A BRIEF COMPARISON OF G-CONTRACTION CONDITIONS AND A GENERALIZED FIXED POINT THEOREM

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Abstract. Let (X, G) be a G-metric space and f denote a self-map on X. A brief comparison of some G-contraction conditions is made, and a new generalized fixed point theorem is obtained by employing a wider inequality.

Keywords: G-metric space, G-Cauchy sequence, fixed point, G-contractive fixed point.
(2010) Mathematical Subject Classification: 54H25.

1. Introduction

Let X be a nonempty set and $G : X \times X \times X \to \mathbb{R}$ be such that

(G1) $G(x, y, z) \geq 0$ for all $x, y, z \in X$ with $G(x, y, z) = 0$ if $x = y = z$,

(G2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, x, z) = G(z, x, y) = G(y, z, x) = G(z, y, x)$ for all $x, y, z \in X$,

(G5) $G(x, y, z) \leq G(x, w, w) + G(w, y, z)$ for all $x, y, z, w \in X$.

Then G is called a G-metric on X and the pair (X, G) denotes a G-metric space. Axiom (G5) is referred to as the rectangle inequality (of G). This notion was introduced by Mustafa and Sims [3] in 2006.

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It is easily seen that

$$G(x, y, y) \leq 2G(x, x, y) \text{ for all } x, y \in X. \tag{1.1}$$

We use the following notions, developed in [3]:

Let (X, G) be a G-metric space. A G-ball in X is defined by

$$B_G(x_0, r) = \{y \in X : G(x_0, y, y) < r\}.$$

It is easy to see that the family of all G-balls forms a base topology, called the G-metric topology $\tau(G)$ on X. A sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ is said to be *G-convergent* with limit $p \in X$, if it converges to p in $\tau(G)$. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a G-metric space (X, G) is said to be G-Cauchy, if $\lim_{n,m\to\infty} G(x_n, x_m, x_m) = 0$. The space X is said to be G-complete, if every G-Cauchy sequence in X converges in it.

Definition 1.1 Let (X, G) be a G-metric space. A set $S \subset X$ is said to be G-bounded or simply bounded, if there exists a positive number M such that $G(x, y, z) < M$ for all $x, y, z \in S$. Note that if S is G-bounded, then its diameter $\delta(S) = \sup\{G(x, y, z) : x, y, z \in S\}$ is finite.

Definition 1.2 A self-map f on a G-metric space (X, G) is G-continuous at a point $p \in X$, if $f^{-1}(B_G(fp, r)) \in \tau(G)$ for all $r > 0$, and f is G-continuous on X, if it is G-continuous at every $p \in X$.

Lemma 1.1 *A self-map f on a G-metric space (X, G) is G-continuous at a point $p \in X$ if and only if the sequence $\langle fp_n \rangle_{n=1}^{\infty} \subset X$ G-converges to fp whenever $\langle p_n \rangle_{n=1}^{\infty}$ is a sequence in X which G-converges to fp.*

2. A brief comparison

Mohanta [1] proved a pair of results which are given below:

Theorem 2.1 *Let (X, G) be a complete G-metric space and f, a self-map on X satisfying*

$$\begin{aligned} G(fx, fy, fz) \leq\ & aG(x, y, z) + bG(x, fx, fx) + cG(y, fy, fy) + dG(z, fz, fz) \\ & + e \max\{G(x, fy, fy), G(y, fx, fx), G(y, fz, fz), G(z, fy, fy), \\ & G(z, fx, fx), G(x, fz, fz)\} \text{ for all } x, y, z \in X, \end{aligned} \tag{2.1}$$

where $a, b, c, d, e \geq 0$ with $a + b + c + d + 2e < 1$. Then f has a unique fixed point p and f is G-continuous at p.

Theorem 2.2 *Let (X, G) be a complete G-metric space and f, a self-map on X satisfying*

$$\begin{aligned} G(fx, fy, fz) \leq\ & a[G(x, fy, fy) + G(y, fx, fx)] + b[G(y, fz, fz) + G(z, fy, fy)] \\ & + c[G(x, fz, fz) + G(z, fx, fx)] + dG(x, y, z) \\ & + e \max\{G(x, fx, fx), G(y, fy, fy), G(z, fz, fz)\} \\ & \text{for all } x, y, z \in X, \end{aligned} \tag{2.2}$$

where $a.b, c, d, e \geq 0$ with $2a + 2b + 2c + d + 2e < 1$. Then f has a unique fixed point p and f is G-continuous at p.

The case $e = 0$ of Theorem 2.1 gives that of Mustafa et al [2], while with $d = e = 0$ and $a+b+c = k$, Theorem 2.2 reduces to the result proved by Mustafa and Sims [4].

