BOUBAKER PIVOTAL ITERATION SCHEME (BPIS)

Amjed Zraiqat

Al-Zaytoonah University of Jordan
Amman
Jordan

Abstract. In this paper, we present a numerical scheme for the solution of fourth-order boundary value problems with two-point boundary conditions. The Boubaker Pivotal Iteration Scheme (BPIS) is applied to construct the numerical solution. This approach provides the solution in the form of analytical function and not at grid points. Some examples are displayed to demonstrate the computational efficiency of the method.

Keywords: fourth-order BVPs; Boubaker polynomials expansion scheme; pivotal function; numerical solution.

1. Introduction

Numerical methods are becoming more and more important in mathematical and engineering applications not only because of the difficulties encountered in finding exact analytical solutions, but also because of the ease with which numerical techniques can be used in conjunction with modern high-speed digital computers. There exist a large number of fourth-order BVPs in science and engineering, whose solutions cannot easily be obtained by the well-known analytical methods. For such problems, we can obtain approximate solutions for the given problems using numerical methods under the given boundary conditions. The objective of numerical methods is to solve complex numerical problems using only the simple operations of arithmetic, to develop and evaluate methods for computing numerical results from given data. The methods of computation are called algorithms. An algorithm is a finite sequence of rules for performing computations on a computer such that at each instant the rules determine exactly what the computer has to do next. Numerical methods tend to emphasize the implementation of the algorithms. Thus, numerical methods are methods for solving problems on computers by numerical calculations, often giving a table of numbers and/or graphical representations or figures.

This work considers numerical approximation for fourth-order nonlinear boundary value problem of the form

$$u^{(4)}(x)+\sum_{i=0}^{3} f_i(x)u^{(i)}(x)=f(x,u(x)),\quad 0\le x\le 1 \tag{1.1}$$

with the boundary conditions

$$u(0) = \alpha_0, \quad u(1) = \alpha_1, \quad u'(0) = \beta_0, \quad u'(1) = \beta_1 \tag{1.2}$$

where α_i, and β_i (i=0, 1) all are real constants, $f(x, u)$ is a continuous real valued function, and $f_i(x)$ (i=0, 1, 2, 3, 4) are all continuous functions on the interval [0, 1].

Two-point boundary value problems have been extensively studied in the literature. These problems generally arise in the mathematical modeling of visco-elastic and inelastic flows, deformation of beams, plate deflection theory, and other branches of mathematical, physical and engineering sciences, see [5]-[7]. Theorems which discuss the conditions for the existence and uniqueness of solutions of such problems can be found in Agarwal's book [8]. Exact solutions of such problems can be found only in very rare cases. Various numerical methods such as Finte difference method [9], Spline techniques [10], [11], B-spline technique [12], [13], and others have been employed to solve fourth-order boundary value problems.

In this paper, we present a novel technique to solve (1.1) and (1.2) by using BPIS where the Boubaker Polynomials Expansion Scheme (BPES) constitutes the base of our method. The Boubaker Polynomials Expansion Scheme BPES is a resolution protocol which has been successfully applied to several applied-physics and mathematics problems. Solutions have been proposed through the BPES in many fields such as numerical analysis [14]-[19], theoretical physics [16]-[21], mathematical algorithms [18], heat transfer [22], [23], and material characterization [24]. The rest of the paper is organized as follows. In the next section some properties of Boubaker Polynomials which gave the fundamentals of BPIS is introduced. The solution of (1.1) and (1.2) using BPIS introduced in Section 3. The numerical Examples are presented in Section 4. Section 5 ends this paper with a brief conclusion.

