ENTROPY OF QUANTUM DYNAMICAL SYSTEMS WITH INFINITE PARTITIONS

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Abstract. In this paper, the concepts of the entropy and relative entropy on quantum logic with countable partitions are defined and some ergodic properties of quantum dynamical systems are investigated. Finally, we show that the entropy is invariant under isomorphic relation.

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1. Introduction and countable partitions

Entropy is a tool to measure the amount of uncertainty in random event. Entropy has been applied in a variety of problem areas including physics, computer science, general systems theory, information theory, statistics, biology, chemistry, sociology and many other fields. The quantum logic approach was introduced by Birkhoff and Von Neumann [1]. Later Riecan and Dvurecenskij proposed a new model for quantum mechanics [3]. Yuan, Khare, Roy and Ebrahimzadeh using the notion of state of quantum logic, were introduced the entropy and logical entropy of finite partitions on quantum logic [2], [4], [5], [7]. In this paper, the entropy with countable partitions is defined and then by using this concept, entropy of

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dynamical systems is defined and some of its properties are investigated. Also, the relative entropy of countable partitions is defined and studied under the relation s-refinement.

At first, some basic definitions are presented that will be useful in further considerations.

Definition 1.1. [7] A quantum logic QL is a σ-orthomodular lattice, i.e., a lattice L $(L, \leq, \vee, \wedge, 0, 1)$ with the smallest element 0 and the greatest element 1, an operation $' : L \to L$ such that the following properties are hold for all $a, b \in L$:

(i) $a'' = a$, $a \leq b \Rightarrow b' \leq a'$, $a \vee a' = 1$, $a \wedge a' = 0$;

(ii) Given any countable sequence $(a_i)_{i\in I}$, $a_i \leq a_j'$, $i \neq j$, the join $\vee_{i\in\mathbb{N}} a_i$ exists in L;

(iii) L is orthomodular: $a \leq b \Rightarrow b = a \vee (b \wedge a')$.

Two elements $a, b \in QL$ are called orthogonal if $a \leq b'$ and denoted by $a \perp b$. A sequence $(a_i)_{i\in I}$ is said orthogonal if $a_i \perp a_j$, $\forall i \neq j$.

Definition 1.2. [7] Let L be a QL. A map $s : L \to [0, 1]$ is a state iff $s(1) = 1$ and for any orthogonal sequence $(a_i)_{i\in I}$, $s(\vee_{i\in I} a_i) = \sum\limits_{i\in I} s(a_i)$.

Definition 1.3. Let $P = \{a_i : i \in \mathbb{N}\}$ be a countable system of elements of the QL, L. P is called to be a $\vee$-orthogonal system iff $\vee_{i=1}^k a_i \perp a_{k+1}$, $\forall k$.

Definition 1.4. We say a system $P = \{a_i : i \in \mathbb{N}\} \subset L$ is the partition of L corresponding to the state s iff:

(i) P is a $\vee$-orthogonal system;

(ii) $s(\vee_{i\in\mathbb{N}} a_i) = \sum\limits_{i=1}^{\infty} s(a_i) = 1$.

2. Entropy and relative entropy of countable partitions

Definition 2.1. Let $P = \{a_i : i \in \mathbb{N}\}$ and $R = \{b_j : j \in \mathbb{N}\}$ be two countable partitions of L. We say R is a s-refinement of P, denoted by $P \leq_s R$, if for each $b_j \in R$ there exists $a_i \in P$ with $s(b_j \wedge a_i) = s(b_j)$.

Let $P = \{a_i : i \in \mathbb{N}\}$ and $R = \{b_j : j \in \mathbb{N}\}$ be two countable partitions of L corresponding to a state s and $s(\vee_{i\in\mathbb{N}}(a_i \wedge b)) = s(b), \forall b \in L$. Then by Definition 1.2, we get $\sum_{i=1}^{\infty} s(a_i \wedge b) = s(b)$. In the remaining of the present paper, s has this property. Then the common refinement of these partitions is the partition

$$P \vee R = \{a_i \wedge b_j : a_i \in P, b_j \in R, i, j \in \mathbb{N}\}.$$

Definition 2.2. Let $P = \{a_i : i \in \mathbb{N}\} \subset L$ be a partition of the QL, L corresponding to a state s. The entropy of P with respect to state s is defined by

$$H_s(P) = -\sum_{i=1}^{\infty} s(a_i) \log s(a_i)$$

such that $0 \log 0 = 0$.

