REGULAR MULTIPLICATIVE TERNARY HYPERRING

Md. Salim
T.K. Dutta

Department of Pure Mathematics
University of Calcutta
35, Ballygunge Circular Road
Kolkata – 700019
India
e-mails: smpmath746@yahoo.in
duttatapankumar@yahoo.co.in

K.P. Shum

Institute of Mathematics
Yunnan University
Kunming, 650091
People Republic of China
e-mail: kpshum@ynu.edu.cn

Abstract. Regular multiplicative ternary hyperring are introduced and considered. Some properties of regular multiplicative ternary hyperring are studied. Several characterization theorems of the above ternary hyperrings in terms of its hyperideals are obtained. In addition, regular hyperideals in a multiplicative ternary hyperring are particularly considered and investigated. Finally, we explore the relationships between the regular multiplicative ternary hyperrings and the hyperideals of a multiplicative ternary hyperring.

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1. Introduction

Algebraic structures play an important role in mathematics with wide applications in many disciplines such as theoretical physics, computer science, information science and coding theory etc. The theory of algebraic hyperstructures (or hypersystem) is a well established branch of classical algebraic theory. The theory of hyperstructure was introduced by F. Marty [6] in 1934. He first studied the hypergroups and analyzed their properties and then applied them to groups and rational algebraic functions. Nowadays there has been a remarkable growth of hyperstructure theory. Many researchers have observed that the theory of hyperstructure has many applications in pure and applied science. In [2], P. Corsini and V. Leoreanu have collected numerous applications of algebraic hyperstructures. The notion of multiplicative hyperring has been introduced by R. Rota [8] in which the addition is a binary operation and multiplication is a binary hyperoperation. M.A. Krasner also introduced the notion of hyperring, called Krasner hyperring [4]. In a Krasner hyperring $(R, +, \cdot)$, '+' is a binary hyperoperation and '$\cdot$' is a binary operation, in which the zero element is absorbing.

In 1971, W.G. Lister [5] introduced the notion of ternary ring and study some important properties of it. According to W.G. Lister [5], a ternary ring is an algebraic system consisting of a nonempty set R together with a binary operation, called addition and a ternary multiplication, which forms a commutative group relative to addition, a ternary semigroup relative to multiplication and left, right, lateral distributive laws hold.

In 2014, J.R. Castillo and Jocelyn S. Paradero-Vilela [1] have introduced a special kind of ternary hyperrings, called the Krasner ternary hyperring. In a Krasner ternary hyperring $(R, +, \cdot)$, '+' is a binary hyperoperation and '$\cdot$' is a ternary multiplication.

In [11], we have studied the multiplicative ternary hyperrings. Our notion of multiplicative ternary hyperring in this paper differs from the notion of Krasner multiplicative ternary hyperring. In our multiplicative ternary hyperring $(R, +, \circ)$, '+' is a binary operation and '$\circ$' is always a ternary hyper operation, in which the zero element is an absorbing zero (i.e., $0_R \circ x \circ y = x \circ 0_R \circ y = x \circ y \circ 0_R = \{0_R\}$ for all $x, y \in R$). We will introduce the notion of regular multiplicative ternary hyperring and to give some characterization theorems for such ternary hyperrings. We prove that if a multiplicative ternary hyperring R is regular, then for an hyperideal I of R, both I and R/I are regular.

Conversely, if R is a multiplicative ternary hyperring and if there exists an hyperideal I of R such that both I and R/I are regular, then R is regular. We also consider the idempotent hyperideals and prove that if a commutative multiplicative ternary hyperring R is regular then every hyperideal I of R is idempotent and conversely.

Finally, we consider the regular hyperideal and prove that a multiplicative ternary hyperring is regular if and only if $\{0\}$ is regular hyperideal of R.

