called a *prime* A-ideal of M, if $am\in N$ implies that $m\in N$ or $a\in(N:M)$, for any $a\in A$ and $m\in M$, where $(N:M)=\{a\in A: aM\subseteq N\}$. Moreover, the set of all prime A-ideals of M is showed by $Spec(M)$.

Definition 2.13 [8] Let M be an A-module. An A-ideal N of M is called a maximal A-ideal of M, if there exist no A-ideal K of M containing N such that $N\subsetneq K\subsetneq M$. The set of all maximal A-ideals of M is showed by $Max(M)$. Let I be a proper A-ideal in M. The intersection of all maximal A-ideals of M which contain I is called the radical of I and is denoted by $Rad(I)$.

Definition 2.14 [7] Let I be a proper $\cdot$-ideal of A. I is called a $\cdot$-prime ideal of A, if $x.y\in I$ implies that $x\in I$ or $y\in I$, for any $x,y\in A$.

Note. From now on, in this paper, we let A be a PMV-algebra, M be an MV-algebra, $\sum_{i=1}^{n}x_i$ means $x_1\oplus x_2\oplus\cdots\oplus x_n$ and in particular case, $\underbrace{x\oplus x\oplus\cdots\oplus x}_{n\ times}=nx$.

3. Prime A-ideals in MV-modules and prime $\cdot$-ideals in PMV-algebras

In this section, we study prime A-ideals in MV-modules and $\cdot$-prime ideals in PMV-algebras. Then we state and prove some conditions to obtain them.

Remark. Let A be unital and I be an ideal of A. Then by Lemma 2.6, since $x \leq 1$, $x.y \leq x.1 = x \in I$ and $y.x \leq y.1 = y \in I$ and so $x.y, y.x \in I$, for every $x, y \in I$. It means that if A is unital, then every ideal of A is a $\cdot$-ideal of A.

Example 3.1 Let $A = \{0, 1, 2, 3\}$ and the operations "$\oplus$" and "." on A be defined as follows:

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

.	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Consider $0' = 3$, $1' = 2$, $2' = 1$ and $3' = 0$. Then it is easy to show that $(A, \oplus, ', ., 0)$ is a PMV-algebra and $(A, \oplus, ', 0)$ is an MV-algebra. Now, let the operation $\bullet : A \times A \longrightarrow A$ be defined by $a \bullet b = a.b$, for every $a, b \in A$. It is easy to show that A is an MV-module on A, $I = \{0, 1\}$, $J = \{0, 2\}$ are prime A-ideals of A and $\{0\}$ is not a prime A-ideal of A. Also, $I = \{0, 1\}$ is a $\cdot$-prime ideal of A and $I = \{0\}$ is not a $\cdot$-prime ideal of A.

Proposition 3.2 *Let M be an A-module and N be an A-ideal of M. Then $(N : M) = \{a \in A : aM \subseteq N\}$ is an ideal of A.*

Proof. It is clear that $0 \in (N : M)$. Let $a, b \in (N : M)$. Then $am, bm \in N$, for every $m \in N$. Similar the proof of Lemma 2.9 (k), we have $am \odot (bm)' \leq (a \odot b')m$, for every $a, b \in A$ and $m \in M$. If we set $a \oplus b$ instead of a, then by Lemma 2.9 (g), we have $(a \oplus b)m \odot (bm)' \leq ((a \oplus b) \odot b')m = (a \wedge b')m \leq am$. Since

$$(a \oplus b)m = (a \oplus b)m \vee bm = (a \oplus b)m \odot (bm)' \oplus bm \leq am \oplus bm \in N,$$

$a \oplus b \in (N : M)$. Now, let $a \leq b$ and $b \in (N : M)$. Then by Lemma 2.9(g), $am \leq bm \in N$ and so $am \in N$, for every $m \in M$. It means that $a \in (N : M)$. ∎

Remark. Similar the Proposition 3.2, $(N : m)$ is an ideal of A, for every $m \in M$.

Proposition 3.3 *Let M be a unitary A-module and N, L be A-ideals of M. Then*
(i) $(N : M)$ is a prime ideal of A or $\frac{A}{(N:M)}$ has at least two elements,
(ii) if N is a prime A-ideal of M and $m \notin N$, then $(N : m)$ is a $\cdot$-prime ideal of A,
(iii) N is a prime A-ideal of M if and only if $(N : m) = (N : M)$, where $m \notin N$,
(iv) $N \subseteq L$ implies that $(N : M) \subseteq (L : M)$,
(v) if N is a prime A-ideal of M, then $(N : M)$ is a $\cdot$-prime ideal of A.