Definition 2.3. Let $P = \{a_i : i \in \mathbb{N}\}$ and $R = \{b_j : j \in \mathbb{N}\}$ be two countable partitions of L corresponding to a state s. The relative entropy of P with respect to R is defined as following:

$$H_s(P||R) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} s(a_i) \log \frac{s(a_i)}{s(b_j)} \tag{2.1}$$

whenever $s(b_j) \neq 0$.

In the next proposition, it is proved subadditivity of entropy of countable partitions on a QL.

Proposition 2.4. *Let P and R be countable partitions of L corresponding to a state s. Then*

(i) $H_s(P) \geq 0$;

(ii) $P \leq_s R$ *implies that* $H_s(P) \leq H_s(R)$;

(iii) $H_s(P \vee R) \leq H_s(P) + H_s(R)$.

Proof. (ii) For each $b_j \in R$, there exists $a_i \in P$ such that $s(b_j) = s(b_j \wedge a_i)$. By definition of state,

$$s(b_j) \leq \sum_{j=1}^{\infty} s(a_i \wedge b_j) = s(a_i).$$

(iii) Let $P = \{a_i : i \in \mathbb{N}\}$ and $Q = \{b_j : j \in \mathbb{N}\}$ be two countable partitions corresponding to a state s, $f(x) = x \log x$, for $x > 0$ and $f(x) = 0$, for $x = 0$. From [8], we have

$$f\left(\sum_{j=1}^{\infty} \alpha_j x_j\right) \leq \sum_{j=1}^{\infty} \alpha_j f(x_j),$$

where $\sum_{j=1}^{\infty} \alpha_j = 1$ and $\alpha_j, x_j \in [0, 1]$. Let $\alpha_j = s(b_j)$ and $x_j = \frac{s(a_i \wedge b_j)}{s(b_j)}$, $s(b_j) \neq 0$, $j \in \mathbb{N}$. We have $\alpha_j, x_j \in [0, 1]$, $\sum_{j=1}^{\infty} \alpha_j = \sum_{j=1}^{\infty} s(b_j) = 1$. By definition of state, we can write, for each $i \in \mathbb{N}$,

$$\sum_{j=1}^{\infty} \alpha_j x_j = \sum_{j=1}^{\infty} s(b_j) \, \frac{s(a_i \wedge b_j)}{s(b_j)} = \sum_{j=1}^{\infty} s(a_i \wedge b_j) = s(a_i)$$

and

$$f\left(\sum_{j=1}^{\infty}\alpha_j x_j\right) = f(s(a_i)).$$

Similarly, we have

$$\begin{aligned}\sum_{j=1}^{\infty}\alpha_j f(x_j) &= \sum_{j=1}^{\infty}s(b_j)\ f\left(\frac{s(a_i\wedge b_j)}{s(b_j)}\right)\\ &= \sum_{j=1}^{\infty}s(a_i\wedge b_j)\ (\log(s(a_i\wedge b_j)-\log s(b_j))\\ &= \sum_{j=1}^{\infty}s(a_i\wedge b_j)\ \log s(a_i\wedge b_j)-\sum_{j=1}^{\infty}s(a_i\wedge b_j)\ \log s(b_j).\end{aligned}$$

So

$$\begin{aligned}f\left(\sum_{j=1}^{\infty}\alpha_j x_j\right) &= f(s(a_i))\leq\sum_{j=1}^{\infty}\alpha_j f(x_j)\\ &= \sum_{j=1}^{\infty}f(s(a_i\wedge b_j))-\sum_{j=1}^{\infty}s(a_i\wedge b_j)\ \log s(b_j).\end{aligned}$$