2. Preliminaries

Definition 2.1. By a ternary hype roperation '$\circ$' on a nonempty set H, we mean a mapping $\circ$: $H \times H \times H \to P^*(H)$, where $P^*(H)$ is the set of all nonempty subsets of H. For $x, y, z \in H$, the image of the element $(x, y, z) \in H \times H \times H$ under the mapping '$\circ$' will be denoted by $x \circ y \circ z$ (which is called the ternary hyperproduct of x, y, z).

Definition 2.2. A multiplicative ternary hyperring $(R, +, \circ)$ is an additive commutative group $(R, +)$ endowed with a ternary hyperoperation '$\circ$' such that the following conditions hold :

(i) $(a \circ b \circ c) \circ d \circ e = a \circ (b \circ c \circ d) \circ e = a \circ b \circ (c \circ d \circ e)$;

(ii) $(a + b) \circ c \circ d \subseteq a \circ c \circ d + b \circ c \circ d$;
$a \circ (b + c) \circ d \subseteq a \circ b \circ d + a \circ c \circ d$;
$a \circ b \circ (c + d) \subseteq a \circ b \circ c + a \circ b \circ d$;

(iii) $(-a) \circ b \circ c = a \circ (-b) \circ c = a \circ b \circ (-c) = -(a \circ b \circ c)$ for all $a, b, c \in R$;

(iv) $0_R \circ x \circ y = x \circ 0_R \circ y = x \circ y \circ 0_R = \{0_R\}$ for all $x, y \in R$ (absorbing property of 0_R),

for all $a, b, c, d, e \in S$. We remark here that, if the inclusions in (ii) are replaced by equalities, then the multiplicative ternary hyperring is called a strongly distributive multiplicative ternary hyperring.

Definition 2.3. A multiplicative ternary hyperring $(R, +, \circ)$ is called commutative if $a_1 \circ a_2 \circ a_3 = a_{\sigma(1)} \circ a_{\sigma(2)} \circ a_{\sigma(3)}$, where σ is a permutation of $\{1, 2, 3\}$ for all $a_1, a_2, a_3 \in R$.

Definition 2.4. A nonempty finite subset $\varepsilon = \{(e_i, f_i)\}_{i=1}^{n}$ of $R \times R$ where R is a multiplicative ternary hyperring is called a left (resp. lateral, right) identity set of R if

$$\text{for any } a \in R,\ a \in \sum_{i=1}^{n} e_i \circ f_i \circ a \ \left(\text{resp. } a \in \sum_{i=1}^{n} e_i \circ a \circ f_i,\ a \in \sum_{i=1}^{n} a \circ e_i \circ f_i\right)$$

An element e of a multiplicative ternary hyperring $(R, +, \circ)$ is called a unital element of R if $a \in (e \circ e \circ a) \cap (e \circ a \circ e) \cap (a \circ e \circ e)$.

Definition 2.5. Let $(S, +, \circ)$ and $(S', +, \circ)$ be two multiplicative ternary hyperrings. Then a mapping$f : S \to S'$ is called a homomorphism(a good homomorphism) if $f(a + b) = f(a) + f(b)$ and $f(a \circ b \circ c) \subseteq f(a) \circ f(b) \circ f(c)$ (resp. $f(a \circ b \circ c) = f(a) \circ f(b) \circ f(c)$).

Definition 2.6. Let $(R, +, \circ)$ be a multiplicative ternary hyperring. An additive subgroup I of R is called

(i) a left hyperideal of R if $r, r' \in R, \Rightarrow r \circ r' \circ x \subseteq I$, for all $x \in I$;

(ii) a right hyperideal of R if $r, r' \in R, \Rightarrow x \circ r \circ r' \subseteq I$, for all $x \in I$;

(iii) a lateral hyperideal of R if $r, r' \in R, \Rightarrow r \circ x \circ r' \subseteq I$, for all $x \in I$;

(iv) a two sided hyperideal of R if I is both a left and a right hyperideal of R;

(v) an hyperideal of R if I is a left, a right, and a lateral hyperideal of R.

Based on the above definitions, we have the following resullts.