Proof. (i) By Proposition 3.2, $(N:M)$ is an ideal of A. Let $(N:M)$ is not a prime ideal of A. Then there exist $x, y \in A$ such that $x \ominus y \notin (N:M)$ and $y \ominus x \notin (N:M)$ and so $d(x,y) = (x \ominus y) \oplus (y \ominus x) \notin (N:M)$. It means that $\frac{x}{(N:M)} \neq \frac{y}{(N:M)}$ and so $\frac{A}{(N:M)}$ has at least two elements. Now, let $\frac{A}{(N:M)}$ have only element $\frac{0}{(N:M)}$. Then $\frac{x}{(N:M)} = \frac{y}{(N:M)}$, for every $x, y \in A$ and so $x \ominus y \leq (x \ominus y) \oplus (y \ominus x) = d(x,y) \in (N:M)$. Hence, $x \ominus y \in (N:M)$ and so $(N:M)$ is a prime ideal of A.
$(ii), (iii), (iv)$ The proofs are easy.
(v) By Proposition 3.2, $(N:M)$ is an ideal of A. If $A = (N:M)$, then $1 \in (N:M)$ and so $M = N$, which is a contradiction. Hence, $(N:M)$ is a proper ideal of A. Since A is unital, $(N:M)$ is a $\cdot$-ideal of A. Let $x.y \in (N:M)$, for any $x, y \in A$. Then $x(ym) = (x.y)m \in N$, for every $m \in M$ and so $ym \in N$ or $x \in (N:M)$. Let $x \notin (N:M)$. Then $ym \in N$ and so $y \in (N:M)$ or $m \in N$. If $m \in N$, then $N = M$, which is a contradiction. Hence, $y \in (N:M)$ and so $(N:M)$ is a $\cdot$-prime ideal of A. ∎

Lemma 3.4 *Let M be a unitary A-module and $m \in M$. Then*

$$I_m = \left\{ \sum_{i=1}^{k} t_i m : \sum_{i=1}^{k} t_i m \leq nm, \ for\ some\ n, k \in \mathbb{N} \cup \{0\}, \right.$$
$$\left. where\ t_i \in A\ and\ t_1 m + \cdots + t_k m\ is\ defined \right\}$$

is an A-ideal of M.

Proof. (I_1) It is clear that $0 \in I_m$.
(I_2) Let $x \leq \sum_{i=1}^{k} t_i m \in I_m$, for some $x \in M$. Then $x = 1x \leq \sum_{i=1}^{k} t_i m \leq nm \in I_m$, for some $n \geq 0$ and so $x \in I_m$.
(I_3) Let $\sum_{i=1}^{k} t_i m, \sum_{i=1}^{w} s_i m \in I_m$. Then there exist $n_1, n_2 \geq 0$ such that $\sum_{i=1}^{k} t_i m \leq n_1 m$ and $\sum_{i=1}^{w} s_i m \leq n_2 m$ and so

$$\begin{aligned} \sum_{i=1}^{k+w} c_i m &\leq \sum_{i=1}^{k} t_i m \oplus \sum_{i=1}^{w} s_i m \\ &\leq n_1 m \oplus n_2 m = \underbrace{m \oplus \cdots \oplus m}_{n_1\ times} \oplus \underbrace{m \oplus \cdots \oplus m}_{n_2\ times} = (n_1 + n_2)m, \end{aligned}$$

where $c_i \in A$, for $1 \leq i \leq k + w$. It means that $\sum_{i=1}^{k} t_i m \oplus \sum_{i=1}^{w} s_i m \in I_m$.
(I_4) Let $a \in A$ and $\sum_{i=1}^{k} t_i m \in I_m$. Then there exists $n \geq 0$ such that $\sum_{i=1}^{k} tm \leq nm$.