Summarizing, we obtain

$$\sum_{i=1}^{\infty}f(s(a_i)\leq\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}f(s(a_i\wedge b_j))-\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}s(a_i\wedge b_j)\ \log s(b_j).$$

From definition of state, we have $\sum_{i=1}^{\infty}s(a_i\wedge b_j)=s(b_j), j\in\mathbb{N}$, therefore

$$\sum_{i=1}^{\infty}f(s(a_i)\leq\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}f(s(a_i\wedge b_j))-\sum_{j=1}^{\infty}f(s(b_j)).$$

This implies that $H_s(P\vee R)\leq H_s(P)+H_s(R)$. ■

Now, the relative entropy of infinite partitions under the relation s-refinement, will be studied.

Proposition 2.5. *Let P, R and M be countable partitions of L. Then*

(i) $H_s(P||P^0)=H_s(P)$ *where* $P^0=\{1\}$;

(ii) $P\leq_s M$ *implies that* $H_s(M||R)\leq H_s(P||R)$;

(iii) $P\leq_s M$ *implies that* $H_s(R||M)\geq H_s(R||P)$.

Proof. (i) $s(1) = 1$.

(ii) Let $P = \{p_i : i \in \mathbb{N}\}$, $R = \{r_j : j \in \mathbb{N}\}$ and $M = \{m_k : k \in \mathbb{N}\}$. Since $P \leq_s M$, for each m_k there exists p_i such that $s(m_k \wedge p_i) = s(m_k)$. Thus

$$s(m_k) = s(m_k \wedge p_i) \leq \sum_{k=1}^{\infty} s(m_k \wedge p_i) = s(p_i),$$

and this implies that

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} s(m_k) \log \frac{s(m_k)}{s(r_j)} \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} s(p_i) \log \frac{s(p_i)}{s(r_j)}.$$

So $H_s(M||R) \leq H_s(P||R)$.

(iii) As we proved in part *ii*), $s(m_k) \leq s(p_i)$. Therefore,

$$\frac{s(m_k)}{s(r_j)} \leq \frac{s(p_i)}{s(r_j)},$$

and this finished the proof. ■

3. Entropy of dynamical systems

Definition 3.1. Let L be a QL and $T : L \to L$ be a map with the following properties:

(i) $T\left(\bigvee_{i=1}^{k} a_i\right) = \bigvee_{i=1}^{k} T(a_i)$, $T\left(\bigvee_{i=1}^{\infty} a_i\right) = \bigvee_{i=1}^{\infty} T(a_i)$, $\forall k \in \mathbb{N}, \forall a_i \in L$;

(ii) $T(a \wedge b) = T(a) \wedge T(b)$, $\forall a, b \in L$;

(iii) $T(a') = (T(a))'$, $\forall a \in L$.

$T : L \to L$ with respect to a state s is called state preserving if $s(T(a)) = s(a)$ for every $a \in L$. Then the triple (L, s, T) is said a quantum dynamical system.

Proposition 3.2. *Let* (L, s, T) *be a quantum dynamical system and* $P = \{a_i : i \in \mathbb{N}\}$ *be a partition of* (L, s)*, then*

(i) *If* $a_i \leq a_j$*, then* $T(a_i) \leq T(a_j)$*;*

(ii) $T(P) = \{T(a_i) : i \in \mathbb{N}\}$ *is a partition;*

(iii) $H_s(T(P)) = H_s(P)$.

Proof. (i) $a_i \leq a_j$ so $a_i \wedge a_j = a_i$. By Definition 3.1, $T(a_i \wedge a_j) = T(a_i) \wedge T(a_j) = T(a_i)$. Thus $T(a_i) \leq T(a_j)$.

(ii) By Definition 3.1 and part i), the proof is obvious.