2. Properties of Boubaker polynomials

In this section, we start with some notations, definitions and basic results that are useful for the proposed method. In this paper, the Boubaker Polynomials will be used to investigate the fourth-order boundary value problems. In recent years, a lot of attention has been devoted to the study of Boubaker Polynomials Expansion Scheme to investigate various scientific models, using only the subsequence $\{B_{4k}(x)\}_{k=1}^{\infty}$. The monic Boubaker polynomials are defined as:

$$B_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} \frac{n-4k}{n-k} x^{n-2k}, \ n \geq 1, \quad \left\lfloor \frac{n}{2} \right\rfloor = \frac{2n + (-1)^n - 1}{4} \tag{2.1}$$

and $B_0(x) = 1$. Such polynomials satisfy the relations

$$\begin{aligned} B_n(-x) &= (-1)^n B_n(x), & n \geq 0 \\ B_n(x) &= x B_{n-1}(x) - B_{n-2}(x), & n = 3, 4, ... \end{aligned} \tag{2.2}$$

In particular, the $4k$-Boubaker polynomials satisfy the relation

$$B_{4(k+1)} = (x^4 - 4x^2 + 2)B_{4k} - \beta_k B_{4(k-1)}(x), \quad k \geq 1 \tag{2.3}$$

with $B_0(x) = 1$ and $B_4(x) = x^4 - 2$ where $\beta_0 = 0, \beta_1 = -2$ and $\beta_k = 1$ for $k \geq 2$, see [**?**]. For example, for $k \leq 4$ we have

$$\begin{aligned} B_4(x) &= x^4 - 2 \\ B_8(x) &= x^8 - 4x^6 + 8x^2 - 2 \\ B_{12}(x) &= x^{12} - 8x^{10} + 18x^8 - 35x^4 + 24x^2 - 2 \\ B_{16}(x) &= x^{16} - 12x^{14} + 52x^{12} - 88x^{10} + 168x^6 - 168x^4 + 48x^2 - 2. \end{aligned} \tag{2.4}$$

As a well known powerful properties of the $4k$-Boubaker polynomials their values at 0 and r_k, where r_k designates the $4k$-Boubaker polynomial minimal positive root. Some of these properties are listed below:

$$\sum_{k=1}^{N} B_{4k}(x)\big|_{x=0} = -2N, \qquad \sum_{k=1}^{N} B_{4k}(x)\big|_{x=r_k} = 0, \tag{2.5}$$

also first derivatives values:

$$\begin{aligned} &\sum_{k=1}^{N} \frac{dB_{4k}(x)}{dx}\Big|_{x=0} = 0, \qquad \sum_{k=1}^{N} \frac{dB_{4k}(x)}{dx}\Big|_{x=r_k} = \sum_{k=1}^{n} H_k, \\ &H_n = B'_{4n}(r_n) = \frac{4r_n(2 - r_n^2) \sum\limits_{k=1}^{n} B_{4k}^2(r_n))}{B_{4(n+1)}(r_n)} + 4r_n^3. \end{aligned} \tag{2.6}$$

Finally, we close this section by stating one the most noticed results, see [25]:

Theorem 2.1 *Every polynomial $B_n(x)$, $n \geq 2$, has two conjugate complex roots $\pm i\sqrt{\gamma_n}$, $\gamma_n > 0$, and other zeros are real and symmetrically distributed in $(-2, 2)$, where $\lim\limits_{n\to\infty} \gamma_n = \frac{4}{3}$.*

3. Analysis of the method

In this section, we employ our technique of **BPIS** to find out an analytical solution of the general fourth-order boundary value problem (1.1) and (1.2). We first formulate and analyze **BPIS** for solving such problems.

Definition 3.1 The boundary conditions $u(1) = \alpha_1$, and $u'(0) = \beta_0$ in (1.2) are called pivotal conditions of the boundary value problem (1.1) and (1.2).

Definition 3.2 A function $\psi(x) \in C^4[0, 1]$ is said to be a pivotal function of the fourth-order boundary value problem (1.1) and (1.2) if it satisfies the conditions $\psi(1) = -\alpha_1$, and $\psi'(0) = -\beta_0$.

Remark 3.3 It can be noted that the boundary value problem (1.1) and (1.2) has infinitely many pivotal functions. For example, each term in the sequence $\{(\beta_0 - \alpha_1)x^n - \beta_0 x\}_{n=2}^{\infty}$ is a pivotal function.