Since $\sum_{i=1}^{k} t_i m \leq nm$, by Proposition 2.9(f) and (h),

$$a\left(\sum_{i=1}^{k} t_i m\right) \leq a(m \oplus \cdots \oplus m) \leq \underbrace{am \oplus \cdots \oplus am}_{n\ times}.$$

By Proposition 2.9(j), since $(am)' \oplus m = a'm \oplus m' \oplus m = 1$, $am \leq m$ and so $a\left(\sum_{i=1}^{k} t_i m\right) \leq \underbrace{m \oplus \cdots \oplus m}_{n\ times} = nm$. It results that $\sum_{i=1}^{k}(a.t_i)m = \sum_{i=1}^{k} a(t_i m) \in I_m$. ■

Definition 3.5 A PMV-algebra A is called *commutative*, if $x.y = y.x$, for every $x, y \in A$.

Example 3.6 In Example 3.1, A is a commutative PMV-algebra.

Notation: For A-module M, $I \subseteq A$ and A-ideal N of M, we let

$$IN = \{xm : x \in I, m \in N\}.$$

Theorem 3.7 *Let A be commutative, M be an A-module, N be a proper A-ideal of M and $x \oplus x = x$, for every $x \in A$. Then N is a prime A-ideal of M if and only if for every ideal I of A and A-ideal D of M, $ID \subseteq N$ implies that $I \subseteq (N : M)$ or $D \subseteq N$.*

Proof. ($\Rightarrow$) Let N be a prime A-ideal of M, I be an ideal of A and D be an A-ideal of M such that $ID \subseteq N$. We show that $I \subseteq (N : M)$ or $D \subseteq N$. Let $I \nsubseteq (N : M)$ and $D \nsubseteq N$. Then there exist $x \in I$ and $d \in D$ such that $xM \nsubseteq N$ and $d \notin N$. On the other hand, $ID \subseteq N$ implies that $xd \in N$. Since N is a prime A-ideal of M and $d \notin N$, $xM \subseteq N$, which is a contradiction.
($\Leftarrow$) Let for every ideal I of A and A-ideal D of M, $ID \subseteq N$ implies that $I \subseteq (N : M)$ or $D \subseteq N$. Let there exist $x \in A$ and $m \in M$ such that $xm \in N$ and $m \notin N$. By Proposition 2.2 and Lemma 3.4, let $I = \prec x \succ$ and $D = I_m$. Then for $y \in I$, by Proposition 2.2, there exists $n \geq 0$ such that $y \leq nx$ and so $y \ominus nx = 0$. Hence, $ym = (y \ominus 0)m = (y \ominus (y \ominus nx))m = (y \odot (y \odot (nx)')')m = (y \odot (y' \oplus nx))m = (y \wedge nx)m$. By Proposition 2.9(g), since $y \wedge nx \leq nx$, $ym = (y \wedge nx)m \leq (nx)m = xm \in N$. Hence, $ym \in N$ and so $ID = \{y(\sum_{i=1}^{k} t_i m) : y, t \in A\} = \{\sum_{i=1}^{k} t_i(ym) : y, t \in A\} \subseteq N$ and so $I \subseteq (N : M)$ or $D \subseteq N$. Since $m \notin N$, $I \subseteq (N : M)$ and so $xM \subseteq N$. Therefore, N is a prime A-ideal of M. ■

Definition 3.8 Let M be an A-module. M is called *torsion free* if $xm = 0$ implies that $x = 0$ or $m = 0$, for any $x \in A$ and $m \in M$.

Example 3.9 (i) Consider $L_2 = \{0, 1\}$, $L_4 = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$, $a \oplus b = min\{1, a + b\}$, $a' = 1 - a$ and $+, -, .$ are ordinary operations in $\mathbb{R}$. Then it is routine to show that $(L_2, \oplus, ', ., 0)$ is a PMV-algebra and $(L_4, \oplus, ', 0)$ is an MV-algebra. Let operation $\bullet : L_2 \times L_4 \longrightarrow L_4$ is defined by $a \bullet b = a.b$, for any $a \in L_2$ and $b \in L_4$. Then it is easy to show that L_4 is a torsion free L_2-module.
(ii) In Example 3.1, A is not a torsion free A-module.