Remark 2.7. Let R be a multiplicative ternary hyperring. If I, J and K are three hyperideals of R, then

$$I \circ J \circ K = \cup \left\{ \sum_{finite} a_i \circ b_i \circ c_i : a_i \in I, b_i \in J, c_i \in K \right\}$$

is a hyperideal of R.

Remark 2.8. If $(R, +, \circ)$ is a multiplicative ternary hyperring and $a \in R$, then

$$\langle a \rangle_r = a \circ R \circ R + na$$

(where n is an integer) is the right hyperideal of R generated by a.

Similarly if $(R, +, \circ)$ is a multiplicative ternary hyperring and $a \in R$, then the following equations hold:

$$\langle a \rangle_m = R \circ a \circ R + R \circ R \circ a \circ R \circ R + na$$

(where n is an integer) is the lateral hyperideal of R generated by a, and

$$\langle a \rangle_l = R \circ R \circ a + na$$

(where n is an integer) is the left hyperideal of R generated by a.

Remark 2.9. If $(R, +, \circ)$ is a multiplicative ternary hyperring $a \in R$ with a unital element e, then the following equation holds:

$$\langle a \rangle_r = a \circ R \circ R$$

is the right hyperideal of R generated by a.

Similarly,

$$\langle a \rangle_m = R \circ a \circ R + R \circ R \circ a \circ R \circ R$$

is the lateral hyperideal of R generated by a. And

$$\langle a \rangle_l = R \circ R \circ a$$

is the left hyperideal of R generated by a.

3. Regular ternary hyperring

Definition 3.1. Let $(R, +, \circ)$ be a multiplicative ternary hyperring. Then, an element $a \in R$ is called a regular element if there exists an element $x \in R$ such that $a \in a \circ x \circ a$The multiplicative ternary hyperring $(R, +, \circ)$ is called regular if all of its elements are regular.

Example 3.2. [13] Let $(R, +, \cdot)$ be a regular ternary ring with a unital element e. Let A be a subset of R containing e. Now we define a multiplicative ternary hyperoperation '$\circ$' on R as follows:

$$a \circ b \circ c = \{a \cdot x \cdot b \cdot y \cdot c : x, y \in A\}.$$

Then $(R, +, \circ)$ is a multiplicative ternary hyperring. We denote this ring by $(R_A, +, \circ)$. Let $a \in R_A = R$. Since R is a regular ring, there exists an element $b \in R$ such that $aba = a$. Now, we derive that

$$a = a \cdot e \cdot b \cdot e \cdot a \in \{a \cdot x \cdot b \cdot y \cdot a : x, y \in A\} = a \circ b \circ a,$$

because $e \in A$. Thus, we have shown that $(R_A, +, \circ)$ is a regular multiplicative ternary hyperring.

Theorem 3.3. *Let f be a homomorphism from a regular multiplicative ternary hyperring R onto a multiplicative ternary hyperring T. Then T is a regular ternary hyperring. we observe that the homomorphic image of a regular multiplicative ternary hyperring is still a regular multiplicative ternary hyperring.*

Proof. The proof is obvious and hence we omit the proof. ∎

In the following proposition, we characterize the regular multiplicative ternary hyperrings.

Proposition 3.4. *A multiplicative ternary hyperring $(R, +, \circ)$ is regular if and only if for each $a \in R$, there exist $x_1, x_2, y_1, y_2, z_1, z_2 \in R$ such that*

$$a \in (a \circ x_1 \circ x_2) \circ (y_1 \circ a \circ y_2) \circ (z_1 \circ z_2 \circ a).$$