For the convenience of the reader, we consider a fourth-order boundary value problem of the form

$$Lu + Nu = g \tag{3.1}$$

with boundary conditions (1.2), where L is the linear differential operator which is given by

$$L = \frac{d^4}{dx^4} + \sum_{i=1}^{3} f_i(x) \frac{d^i}{dx^i}, \tag{3.2}$$

Nu represents the nonlinear term, and $g(x)$ is a source term. The BPIS starts by transforming the above problem using the transformation

$$v(x) = u(x) + \psi(x) \tag{3.3}$$

where $\psi(x)$ is an arbitrary pivotal function. Thus we obtain a new fourth-order boundary value problem of the form

$$Lv + N(v - \psi) = g(x) + L\psi \tag{3.4}$$

with boundary conditions

$$v(0) = \alpha_0 + \psi(0), \quad v(1) = 0, \quad v'(0) = 0, \quad v'(1) = \beta_1 + \psi'(1) \tag{3.5}$$

Assume that the approximation v_N of the solution of (3.4) and (3.5) has the form

$$v_N(x) = \frac{1}{2N} \sum_{k=1}^{N} \lambda_k B_{4k}(r_k x) \tag{3.6}$$

N is a preassigned integer, $r_k|_{k=1,\dots,N}$ are $4k$-Boubaker minimal positive roots, and the constants $\lambda_k|_{k=1,\dots,N}$ are to be determined. It is worth noting that the function $v_N(x)$ satisfies the transformed pivotal conditions.

For the determination of the constants $\lambda_k|_{k=1,\dots,N}$, an appropriate N-sampling is carried out inside the interval $[0,1]$ by setting

$$x_k = \frac{k-1}{N-1}, \quad k = 1, 2, \dots, N. \tag{3.7}$$

Then we seek for the values of $\lambda_k|_{k=1,\dots,N}$ that minimizes the objective function

$$\sum_{k=1}^{N} \left(L(v_N)|_{x_k} - F(x_k) \right)^2, \tag{3.8}$$

where

$$F(x_k) = g(x_k) + L(\psi)|_{x_k} - N\left(\frac{1}{2N}B_{4k}(r_k x) - \psi\right)\Big|_{x_k} \tag{3.9}$$

subject to the constraints

$$\sum_{k=1}^{N}\lambda_k = -N(\alpha_0 + \psi(0)), \qquad \frac{dv_N}{dx}\Big|_{x=1} = \beta_1 + \psi'(1). \tag{3.10}$$

Thus $u_N(x) = v_N(x) - \psi(x)$ is an approximate solution of the solution of the boundary value problem (3.1) and (1.2). This algorithm will be clarified with a famous example in the next section where the approximate solution is in good agreement with the exact one even for small values of N. The above analysis leads us to a very important linearization process which comes from the flexibility in choosing the pivotal functions. The second example in the next section illustrate the linearization process.

4. Numerical examples

In this section, we present and discuss the numerical results by employing the BPIS for two examples. Results demonstrate that the present method is remarkably effective.

Example 1. Scott and Watts [1] considered the following special fourth-order boundary value problem:

$$u^{(4)}(x) - (1+c)u'' + cu(x) = \frac{1}{2}cx^2 - 1 \tag{4.1}$$

with the boundary conditions

$$u(0) = 1, \quad u'(0) = 1, \quad u(1) = 1.5 + \sinh(1), \quad u'(1) = 1 + \cosh(1). \tag{4.2}$$

The exact solution of the above problem is

$$u(x) = 1 + \frac{1}{2}x^2 + \sinh(x). \tag{4.3}$$

The solution of (4.1) is independent of the parameter c. Scott and Watts have solved (4.1) with boundary conditions (4.2) for large values of c with orthonormalization process. It has been shown that each time the solutions started to lose their linear independence, one has to perform orthonormalization. In fact, as c got bigger it required more normalization. Momani and Noor obtained the solution of problem (4.1) with boundary conditions (4.2) by differential transform method (DTM) and the solution is accurate for $c < 10^6$ [2]. Noor and Mohyud-Din solved the same boundary value problem by using variational iteration method [3]. A new reproducing kernel Hilbert space method (RKHSM) introduced in [4] to solve (4.1) and (4.2) for large values of c. Equation (4.1) can be rewritten as

$$[u^{(4)}(x) - u^{(2)}(x) + 1] - c\left[u^{(2)}(x) - u(x) + \frac{1}{2}x^2\right] = 0. \tag{4.4}$$