Theorem 2.3 *Let (X, G) be a complete G-metric space, f a self-map on X satisfying*

$$G(fx, fy, fz) \leq k \max\big\{[G(x, fy, fy) + G(y, fx, fx) + G(z, fz, fz)], [G(y, fz, fz) + G(z, fy, fy) + G(x, fx, fx)], [G(z, fx, fx) + G(x, fz, fz) + G(y, fy, fy)]\big\} \quad \text{for all } x, y, z \in X, \tag{2.3}$$

where $0 < k < 1/3$. Then f will have a unique fixed point p and f will be G-continuous at p.

and Vats et al [5] proved

Theorem 2.4 *Suppose that (X, G) is a complete G-metric space, f a self-map on X satisfying the condition*

$$G(fx, fy, fz) \leq k \max\big\{ G(x, fx, fx) + G(y, fy, fy) + G(z, fz, fz), G(x, fy, fy) + G(y, fx, fx) + G(z, fy, fy), G(x, fz, fz) + G(y, fz, fz) + G(z, fx, fx)\big\} \quad \text{for all } x, y, z \in X, \tag{2.4}$$

where $0 < k < 1/4$. Then f will have a unique fixed point p and f is continuous at p.

Since $\alpha + \beta + \gamma \leq 3 \max\{\alpha, \beta, \gamma\}$ for $\alpha \geq 0$, $\beta \geq 0$ and $\gamma \geq 0$, one can obtain the following generalizations respectively:

Theorem 2.5 *Let (X, G) be a complete G-metric space and f denote a self-map on X satisfying*

$$G(fx, fy, fz) \leq k \max\big\{G(x, fy, fy), G(y, fx, fx), G(z, fz, fz), G(y, fz, fz), G(z, fy, fy), G(x, fx, fx), G(z, fx, fx), G(x, fz, fz), G(y, fy, fy)]\big\} \quad \text{for all } x, y, z \in X, \tag{2.5}$$

where $0 \leq k < 1/9$. Then f has a unique fixed point p, and f is G-continuous at p.

Theorem 2.6 *Let (X, G) be a complete G-metric space, and f denote a self-map on X satisfying*

$$G(fx, fy, fz) \leq k \max\big\{ G(x, fx, fx), G(y, fy, fy), G(z, fz, fz), G(x, fy, fy), G(y, fx, fx), G(z, fy, fy), G(x, fz, fz), G(y, fz, fz), G(z, fx, fx)\} \quad \text{for all } x, y, z \in X, \tag{2.6}$$

where $0 \leq k < 1/12$. Then $f$has a unique fixed point p, and f is G-continuous at p.

In this paper, a nice generalization of Theorem 2.5 and Theorem 2.6 is obtained by employing a wider inequality.

3. Main result

Given $x, y, z \in X$, define $D(x,y,z) = \max E_f(x,y,z)$, where

$$E_f(x,y,z) = \left\{ G(f^i p, f^j q, f^k r) : 0 \le i, j, k \le 1; p, q, r \in \{x, y, z\} \right\}. \tag{3.1}$$

It may be noted that $E_f(x,y,z)$ has 36 elements.

Our main result is

Theorem 3.1 *Let (X, G) be a G-metric space and $f : X \to X$ satisfying the following inequality:*

$$G(fx, fy, fz) \le c \max D(x,y,z) \text{ for all } x, y, z \in X, \tag{3.2}$$

where $0 < c < 1/3$. If X is G-complete, then f has a unique fixed point p and f is G-continuous at p.

Proof. Writing $x = x_{n-1}$, $y = z = x_n$ in (3.2) and then simplifying, we get

$$G(x_n, x_{n+1}, x_{n+1}) = G(fx_{n-1}, fx_n, fx_n) \le cD(x_{n-1}, x_n, x_n) = cM, \tag{3.3}$$

where

$$\begin{aligned} M = \max\{ & G(x_{n-1}, x_{n-1}, x_n), G(x_{n-1}, x_{n-1}, x_{n+1}), \\ & G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_{n+1}), G(x_{n-1}, x_{n+1}, x_{n+1}) \\ & G(x_n, x_n, x_{n+1}), G(x_n, x_{n+1}, x_{n+1})\}. \end{aligned} \tag{3.4}$$

On the other hand, taking $x = y = x_{n-1}$, $z = x_n$ in (3.2), one obtains similarly

$$G(x_n, x_n, x_{n+1}) \le cM, \tag{3.5}$$

Therefore from (3.3) and (3.5), it follows that

$$\max\{G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_n, x_{n+1})\} \le cM \text{ for all } n \ge 1. \tag{3.6}$$

Now the following cases arise:

Case (a). $M = G(x_{n-1}, x_{n-1}, x_n)$. Then (3.6) implies that

$$G(x_n, x_n, x_{n+1}) \le cG(x_{n-1}, x_{n-1}, x_n)$$

which, by induction, gives

$$G(x_n, x_n, x_{n+1}) \le c^n G(x_0, x_0, x_1) \text{ for all } n \ge 1.$$

Then for $m > n$, from the repeated application of the rectangle inequality, it follows that

$$\begin{aligned} G(x_n,&x_n,x_m) \\ &\leq \underbrace{G(x_n,x_n,x_{n+1}) + G(x_{n+1},x_{n+1},x_{n+2}) + ... + G(x_{m-1},x_{m-1},x_m)}_{m-n \text{ terms}} \\ &\leq c^n \left(1 + c + \cdots + c^{m-n-1}\right) G(x_0,x_0,x_1) \\ &\leq \frac{c^n}{1-c} \cdot G(x_0,x_0,x_1) \to 0 \text{ as } m,n \to \infty. \end{aligned}$$

Case (b). $M = G(x_{n-1},x_{n-1},x_{n+1})$. Then (3.6) implies that

$$\begin{aligned} G(x_n,x_n,x_{n+1}) &\leq cG(x_{n-1},x_{n-1},x_{n+1}) \\ &\leq c[G(x_{n-1},x_{n-1},x_n) + G(x_n,x_n,x_{n+1})] \\ &\leq \left(\tfrac{c}{1-c}\right) G(x_{n-1},x_{n-1},x_n) \\ &\ldots \\ &\leq \left(\tfrac{c}{1-c}\right)^n G(x_0,x_0,x_1) \text{ for all } n \geq 1, \end{aligned}$$

Then for $m > n$, by the rectangle inequality, it follows that

$$G(x_n,x_n,x_m) \leq \tfrac{1}{1-c} \left(\tfrac{c}{1-c}\right)^n G(x_0,x_0,x_1) \to 0 \text{ as } m,n \to \infty.$$

Case (c). $M = G(x_{n-1},x_n,x_n)$. Then (3.6) implies that

$$G(x_n,x_{n+1},x_{n+1}) \leq cG(x_{n-1},x_n,x_n) \leq \cdots \leq c^n G(x_0,x_1,x_1) \text{ for all } n \geq 1.$$

Then for $m > n$, we have

$$G(x_n,x_m,x_m) \leq \left(\tfrac{c}{1-c}\right)^n \cdot G(x_0,x_1,x_1) \to 0 \text{ as } m,n \to \infty.$$

Case (d). $M = G(x_{n-1},x_n,x_{n+1})$. Then (3.6) and induction imply that

$$G(x_n,x_n,x_{n+1}) \leq \left(\tfrac{2c}{1-c}\right)^n G(x_0,x_0,x_1) \text{ for all } n \geq 1,$$

Then for $m > n$, it follows that

$$G(x_n,x_n,x_m) \leq \tfrac{1}{1-c} \left(\tfrac{2c}{1-c}\right)^n G(x_0,x_0,x_1) \to 0 \text{ as } m,n \to \infty.$$

Case (e). $M = G(x_{n-1},x_{n+1},x_{n+1})$ is similar to Case (b).
For $M{=}G(x_n,x_n,x_{n+1})$, we see that $G(x_n,x_n,x_{n+1}){=}0$, while $G(x_n,x_{n+1},x_{n+1}){=}0$ if $M = G(x_n,x_{n+1},x_{n+1})$ for $n \geq 1$. Thus, from all these cases, we find that $\langle x_n \rangle_{n=1}^{\infty}$ is G-Cauchy.

Since X is G-complete. We can find a point $p \in X$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} f^n x_0 = p. \tag{3.7}$$

Now, writing $x = x_{n-1}$, $y = z = p$ in (3.2), we obtain

$$G(x_n, fp, fp) = G(fx_{n-1}, fp, fp) \leq cD(x_{n-1}, p, p)\cdot \tag{3.8}$$

Applying the limit as $n \to \infty$ in (3.8) and then using (3.7) and (1.1), it follows that

$$G(p, fp, fp) \leq c \max\{G(p, p, fp), G(p, fp, fp)\} \leq 2c\, G(p, fp, fp)$$

If $p \neq fp$ in this, then (G2) would give

$$0 < G(p, fp, fp) \leq 2c\, G(p, fp, fp) < G(p, fp, fp),$$

which is a contradiction, since $0 < 2c < 1$. Therefore $fp = p$, showing that p is a fixed point of f. The uniqueness of the fixed point follows easily from (3.2), (G2) and (1.1).