Lemma 3.10 *Let M be an A-module. Then $d(\alpha m, \beta m) \leq d(\alpha, \beta)m$, for every $\alpha, \beta \in A$ and $m \in M$.*

Proof. The proof is similar to the proof of Lemma 2.9 (i) in [5]. ∎

Theorem 3.11 *Let M be a unitary A-module and K be an A-ideal of M. Then K is a prime A-ideal of M if and only if $P = (K : M)$ is a $\cdot$-prime ideal of A and $\frac{M}{K}$ is a torsion free $\frac{A}{P}$-module.*

Proof. $(\Rightarrow)$ Let K be a prime A-ideal of M. By Proposition 3.3 (v), $P = (K : M)$ is a $\cdot$-prime ideal of A. Now, we show that $\frac{M}{K}$ is a torsion free $\frac{A}{P}$-module. Let operation $\bullet : \frac{A}{P} \times \frac{M}{K} \longrightarrow \frac{M}{K}$ is defined by $\frac{x}{P}\frac{m}{K} = \frac{x}{P} \bullet \frac{m}{K} = \frac{xm}{K}$, for every $x \in A$ and $m \in M$. If $(\frac{x_1}{P}, \frac{m_1}{K}) = (\frac{x_2}{P}, \frac{m_2}{K})$, then $d(x_1, x_2) \in P$ and $d(m_1, m_2) \in K$, for any $x_1, x_2 \in A$ and $m_1, m_2 \in M$. By Proposition 2.9(i), $d(x_1m_1, x_1m_2) \leq x_1 d(m_1, m_2) \in K$ and so $d(x_1m_1, x_1m_2) \in K$. Also, by Lemma 3.10, $d(x_1m_2, x_2m_2) \leq d(x_1, x_2)m_2 \in PM \subseteq K$. Since $d(x_1m_1, x_2m_2) \leq d(x_1m_1, x_1m_2) \oplus d(x_1m_2, x_2m_2) \in K$, $d(x_1m_1, x_2m_2) \in K$ and so $\frac{x_1m_1}{K} = \frac{x_2m_2}{K}$. Hence, $\bullet$ is well defined. Let $x_1, x_2, x \in A$ and $m_1, m_2, m \in M$.

$(\frac{A}{P}\frac{M}{K}1)$: If $\frac{m_1}{K} + \frac{m_2}{K}$ is defined in $\frac{M}{K}$, then $\frac{m_1}{K} \leq \frac{m_2'}{K}$ and so by Proposition 2.9 (f, c), $\frac{xm_1}{K} \leq \frac{xm_2'}{K} \leq \frac{(xm_2)'}{K}$. It results that $\frac{xm_1}{K} + \frac{xm_2}{K}$ is defined and so $\frac{x}{P}(\frac{m_1}{K} + \frac{m_2}{K}) = \frac{xm_1}{K} + \frac{xm_2}{K} = \frac{x}{P}\frac{m_1}{K} + \frac{x}{P}\frac{m_2}{K}$.
$(\frac{A}{P}\frac{M}{K}2)$: if $\frac{x1}{P} + \frac{x_2}{P}$ is defined in $\frac{A}{P}$, then $\frac{x1}{P} \leq \frac{x_2'}{P}$ and so $\frac{x_1' \oplus x_2'}{P} = \frac{x_1'}{P} \oplus \frac{x_2'}{P} = \frac{1}{P}$. It means that $x_1 \ominus x_2' = d(x_1' \oplus x_2', 1) \in P = (K : M)$ and so $(x_1 \ominus x_2')m \in K$, for every $m \in M$. Since $x_1 \ominus x_2' \leq x_1$, $(x_1 \ominus x_2')m \leq x_1 m$, for every $m \in M$. By Propositions 2.4 (ii) and 2.9 (d),

$$\begin{aligned}
(x_1 \ominus x_2')m &= (x_1 \ominus x_2')m \wedge x_1m \\
&= ((x_1 \ominus x_2')m \oplus (x_1m)') \odot (x_1m) = ((x_1 \ominus x_2')m + (x_1m)') \odot (x_1m) \\
&\geq ((x_1 \ominus x_2')m \oplus x_1'm) \odot (x_1m) = ((x_1 \ominus x_2') + x_1')m \odot (x_1m) \\
&= ((x_1 \odot x_2) + x_1')m \odot (x_1m)l = (x_1' \vee x_2)m \odot x_1m \\
&\geq x_2m \odot x_1m = x_2m \ominus (x_1m)'.
\end{aligned}$$

Then $x_2m \ominus (x_1m)' \in K$ and so $d((x_2m)' \oplus (x_1m)', 1) = x_2m \ominus (x_1m)' \in K$. It results that $\frac{(x_1m)'}{K} \oplus \frac{(x_2m)'}{K} = \frac{1}{K}$ and so $\frac{x1m}{K} \leq \frac{(x_2m)'}{K}$. Hence, $\frac{x_1m}{K} + \frac{x_2m}{K}$ is defined and so $(\frac{x1}{P} + \frac{x_2}{P})\frac{m}{K} = \frac{x1}{P}\frac{m}{K} + \frac{x_2}{P}\frac{m}{K}$.
$(\frac{A}{P}\frac{M}{K}3)$: The proof is routine.