(iii) $s(T(a_i)) = s(a_i)$. ■

Theorem 3.3. [6] *Let* $\{(a_i)\}_{i=1}^{\infty}$ *be sequence of nonnegative numbers such that* $a_{r+s} \leq a_r + a_s$ *for each* $r, s \in \mathbb{N}$. *Then* $\lim\limits_{n\to\infty} \frac{1}{n} a_n$ *exists.*

Proposition 3.4. $\lim\limits_{n\to\infty} \dfrac{1}{n} H_s(\vee_{i=1}^{n} T^i P)$ *exists.*

Proof. Let $a_n = H_s(\vee_{i=1}^{n} T^i P) \geq 0$. Then

$$H_s(\vee_{i=1}^{n+p} T^i P) \leq H_s(\vee_{i=1}^{n} T^i P) + H_s(\vee_{i=n+1}^{n+p} T^i P) = a_n + a_p.$$

So $a_{n+p} \leq a_n + a_p$, $\forall n, p$. Hence $\lim\limits_{n\to\infty} \frac{a_n}{n}$ exists. ∎

Definition 3.5. Let (L, s, T) be a quantum dynamical system and P be a countable partition of (L, s). The entropy of T respect to P is defined by

$$h_s(T, P) = \lim_{n\to\infty} \frac{1}{n} H_s(\vee_{i=1}^{n} T^i P). \tag{3.1}$$

Definition 3.6. Let (L, s, T) be a quantum dynamical system. The entropy of T is defined by

$$h_s(T) = \sup_P h_s(T, P) \tag{3.2}$$

where the supremum is taken over all countable partitions of (L, s).

In the following propositions, some ergodic properties of $h_s(T, P)$ and $h_s(T)$ will be studied.

Proposition 3.7. *If* (L, s, T) *is a quantum dynamical system, then*

(i) $h_s(T, P) \geq 0$;

(ii) $P \leq_s R$ *implies that* $h_s(T, P) \leq h_s(T, R)$;

(iii) $h_s(T, P) \leq H_s(P)$;

(iv) $h_s(T, P \vee R) \leq h_s(T, P) + h_s(T, R)$;

(v) $h_s(T, P) = h_s\left(T, \vee_{i=1}^{k} T^i P\right)$, *for* $k \in \mathbb{N}$.

Proof. (i) Obvious.
(ii) We have for each $i \in \mathbb{N}$, $\vee_{i=1}^{n} T^i P \leq_s \vee_{i=1}^{n} T^i R$. So, by definition and Proposition 2.4, it holds.
(iii) From Propositions 2.4, 3.2 we have for each $n \in \mathbb{N}$,

$$\frac{1}{n} H_s(\vee_{i=1}^{n} T^i P) \leq \frac{1}{n} \sum_{i=1}^{n} H_s(T^i P) = \frac{1}{n} \sum_{i=1}^{n} H_s(P) = H_s(P).$$

(iv) By Definition 3.1, $T(a \wedge b) = T(a) \wedge T(b)$, $\forall a, b \in B$. So, by using Proposition 2.4, we can write

$$\begin{aligned} H_s(\vee_{i=1}^n T^i(P \vee R)) &= H_s((\vee_{i=1}^n T^i P) \vee (\vee_{i=1}^n T^i R)) \\ &\leq H_s(\vee_{i=1}^n T^i P) + H_s(\vee_{i=1}^n T^i R). \end{aligned}$$

(v)

$$\begin{aligned} h_s(T, \vee_{i=1}^k T^i P) &= \lim_{n\to\infty} \frac{1}{n} H_s(\vee_{j=1}^n T^j (\vee_{i=1}^k T^i P)) \\ &= \lim_{n\to\infty} \frac{1}{n} H_s(\vee_{i=1}^{k+n-1} T^i P) \\ &= \lim_{n\to\infty} \left(\frac{k+n-1}{n}\right) \frac{1}{k+n-1} H_s(\vee_{i=1}^{k+n-1} T^i P) \\ &= h_s(T, P). \end{aligned}$$