Continuity at p: Suppose that $\langle p_n \rangle_{n=1}^{\infty}$ is such that

$$\lim_{n\to\infty} p_n = p. \tag{3.9}$$

Writing $x = p_n$ with $y = z = p$ in (3.2), and using (G5), we get

$$G(fp_n, p, p) = G(fp_n, fp, fp) \leq cD(p_n, p, p) \leq cP, \tag{3.10}$$

where

$$P = \max\{G(p_n, p_n, fp_n), G(p_n, p_n, p), G(p_n, fp_n, fp_n), \\ G(p_n, fp_n, p), G(p_n, p, p), G(fp_n, fp_n, p), G(fp_n, p, p)\}\cdot$$

Here again we discuss different cases:

Case (a). Suppose that $P = G(p_n, p_n, fp_n)$. Then (3.10) and (G5) imply that

$$G(fp_n, fp, fp) \leq cG(p_n, p_n, fp_n) \leq c[G(p_n, p_n, p) + G(p, p, fp_n)] \leq \tfrac{c}{1-c} G(p_n, p_n, p)$$

Proceeding the limit as $n \to \infty$ in this, it follows that $G(fp_n, fp, fp) \to 0$. In other words, $\lim_{n\to\infty} fp_n = fp = p$, proving that f is continuous at p.

Case (b). Suppose that $P = G(p_n, fp_n, fp_n)$. Then

$$G(fp_n, fp, fp) \leq cG(p_n, fp_n, fp_n) \leq 2cG(p_n, p_n, fp_n)$$

and the continuity follows from Case (a).

Case (c). Suppose that $P = G(p_n, fp_n, p)$. Then

$$G(fp_n, fp, fp) \leq cG(p_n, fp_n, p) \leq c[G(p_n, p, p) + G(p, fp_n, p)] \leq \tfrac{c}{1-c} G(p_n, p, p).$$

and the continuity follows as in Case (a).

Case (d). Suppose that $P = G(fp_n, fp_n, p)$. Then

$$G(fp_n, fp, fp) \le cG(fp_n, fp_n, p) \le 2cG(fp_n, p, p)$$

and the continuity follows again from Case (a).

If $P = G(p_n, p_n, p)$ or $G(p_n, p, p)$, it easily follows that $G(fp_n, fp, fp) \to 0$ as $n \to \infty$, in view of (3.9). Finally, if $P = G(fp_n, p, p)$, the continuity is obvious. ∎

4. Corollaries and discussion

First, we have

Corollary 4.1 (Mohanta [1], Theorem 3.7) *Let (X, G) be a complete G-metric space and f be a self-map on X satisfying*

$$G(fx, fy, fz) \le k \max A_f(x, y, z) \text{ for all } x, y, z \in X, \tag{4.1}$$

where

$$\begin{aligned} A_f(x,y,z) = \{&G(x,fx,fx), G(y,fy,fy), G(z,fz,fz), G(x,fy,fy),\\ &G(y,fz,fz), G(z,fx,fx), G(x,fz,fz), G(y,fx,fx),\\ &G(z,fy,fy), G(x,fy,fz), G(y,fz,fx), G(z,fx,fy),\\ &G(x,y,fz), G(y,z,fx), G(z,x,fy), G(x,y,z)\}\end{aligned}$$

and $0 < k < 1/3$. Then f has a unique fixed point p and f is G-continuous at p

Since $A_f(x, y, z) \subset E_f(x, y, z)$, the corollary follows from Theorem 3.1.

Theorem 4.1 (Vats et al. [5]) *Suppose that (X, G) is a complete G-metric space and f, a self-map on X satisfying*

$$G(fx, fy, fz) \le k \max B_f(x, y, z) \text{ for all } x, y, z \in X, \tag{4.2}$$

where

$$\begin{aligned} B_f(x,y,z) = \{&G(x,fx,fx), G(x,fy,fy),\\ &G(x,fz,fz), G(y,fy,fy), G(y,fx,fx),\\ &G(y,fz,fz), G(z,fz,fz), G(z,fx,fx)\end{aligned}$$

and $0 \le k < 1/2$. Then f will have a unique fixed point p and f is G-continuous at p.

Since $B_f(x, y, z) \subset E_f(x, y, z)$, this corollary also follows from Theorem 3.1.

We conclude the paper with the following remarks:

Remark 4.1 The right hand side terms in (2.5) and (2.6) are some among those of $E_f(x, y, z)$. Hence (2.5) and (2.6) imply (3.2). Thus the conclusions of Theorem 2.5 and Theorem 2.6 follow from that of Theorem 3.1.

Remark 4.2 If we choose a, b, c, d and e such that $a+b+c+d+2e < 1/3$, then the right hand side of (2.1) is less than or equal to the right hand side of (3.2). Thus the conclusion of Theorem 2.1 follows from that of Theorem 3.1.

Remark 4.3 If we choose a, b, c, d and e such that $2a + 2b + 2c + d + 2e < 1/3$, then the right hand side of (2.2) is less than or equal to the right hand side of (3.2). Thus the conclusion of Theorem 2.2 follows from that of Theorem 3.1.

Acknowledgements. The authors would like to express their sincere thanks to the referee for his valuable suggestions in greatly improving the paper.

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Accepted: 21.01.2016