Proof. Let R be a regular multiplicative ternary hyperring and $a \in R$. Then there exists $x \in R$ such that $a \in a \circ x \circ a$ which implies that $a \in a \circ x \circ a \circ x \circ a \subseteq a \circ x \circ a \circ x \circ a \circ x \circ a \subseteq a \circ x \circ a \circ x \circ a \circ x \circ a \circ x \circ a$. Consequently, $a \in (a \circ x_1 \circ x_2) \circ (y_1 \circ a \circ y_2) \circ (z_1 \circ z_2 \circ a)$ for some $x_1, x_2, y_1, y_2, z_1, z_2 \in R$. Conversely, let a an arbitrary element of a multiplicative ternary hyperring $(R, +, \circ)$ and $a \in (a \circ x_1 \circ x_2) \circ (y_1 \circ a \circ y_2) \circ (z_1 \circ z_2 \circ a)$ for some $x_1, x_2, y_1, y_2, z_1, z_2 \in R$. This implies that $a \in a \circ (x_1 \circ x_2 \circ y_1 \circ a \circ y_2 \circ z_1 \circ z_2) \circ a \Rightarrow a \in a \circ s \circ a$ for some $s \in x_1 \circ x_2 \circ y_1 \circ a \circ y_2 \circ z_1 \circ z_2$. This proves that a and hence R is regular. ∎

It is noted that a left and a right hyperideal of a regular multiplicative ternary hyperring may not be regular; however, for a lateral hyperideal of a regular multiplicative ternary hyperring, we have the following results.

Proposition 3.5. *If I is a lateral hyperideal of a regular multiplicative ternary hyperring, then I is regular as a multiplicative ternary hyperring.*

Proof. Let I be a lateral hyperideal of a multiplicative ternary hyperring R. Let $a \in I$. Since R is regular, there exists $b \in R$ such that $a \in a \circ b \circ a \subseteq (a \circ b \circ a) \circ b \circ a = a \circ (b \circ a \circ b) \circ a$ where $b \circ a \circ b \subseteq I$. Thus I is regular. ∎

Corollary 3.6. *Every hyperideal of a regular multiplicative ternary hyperring is regular as a multiplicative ternary hyperring.*

Theorem 3.7. *If I and J are regular hyperideals of a multiplicative ternary hyperring $(R,+,\circ)$, then $I \cap J$ is regular as a multiplicative ternary hyperring.*

Proof. Obviously $I \cap J$ is an hyperideal of R. Let $a \in I \cap J$. Then there exist $x \in I$ and $y \in J$ such that $a \in a \circ x \circ a$ and $a \in a \circ y \circ a$. we hence deduce that $a \in a \circ x \circ a \subseteq (a \circ x \circ a) \circ x \circ (a \circ y \circ a) = a \circ (x \circ a \circ x \circ a \circ y) \circ a$. Now, $x \circ a \circ x \circ a \circ y \subseteq I \cap J$. Consequently, $I \cap J$ is regular as a multiplicative ternary hyperring. ∎

Definition 3.8. [Proposition 3.8, [12]] Let $(R,+,\circ)$ be a multiplicative ternary hyperring and I be an hyperideal of R. Then the multiplicative ternary hyperring $R/I = \{a+I : a \in R\}$ is called a quotient multiplicative ternary hyperring of $(R,+,\circ)$ by I, where $(a+I)+(b+I) = (a+b)+I$ and $(a+I)\circ(b+I)\circ(c+I) = \{p+I : p \in a \circ b \circ c\}$ for any $a,b,c \in R$.

For the hyperideals in a multiplicative ternary hyperring, we have the following results.

Theorem 3.9. *Let R be a regular strongly distributive multiplicative ternary hyperring and I an hyperideal of R. Then I (as a multiplicative ternary hyperring) and the quotient multiplicative ternary hyperring R/I are regular. Conversely, if $(R,+,\circ)$ is a multiplicative ternary hyperring and if there exists an hyperideal I of R such that both I (as a multiplicative ternary hyperring) and R/I are regular, then R is regular.*

Proof. The first part of the theorem follows directly from Proposition 3.5 and Theorem 3.3, since R/I is a homomorphic image of R.