Now, let $\frac{x}{P}\frac{m}{K} = \frac{0}{K}$, for any $x \in A$ and $m \in M$. Then $xm = d(xm, 0) \in K$ and so $m \in K$ or $x \in (K : M) = P$. Hence, $\frac{m}{K} = \frac{0}{K}$ or $\frac{x}{P} = \frac{0}{P}$ and so $\frac{M}{K}$ is a torsion free $\frac{A}{P}$-module.

$(\Leftarrow)$ Let P be a prime A-ideal of M and $\frac{M}{K}$ be a torsion free $\frac{A}{P}$-module. If $K = M$, then $P = (K : M) = (M : M) = A$, which is a contradiction. Now, let $xm \in K$, for any $x \in A$ and $m \in M$. Then $\frac{x}{P}\frac{m}{K} = \frac{xm}{K} = \frac{0}{K}$. Since $\frac{M}{K}$ is torsion free, $\frac{x}{P} = \frac{0}{P}$ or $\frac{m}{K} = \frac{0}{K}$ and so $x \in P = (K : M)$ or $m \in K$. Therefore, K is a prime A-ideal of M. ∎

4. On radical of A-ideals and $\cdot$-ideals

In this section, we present the definition of radical of an A-ideal ($\cdot$-ideal) in MV-modules (MV-algebras) and obtain some properties on it. Also, we characterize radical of a $\cdot$-ideal via elements of A.

Definition 4.1 Let M be an A-module and N be an A-ideal of M. The intersection of all prime A-ideals of M, including N, is called *radical* of N and it is shown by $rad_M(N)$ or $rad(N)$. If N is a prime A-ideal of M, then it is clear that $rad(N) = N$. If there exist no prime A-ideal of M including N, then we let $rad_M(N) = M$.

Example 4.2 In Example 3.1, $rad(I) = \{0, 1\}$ and $rad(\{0\}) = \{0\}$.

Lemma 4.3 *Let M be an A-module and N be an A-ideal of M. Then $x \in (N : M)$ if and only if $x1 \in N$, for every $x \in A$.*

Proof. Let $x \in (N : M)$. Then it is clear that $x1 \in N$. Now, let $x1 \in N$ and $m \in M$. Since $m \leq 1$, by Proposition 2.9(f), $xm \leq x1 \in N$ and so $xm \in N$, for every $m \in M$. Hence, $x \in (N : M)$. ∎

Lemma 4.4 *Let M be a unitary A-module, K be an A-ideal of M and $m \in M$. Then*
(i) $\prec K \cup \{m\} \succ = \{x \in M : x \leq nm \oplus s,\ for\ some\ n \in \mathbb{N}\ and\ s \in K\}$ is an A-ideal of M.
Moreover, if K is a maximal A-ideal of M. Then
(ii) K is a prime A-ideal of M,
(iii) $(K : M)$ is a maximal ideal of A.

Proof. (i) By Proposition 2.3, $\prec K \cup \{m\} \succ$ is an ideal of M. Now, let $a \in A$ and $t \in \prec K \cup \{m\} \succ$. Then $t \leq nm \oplus s$, for some $n \in \mathbb{N}$ and $s \in K$. Since $t \leq nm \oplus s$ and $am \leq m$, by Proposition 2.9(f) and (h),

$$at \leq a(nm \oplus s) \leq \underbrace{am \oplus \cdots \oplus am}_{n\ times} \oplus as \leq \underbrace{m \oplus \cdots \oplus m}_{n\ times} \oplus as = nm \oplus as$$

and so $at \in \prec K \cup \{m\} \succ$. Therefore, $\prec K \cup \{m\} \succ$ is an A-ideal of M.