The solution of the fourth-order boundary value problem is also a solution of

$$u^{(2)}(x) - u(x) + \frac{1}{2}x^2 = 0, \tag{4.5}$$

which is just the term in brackets multiplying c in (4.4). The remaining term in brackets in (4.4) is just the second derivative of (4.5). This unusual behavior results in the solution of the original problem being independent of the constant c. Using the new technique, we discuss the solution of this problem from different points of view. First, we show the vital rule played by pivotal functions in obtaining the approximation $u_N(x)$ of $u(x)$. Second, We examine our approach using different values of c. Finally, we compare our results with previous methods. Consider the following pivotal functions:

$$\psi_1(x) = (-.5 - \sinh(1))x^2 - x \tag{4.6}$$

$$\psi_2(x) = e^{-x} - e^{-1} - 1.5 - \sinh(1) \tag{4.7}$$

$$\psi_3(x) = -\sinh(x) - 1.5 - \sinh(1) \tag{4.8}$$

According to BPIS, one can obtain the approximation $u_N(x)$ of $u(x)$. We have computed the absolute errors $\psi_i^{(4)}$ $(i = 1, 2, 3)$ corresponding to the proposed pivotal functions ψ_1, ψ_2, and ψ_3 with $N = 4$. Numerical results are given in Table 1.

Table 1. Absolute errors $\psi_i^{(4)}$ $(i = 1, 2, 3)$ corresponding to the proposed pivotal functions ψ_1, ψ_2, and ψ_3.

x	True Solution	$\psi_1^{(4)}$	$\psi_2^{(4)}$	$\psi_3^{(4)}$
0.0	1.0000000000	0.0000	0.0000	0.0000
0.1	1.1051667500	5.4e-4	5.7e-5	1.6e-7
0.2	1.2213360025	1.6e-3	2.2e-4	5.5e-7
0.3	1.3495202934	2.7e-3	4.5e-4	9.4e-7
0.4	1.4907523258	3.4e-3	6.9e-4	1.0e-6
0.5	1.6460953055	3.5e-3	8.8e-4	7.5e-7
0.6	1.8166535821	3.1e-3	9.5e-4	1.6e-7
0.7	2.0035837018	2.3e-3	8.5e-4	4.2e-7
0.8	2.2081059822	1.3e-3	5.7e-4	6.1e-7
0.9	2.4315167257	4.1e-4	2.1e-4	3.2e-7
1.0	2.6752011936	0.0000	0.0000	0.0000

Table 1 clearly shows the improvement we achieved using different pivotal functions.

Next, we consider the pivotal function $\psi_3(x)$ and calculate the absolute errors associated with $c = 10^{-6},\ c = 10,\ c = 10^6$.

The table below exhibits the computed results and shows that the absolute errors were not affected by changing the value of the constant c even for very small values which is never discussed before.

Table 2. Absolute errors for $c = 10^{-6},\ c = 10,\ c = 10^6$ when $N = 7$.

x	True Solution	$\psi_3^{(7)}(c = 10^{-6})$	$\psi_3^{(7)}(c = 10)$	$\psi_3^{(7)}, (c = 10^6)$
0.0	1.0000000000	0.0000	0.0000	0.0000
0.1	1.1051667500	1.2e-12	2.3e-12	6.6e-13
0.2	1.2213360025	1.6e-12	2.5e-12	9.2e-13
0.3	1.3495202934	1.8e-12	2.5e-12	3.6e-13
0.4	1.4907523258	3.3e-12	3.6e-12	2.0e-13
0.5	1.6460953055	6.9e-12	6.8e-12	1.5e-12
0.6	1.8166535821	1.1e-11	1.0e-11	2.5e-12
0.7	2.0035837018	1.3e-11	1.3e-11	2.1e-12
0.8	2.2081059822	1.7e-11	1.7e-11	3.5e-12
0.9	2.4315167257	1.7e-11	1.7e-11	6.8e-12
1.0	2.6752011936	0.0000	0.0000	0.0000

Table 3 reveals a comparison between the errors obtained by using the methods mentioned in [4] and our approach. Examining this table closely shows the improvement obtained by the proposed scheme.