Conversely, suppose that R is a multiplicative ternary hyperring and there exists an hyperideal I of R such that both I and R/I are regular. Let $a \in R$. Then $a+I \in R/I$. Since R/I is regular, there exists an element $b+I \in R/I$ where $b \in R$, such that $a+I \in (a+I)\circ(b+I)\circ(a+I)$. Then $a+I = z+I$ for some $z \in a \circ b \circ a$. Since $a+I = z+I$, again $a-z \in I$. Since I is regular

$$\begin{aligned} a-z &\in (a-z)\circ y \circ (a-z) \text{ for some } y \in I \\ &= a\circ y\circ a - a\circ y\circ z - z\circ y\circ a + z\circ y\circ z \\ &\subseteq a\circ y\circ a - a\circ y\circ a\circ b\circ a - a\circ b\circ a\circ y\circ a + a\circ b\circ a\circ y\circ a\circ b\circ a \\ &= a\circ(y - y\circ a\circ b - b\circ a\circ y + b\circ a\circ y\circ a\circ b)\circ a \end{aligned}$$

i.e., $a-z \in a\circ s\circ a$ for some $s \in y - y\circ a\circ b - b\circ a\circ y + b\circ a\circ y\circ a\circ b$. Thus $a-z = t$ for some $t \in a\circ s\circ a$. Then $a = t+z \subseteq a\circ s\circ a + a\circ b\circ a = a\circ(s+b)\circ a$. So a and hence R is regular. ∎

Theorem 3.10. [Theorem 3.18, [12]] *Let I and J be two hyperideals of a multiplicative ternary hyperring $(R,+,\circ)$. Then $I/(I\cap J) \cong (I+J)/J$.*

Theorem 3.11. *Let* $(R, +, \circ)$ *be a strongly distributive multiplicative ternary hyperring and* I *and* J *are hyperideals of* R *as multiplicative ternary hyperring. If* I *and* J *are regular, then* $I + J$ *is regular.*

Proof. Since I and J are regular, by Theorem 3.7 $I \cap J$ is regular. Again by Theorem 3.9, $I/(I \cap J)$ is regular. Since $(I + J)/J$ is a homomorphic image of $I/(I \cap J)$ and so $(I + J)/J$ is regular. Since J and $(I + J)/J$ are regular, by Theorem 3.9, the hyperideal $I + J$ is regular. ■

In the following theorems, we further investigate the properties of the multiplicative ternary hyperrings.

Theorem 3.12. *Let* $(R, +, \circ)$ *be a multiplicative ternary hyperring. Then the following statements are equivalent:*

(i) *R is a regular multiplicative ternary hyperring;*

(ii) *For any right hyperideal I, lateral hyperideal J, and left hyperideal K of R, $I \circ J \circ K = I \cap J \cap K$;*

(iii) *For $a, b, c \in R$, $\langle a\rangle_r \circ \langle b\rangle_m \circ \langle c\rangle_l = \langle a\rangle_r \cap \langle b\rangle_m \cap \langle c\rangle_l$;*

(iv) *$\langle a\rangle_r \circ \langle a\rangle_m \circ \langle a\rangle_l = \langle a\rangle_r \cap \langle a\rangle_m \cap \langle a\rangle_l$, for each element $a \in R$.*

Proof. (i) $\Rightarrow$ (ii). Suppose that R is a regular multiplicative ternary hyperring. Let I, J and K be a right hyperideal, a lateral hyperideal and a left hyperideal of R respectively. Obviously $I \circ J \circ K \subseteq I \cap J \cap K$ (1). Now let $a \in I \cap J \cap K$. Then we have $a \in a \circ x \circ a$ for some $x \in R$. This implies that $a \in a \circ x \circ a \subseteq (a \circ x \circ a) \circ (x \circ a \circ x) \circ (a \circ x \circ a) \subseteq I \circ J \circ K$. Thus we have $I \cap J \cap K \subseteq I \circ J \circ K$ (2). From (1) and (2), it follows that $I \circ J \circ K = I \cap J \cap K$.