(ii) Let $xm \in K$, where $x \in A$ and $m \in M$. If $x \notin (K : M)$, then by Lemma 4.3, $x1 \notin K$. Let $m \notin K$. Then we consider $\prec K \cup \{m\} \succ$. By (i), $\prec K \cup \{m\} \succ$ is an A-ideal of M. Since K is maximal, $\prec K \cup \{m\} \succ = M$ and so $1 \in \prec K \cup \{m\} \succ$. Hence, $1 \leq nm \oplus t$, for some $n \in \mathbb{N}$ and $t \in K$. By Proposition 2.9(f) and (h), $x1 \leq x(nm \oplus t) \leq \underbrace{xm \oplus \cdots \oplus xm}_{n\ times} \oplus xt \in K$ and so $x1 \in K$, which is a contradiction. It results that $m \in M$. Therefore, K is a prime A-ideal of A.

(iii) Let I be an ideal of A such that $(K : M) \subsetneq I \subsetneq A$. Let $a \in A$ such that $a \notin (K : M)$. Then there exists $m \in M$ such that $am \notin K$. Now, similar to the proof of (ii), the result will be obtain. ∎

Theorem 4.5 *Let $J(A)$ be the intersection of all maximal ideals of A, $N(A)$ be the intersection of all prime $\cdot$-ideals of A and M be an A-module. Then*
(i) $J(A)M \subseteq Rad(M)$,
(ii) $rad_M(0) \subseteq Rad(M)$
(iii) $N(A)M \subseteq rad_M(0)$.

Proof. (i) Let N be a maximal A-ideal of M. Then by Lemma 4.4 (ii, iii), N is a prime A-ideal of M and $(N : M)$ is a maximal ideal of A. Hence,

$$(Rad(M) : M) = (\bigcap_{N \in Max(M)} N : M) = \bigcap_{N \in Max(M)} (N : M) \supseteq J(A)$$

and so $J(A)M \subseteq Rad(M)$.
(ii) By Lemma 4.4(ii), it is clear that $rad_M(0) \subseteq Rad(M)$.
(iii) If $rad_M(0) = M$, then $N(A)M \subseteq rad_M(0)$. Let $rad_M(0) \neq M$ and N be a prime A-ideal of M. If $a \in N(A)$, then by Proposition 3.3 (v), $a \in (P : M)$ and so $aM \subseteq P$, for every prime A-ideal P of M. Hence, $N(A)M \subseteq P$, for every prime A-ideal P of M. Therefore, $N(A)M \subseteq rad_M(0)$. ∎

Theorem 4.6 *Let M be an A-module and L, N be A-ideals of M. Then*
(i) $N \subseteq rad(N)$,
(ii) if $L \subseteq N$, *then* $rad(L) \subseteq rad(N)$,
(ii) $rad(rad(N) = rad(N)$,
(iv) $rad(N \cap L) \subseteq rad(N) \cap rad(L)$.

Proof. The proof is routine. ∎

Definition 4.7 Let I be an ideal of A. The intersection of all $\cdot$-prime ideals of A including I is denoted by $r_A(I)$ or $r(I)$. If there exists no $\cdot$-prime ideal of A including I, then we let $r_A(I) = A$.

Example 4.8 In Example 3.1, $r(I) = \{0, 1\}$ and $r(\{0\}) = \{0\}$.
(ii) Let $M_2(\mathbb{R})$ be the ring of square matrixes of order 2 with real elements and let 0 be the matrix with all elements 0. If we define the order relation on components

$$A = (a_{ij})_{i,j=1,2} \geq 0 \; if \; and \; only \; if \; a_{ij} \geq 0 \; for \; any \; i, j,$$

then $M_2(\mathbb{R})$ is an l-ring. If $v = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, then $(M_2(\mathbb{R}), v)$ is an lu-ring and so $A = \Gamma(M_2(\mathbb{R}), v)$ is a PMV-algebra. It is easy to see that $Id(A) = \{\{0\}, A\}$ and $\{0\}$ is not a $\cdot$-prime ideal of A. Then $r(\{0\}) = A$.

Lemma 4.9 $(\alpha \oplus \beta)a \leq \alpha a \oplus \beta a$, *for every* $\alpha, \beta, a \in A$.