Table 3. Numerical results for Example 1 when $c = 10^6$.

x	True Solution	ADM[4]	HPM[4]	DTM[4]	RHKSM[4]	BPIS($\psi_3^{(7)}$)
0.0	1.0000000000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	1.1051667500	6.5E+3	6.5E+3	1.5E-10	7.5E-11	6.6e-13
0.2	1.2213360025	8.8E+4	8.8E+4	3.7E-8	2.1E-10	9.2e-13
0.3	1.3495202934	3.6E+5	3.6E+5	9.0E-7	3.4E-10	3.6e-13
0.4	1.4907523258	9.1E+5	9.1E+5	8.5E-6	4.1E-10	2.0e-13
0.5	1.6460953055	1.6E+6	1.6E+6	4.8E-5	4.1E-10	1.5e-12
0.6	1.8166535821	2.3E+6	2.3E+6	1.9E-4	3.5E-10	2.5e-12
0.7	2.0035837018	2.6E+6	2.6E+6	6.4E-4	2.3E-10	2.1e-012
0.8	2.2081059822	2.1E+6	2.1E+6	1.7E-3	1.1E-10	3.5e-12
0.9	2.4315167257	9.1E+5	9.1E+5	4.2E-3	2.6E-11	6.8e-12
1.0	2.6752011936	0.0000	0.0000	0.0000	0.0000	0.0000

Example 2. We next consider the nonlinear boundary value problem:

$$u^{(4)} - e^{x}u'' + u + \sin(u) = f(x) \tag{4.9}$$

subject to the boundary conditions

$$u(0) = 1, \quad u'(0) = 1, \quad u(1) = 1 + \sinh(1), \quad u'(1) = \cosh(1) \tag{4.10}$$

where

$$f(x) = 1 + \sin(1 + \sinh(x)) - (-2 + e^{x})\sinh(x) \tag{4.11}$$

The exact solution of (4.9) and (4.10) is given by

$$u(x) = 1 + \sinh(x) \tag{4.12}$$

This is an interesting problem considered by Fashan [4]. A linearization process applied to the problem above using two pivotal functions and the results compared with [4], see table 5. Consider the pivotal functions

$$\psi_1(x) = -x - \sinh(1) \tag{4.13}$$

$$\psi_2(x) = -e^{x} + e - 1 - \sinh(1). \tag{4.14}$$

Let $N = 4$. Applying BPIS with respect to the pivotal function ψ_1, we obtain the approximation $U_4(x)$ of $u(x)$, where

$$\begin{aligned} U_4(x) = &-0.0000000000009803538503532164759113909868325x^{16} \\ &+0.0000000023542610058198409258849496537761x^{14} \\ &+0.0000000425293995028127985729754006246 82x^{12} \\ &-0.00001343382023805487587006827748218 6x^{10} \\ &+0.0004500586062610541020106531310260 6x^{8} \\ &-0.0049453446305244210781767729908711x^{6} \\ &+0.1049331457830995431198063126805x^{4} \\ &+0.0747767228313464624723153477993 07x^{2} + x \\ &+1.00000000000000000693888939039072. \end{aligned} \tag{4.15}$$

Substituting (4.15) in the nonlinear term $\sin(u(x))$, we obtain the linearized form of the boundary value problem (4.9) and (4.10) as follows:

$$u^{(4)}(x) - e^{x}u''(x) + u(x) = f(x) - \sin(u_4(x)) \tag{4.16}$$

with boundary conditions

$$u(0) = 1, \quad u'(0) = 1, \quad u(1) = 1 + \sinh(1), \quad u'(1) = \cosh(1) \tag{4.17}$$

Applying BPIS to the linearized problem using the pivotal function $\psi_2(x)$, we have the approximation

$$
\begin{aligned}
u_4^{(2)}(x) = e^x &- 0.500004175740414298700905557455166x^2 \\
&-0.041638055568834480012148060896809x^4 \\
&-0.0014425473149225348443037016434657x^6 \\
&+0.000013311147131641983031454006923302x^8 \\
&-0.0000090876855913124668338091573609605x^{10} \\
&-0.00000083424445094660028827396146693082x^{12} \\
&+0.000000037876040459617077946991317998246x^{14} \\
&-0.000000000015772219821391087485678341815356x^{16} \\
&+0.000000000000000013877787807814456755295395851135
\end{aligned} \tag{4.18}
$$