■

Proposition 3.8. *If* (L, s, T) *is a quantum dynamical system, then*

(i) $h_s(id) = 0$*;*

(ii) *For* $k \geq 1$*,* $h_s(T^k) = kh_s(T)$*.*

Proof. (i) By Definition 2.1, we have $\vee_{i=1}^n T^i P = P$, for all $n \in \mathbb{N}$. Therefore,

$$h_s(id, P) = \lim_{n\to\infty} \frac{1}{n} H_s(\vee_{i=1}^n T^i P) = 0.$$

(ii) Let P be an arbitrary countable partition of (L, s). We have

$$\begin{aligned} h_s(T^k, \vee_{i=1}^n T^i P) &= \lim_{n\to\infty} \frac{1}{n} H_s(\vee_{j=1}^n (T^k)^j (\vee_{i=1}^n T^i P)) \\ &= \lim_{n\to\infty} \frac{1}{n} H_s(\vee_{j=1}^n \vee_{i=1}^k T^{kj+i} P) \\ &= \lim_{n\to\infty} \frac{1}{n} H_s(\vee_{i=1}^{nk-1} T^i P) \\ &= \lim_{n\to\infty} \frac{nk}{n} \frac{1}{nk} H_s(\vee_{i=1}^{nk-1} T^i P) \\ &= kh_s(T, P) \end{aligned}$$

So, $\sup_P h_s(T, P) = \sup_P h_s(T^k, \vee_{i=1}^n T^i P) \leq \sup_P h_\eta(T^k, P) = h_s(T^k)$
Since $P \leq_s \vee_{i=1}^k T^i P$, we have $h_s(T^k, P) \leq h_s(T^k, \vee_{i=1}^n T^i P) = kh_s(T, P)$. ■

Corollary 3.9. *If* (L, s, T) *is a quantum dynamical system with* $T^k = id$ *for some* $k \in \mathbb{N}$*, then* $h_s(T) = 0$*.*

Proof. Since $T^k = id$, by Proposition 3.8, $h_s(T^k) = h_s(id) = 0$, and so, $h_s(T) = \frac{1}{k} h_s(T^k) = 0$. ■

Definition 3.10. Two quantum dynamical systems (L_1, s_1, T_1) and (L_2, s_2, T_2) are called to be isomorphic if there exists a bijective map $g : L_1 \to L_2$ satisfying the following properties. For every $a, b \in L_1$,

(i) $g(a \vee b) = g(a) \vee g(b)$;
(ii) $g(a^{'}) = (g(a))^{'}$;
(iii) $s_1(a) = s_2(g(a))$;
(iv) $g(T_1(a)) = T_2(g(a))$.

In the following proposition, we prove that the entropy of quantum dynamical systems is invariant under isomorphism.

Proposition 3.11. *If (L_1, s_1, T_1) and (L_2, s_2, T_2) are isomorphic quantum dynamical systems, then $h_s(T_1) = h_s(T_2)$.*

Proof. By Definition 3.10, we have

$$\begin{aligned} h_s(T_1, P) &= \lim_{n\to\infty} \frac{1}{n} H_s\left(\bigvee_{i=1}^{n} T_1^i P\right) = \lim_{n\to\infty} \frac{1}{n} H_s\left(g\left(\bigvee_{i=1}^{n} T_1^i P\right)\right) \\ &= \lim_{n\to\infty} \frac{1}{n} H_s\left(\bigvee_{i=1}^{n} g(T_1^i P)\right) = \lim_{n\to\infty} \frac{1}{n} H_s\left(\bigvee_{i=1}^{n} T_2(g^i P)\right) = h_s(T_2, g(P)). \end{aligned}$$

So, $h_s(T_1) = \sup_P h_s(T_1, P) = \sup_P h_s(T_2, g(P)) \leq \sup_P h_s(T_2, P) = h_s(T_2)$. Therefore, $h_s(T_1) \leq h_s(T_2)$. Similarly, we obtain $h_s(T_2) \leq h_s(T_1)$. ∎

4. Conclusion

This paper has defined entropy and relative entropy of infinite partitions on a quantum logic. Also entropy of a dynamical system with infinite partitions was studied and some of its properties were proved.

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