Proof. Since $\beta a \leq (\alpha a)' \oplus \beta a$, by Proposition 2.4(i), $(\alpha a) \odot (\beta a)' = ((\alpha a)' \oplus \beta a)' \leq (\beta a)'$ and so $(\alpha a) \odot (\beta a)' + \beta a$ is defined, where " $+$ " is the partial addition on A. Similarly, $\alpha \odot \beta' + \beta$ is defined, too. Consider A as A-module, where $ab = a.b$, for every $a, b \in A$. Then by Lemma 2.9(d) and (g), since $\alpha \odot \beta' \leq \beta'$,

$(\alpha\odot\beta')a \leq \beta' a \leq (\beta a)'$ and so $(\alpha\odot\beta')a+\beta a$ is defined. Now, $\alpha \leq \alpha\vee\beta$ implies that $\alpha a \leq (\alpha\vee\beta)a$. Then $\alpha a\vee\beta a \leq (\alpha\vee\beta)a$ and so by Lemma 2.7(c),

$$(\alpha a)\odot(\beta a)'+\beta a = \alpha a\vee\beta a \leq (\alpha\vee\beta)a = (\alpha\odot\beta'\oplus\beta)a = (\alpha\odot\beta'+\beta)a = (\alpha\odot\beta')a+\beta a.$$

By Lemma 2.7(f), $\alpha a\odot(\beta a)' \leq (\alpha\odot\beta')a$. If we set $\alpha\oplus\beta$ instead of α, then by Lemma 2.9 (g), we have $(\alpha\oplus\beta)a\odot(\beta a)' \leq ((\alpha\oplus\beta)\odot\beta')a = (\alpha\wedge\beta')a \leq \alpha a$ and so

$$(\alpha\oplus\beta)a = (\alpha\oplus\beta)a\vee\beta a = (\alpha\oplus\beta)a\odot(\beta a)'\oplus\beta a \leq \alpha a\oplus\beta a. \quad ■$$

.

Definition 4.10 $S \subseteq A$ is called a $\cdot$-closed subset of A, if $x.y \in S$, for every $x, y \in S$.

Example 4.11 In Example 3.1, $S = \{1, 3\}$ is a $\cdot$-closed subset of A.

Theorem 4.12 *Let I be an ideal of A and S be a $\cdot$-closed subset of A such that $I\cap S = \emptyset$. Then there exists a $\cdot$-prime ideal P of A such that $I \subseteq P$ and $P\cap S = \emptyset$.*

Proof. Let $T = \{J : J\ is\ an\ ideal\ of\ A, I \subseteq J\ and\ J\cap S = \emptyset\}$. Since $I \in T$, $T \neq \emptyset$. By *Zorn's* Lemma, T has a maximal element P. We show that P is a $\cdot$-prime ideal of A. Let $x.y \in P$ and $x, y \notin P$. Consider $\prec P\cup\{x\}\succ$ and $\prec P\cup\{y\}\succ$. By maximality P, $\prec P\cup\{x\}\succ\cap S \neq \emptyset$ and $\prec P\cup\{y\}\succ\cap S \neq \emptyset$ and so there exist $\alpha \in\prec P\cup\{x\}\succ\cap S$ and $\beta \in\prec P\cup\{y\}\succ\cap S$. Then by Proposition 2.3, there exist $a, b \in P$ and $n, m \in \mathbb{N}\cup\{0\}$ such that $\alpha \leq nx\oplus a$ and $\beta \leq my\oplus b$. By Proposition 2.9 (f, g), $\alpha.\beta \leq (nx\oplus a).(my\oplus b)$. If we consider A as A-module, where $xy = x.y$, for every $x, y \in A$, then the same as Lemma 4.9 for PMV-algebras and by Proposition 2.9 (h),

$$\begin{aligned}\alpha.\beta &\leq (nx\oplus a).my\oplus(nx\oplus a).b \leq nx.my\oplus a.my)\oplus nx.b\oplus a.b\\ &\leq \underbrace{x.y\oplus\cdots\oplus x.y}_{mn\ times}\oplus\underbrace{a.y\oplus\cdots\oplus a.y}_{m\ times}\oplus\underbrace{x.b\oplus\cdots\oplus}_{n\ times}\oplus a.b \in P\end{aligned}$$

and so $\alpha.\beta \in P$. Since S is a $\cdot$-closed subset of A, $\alpha.\beta \in P\cap S$, which is a contradiction. Therefore, P is a $\cdot$-prime ideal of A. ■

Proposition 4.13 *Let I be an ideal of A and $c \in I$. Then $a.c \in I$, for every $a \in A$.*

Proof. The proof is easy. ■

Theorem 4.14 *Let I be an ideal of A. Then*

$$r(I) = \{x \in A : x^n = \underbrace{x.x.\cdots.x}_{n\ times} \in I, for\ some\ n \in \mathbb{N}\}.$$