Again, one can get the approximation $u_5^{(3)}(x)$ of $u(x)$ by substituting $u_4^{(2)}(x)$ in the nonlinear term $\sin(u(x))$, which is given by

$$
\begin{aligned}
u_5^{(3)}(x) = e^x &- 0.49999989830481488172581934346248x^2 \\
&-0.041668056606074833445932563442428x^4 \\
&-0.0013839852672572706244782813768148x^6 \\
&-0.000031972982740266785270422060761747x^8 \\
&+0.0000044279090765597957835121606922403x^{10} \\
&-0.0000011698051952654471291813237324335x^{12} \\
&+0.00000020219576882570412793447981920844x^{14} \\
&+0.000000000023074475795622385833449540653001x^{16} \\
&-0.00000000000089655271029356142341740348491118x^{18} \\
&+0.000000000000001699758430998214551186562586755x^{20} \\
&+0.0000000000000003386180225106727448292076587677
\end{aligned} \tag{4.19}
$$

Continuing the linearization process, the absolute errors associated with the approximation $u_4^{(2)}(x)$, $u_5^{(3)}(x)$, $u_6^{(4)}(x)$, and $\bar{\phi}_7^{(5)}(x)$ are listed in Table 4.

Table 4. Absolute errors associated with $u_4^{(2)}(x)$, $u_5^{(3)}(x)$, $u_6^{(4)}(x)$, and $u_7^{(5)}(x)$ for Example 2.

x	True Solution	$u_4^{(2)}(x)$	$u_5^{(3)}(x)$	$u_6^{(4)}(x)$	$u_7^{(5)}(x)$
0.0	1.0000000000	0.0000	0.0000	0.0000	0.0000
0.1	1.1051667500	3.9e-8	8.8e-10	1.3e-11	5.5e-13
0.2	1.2213360025	1.2e-7	2.1e-9	1.1e-10	7.2e-13
0.3	1.3495202934	1.8e-7	1.0e-9	3.6e-10	8.2e-13
0.4	1.4907523258	1.3e-7	3.5e-9	6.8e-10	1.5e-12
0.5	1.6460953055	4.6e-8	8.5e-9	8.4e-10	3.1e-12
0.6	1.8166535821	2.9e-7	9.3e-9	7.9e-10	4.7e-12
0.7	2.0035837018	4.6e-7	3.6e-9	7.3e-10	5.9e-12
0.8	2.2081059822	4.2e-7	3.7e-9	7.9e-10	7.7e-12
0.9	2.4315167257	1.8e-7	4.2e-9	5.3e-10	7.4e-12
1.0	2.6752011936	0.0000	0.0000	0.0000	0.0000

Finally, Table 5 shows that is powerful and effective compared with [4].

Table 5. Numerical results for Example 2.

x	True Solution	RHKSM [4]	BPEI($u_7^{(5)}(x)$)
0.0	1.000000000	0.0000	0.0000
0.1	1.1051667500	2.78E-8	5.5E-13
0.2	1.2213360025	8.09E-8	7.2E-13
0.3	1.3495202934	1.20E-7	8.2E-13
0.4	1.4907523258	1.25E-7	1.5E-12
0.5	1.6460953055	9.56E-8	3.1E-12
0.6	1.8166535821	4.82E-8	4.7E-12
0.7	2.0035837018	7.38E-9	5.9E-12
0.8	2.2081059822	1.07E-8	7.7E-12
0.9	2.4315167257	7.08E-9	7.4E-12
1.0	2.6752011936	0.0000	0.0000

5. Conclusion

The computations associated with the two examples discussed above were performed by using Matlab R2012a. The existence and uniqueness of the solution is guaranteed by Agarwals book [8]. The proposed algorithm using BPIS, produced a reliable computational method for handling boundary value problems. Comparing the obtained results with other works, the BPIS was clearly reliable if compared with grid points techniques where the solution is defined at grid points only. Moreover, numerical methods based on the approach we used would require considerably less computational effort.

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