Proof. Let $T = \{x \in A : x^n = \underbrace{x.x.\cdots.x}_{n\ times} \in I, for\ some\ n \in \mathbb{N}\}$. It is easy to show that $T \subseteq r(I)$. Let $x \in r(I)$. If $x \notin T$, then $x^n \notin I$, for every $n \in \mathbb{N}$. Consider $S = \{x^n\oplus a : n \in \mathbb{N}\cup\{0\}, a \in I\ and\ x^n \leq a'\}$. Let $x^n\oplus a,\ x^m\oplus b \in S$, for $a, b \in I$ and $n, m \in \mathbb{N}$. Since $x^n \leq a'$ and $x^m \leq a'$, $x^n + a$ and $x^m + a$ are defined in A. Then

$$\begin{aligned}(x^n \oplus a).(x^m \oplus b) &= (x^n + a).(x^m + b) \\ &= x^{m+n} + a.x^m + x^n.b + a.b \\ &= x^{n+m} \oplus t \in S,\end{aligned}$$

where by Proposition 4.13, $t \in I$. It results that S is a $\cdot$-closed subset of A. It is easy to see that $S \cap I = \emptyset$. Then by Theorem 4.12, there is a $\cdot$-prime ideal of A such that $I \subseteq P$ and $S \cap P = \emptyset$. Now, since $x \in r(I)$ and $x = x^1 \oplus 0 \in S$, $x \in P \cap S$, which is a contradiction. Therefore, $x \in T$ and therefore $T = r(I)$. ■

Theorem 4.15 *Let A be unital and $I, I_1, \cdots, I_n$ be ideals of A. Then*
(i) $r(r(I)) = r(I)$,
(ii) $r\left(\bigcap_{k=1}^{n} I_k\right) = \bigcap_{k=1}^{n} r(I_k)$.
(iii) $r(N : M) \subseteq (rad(N) : M)$.

Proof. *(i)* The proof is routine.
(ii) By Theorem 4.14, it is easy to show that $r\left(\bigcap_{k=1}^{n} I_k\right) \subseteq \bigcap_{k=1}^{n} r(I_k)$. Let $x \in \bigcap_{k=1}^{n} r(I_k)$. Then $x \in r(I_k)$, for every $1 \leq i \leq n$ and so there exists $m_k \in \mathbb{N}$ such that $x^{m_k} \in I_k$. Let $m = \max\{m_1, ..., m_n\}$. If we consider A as A-module, where $xy = x.y$, for every $x, y \in A$, then by Proposition 2.9 (j), $x^{m-m_k}.x^{m_k} \leq x^{m_k}$. Since $x^m = x^{m-m_k}.x^{m_k} \leq x^{m_k} \in I_k$, $x^m \in I_k$, for every $1 \leq k \leq n$ and so $x^m \in \bigcap_{k=1}^{n} I_k$. Hence, by Theorem 4.14, $x \in r\left(\bigcap_{k=1}^{n} I_k\right)$. Therefore, $r\left(\bigcap_{k=1}^{n} I_k\right) = \bigcap_{k=1}^{n} r(I_k)$.
(iii) Let P be an arbitrary prime A-ideal of M containing N. Since $N \subseteq P$, by Proposition 3.3 *(iv)*, $(N : M) \subseteq (P : M)$. Hence, $r(N : M) \subseteq (P : M)$ and so

$$r(N : M) \subseteq \bigcap_{N \subseteq P \in Spec(M)} (P : M) = \left(\bigcap_{N \subseteq P \in Spec(M)} P : M\right) = (rad(N) : M). \quad ■$$

5. Conclusion

The categorical equivalence between MV-algebras and lu-groups leads to the problem of defining a product operation on MV-algebras, in order to obtain structures corresponding to l-rings. In fact, by defining MV-modules, MV-algebras were extended. PMV-algebras are MV-algebras whose product operation "." is defined on the whole MV-algebra. we studied $\cdot$-prime ideals in PMV-algebras, prime A-ideals in MV-modules and presented the definition of radical of a $\cdot$-ideal in PMV-algebra A and characterize it via elements of A. Also, we presented definition of the radical of an A-ideal in MV-modules by prime A-ideals that in [8], was defined by maximal A-ideals. The obtained results in the last sections encouraged us to continue this way in order to introduce the notion of primary decomposition of an A-ideal in MV-modules by prime A-ideals, primary decomposition of a $\cdot$-ideal in PMV-algebras and other results.

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