Clearly, (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (i). Let $a \in R$. Then $a \in \langle a\rangle_r \cap \langle a\rangle_m \cap \langle a\rangle_l = \langle a\rangle_r \circ \langle a\rangle_m \circ \langle a\rangle_l = (a \circ R \circ R + na) \circ (R \circ a \circ R + R \circ R \circ a \circ R \circ R + na) \circ (R \circ R \circ a + na) \subseteq a \circ R \circ a$ i.e. $a \in a \circ y \circ a$ for some $y \in R$. Hence, we have proved that a and R are regular. ■

Theorem 3.13. *The following conditions on a multiplicative ternary hyperring R are equivalent:*

(i) *R is regular,*

(ii) *$I \cap J = I \circ R \circ J$ for every right hyperideal I and every left hyperideal J of R,*

(iii) *For $a, b \in R$, $\langle a\rangle_r \cap \langle b\rangle_l = \langle a\rangle_r \circ R \circ \langle b\rangle_l$,*

(iv) *For $a \in R$, $\langle a\rangle_r \cap \langle a\rangle_l = \langle a\rangle_r \circ R \circ \langle a\rangle_l$.*

Proof. Since R is a lateral hyperideal of itself, by (ii) of Theorem 3.12, (i) $\Rightarrow$ (ii) follows.

Clearly, (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv).

(iv) $\Rightarrow$ (i). Let $a \in R$. Then $a \in \langle a\rangle_r \cap \langle a\rangle_l = \langle a\rangle_r \circ R \circ \langle a\rangle_l = (a \circ R \circ R + na) \circ R \circ (R \circ R \circ a + na) \subseteq a \circ R \circ a$. Thus $a \in a \circ R \circ a$. Thus there exists an element $x \in R$ such that $a \in a \circ x \circ a$. Hence, a and R are regular. ∎

Definition 3.14. An element a of a multiplicative ternary hyperring $(R, +, \circ)$ is called idempotent if $a \circ a \circ a = a$.

Definition 3.15. An hyperideal I of a multiplicative ternary hyperring $(R, +, \circ)$ is called idempotent if $I \circ I \circ I = I$.

In the following theorem, we characterize the idempotent multiplicative ternary hyperrings.

Theorem 3.16. *Let $(R, +, \circ)$ be a commutative multiplicative ternary hyperring. Then R is regular if and only if every hyperideal I of R is idempotent.*

Proof. Let R be a regular multiplicative ternary hyperring and I be any hyperideal of R. Then $I \circ I \circ I \subseteq R \circ R \circ I \subseteq I$. Let $a \in I$. Then there exists $x \in R$ such that $a \in a \circ x \circ a \subseteq (a \circ x \circ a) \circ x \circ a = a \circ (x \circ a \circ x) \circ a \subseteq I \circ I \circ I$. Thus $I \subseteq I \circ I \circ I$ and hence $I \circ I \circ I = I$, i.e., I is idempotent.

Conversely, suppose that every hyperideal of R is idempotent. Let I, J and K be three hyperideals of R. Obviously $I \circ J \circ K \subseteq I \cap J \cap K$. Again, we have $I \cap J \cap K = (I \cap J \cap K) \circ (I \cap J \cap K) \circ (I \cap J \cap K) \subseteq I \circ J \circ K$. Thus $I \cap J \cap K = I \circ J \circ K$. Therefore, by Theorem 3.12, we have proved that R is a regular multiplicative ternary hyperring. ∎

4. Regular hyperideal

Let $(R, +, \circ)$ be a regular multiplicative ternary hyperring. Then for any right hyperideal S, lateral hyperideal M and left hyperideal L we have $S \circ M \circ L = S \cap M \cap L$. Hence, we have $(0) + S \circ M \circ L = S \cap M \cap L$. Also for an hyperideal T of R, we have $T + S \circ M \circ L = S \cap M \cap L$ ....(1), where S is a right hyperideal containing T, M is a lateral hyperideal containing T and L is a left hyperideal containing T. Based on relation (1), we now give the following definition of a regular hyperideal.

Definition 4.1. An hyperideal I of a multiplicative ternary hyperring $(R, +, \circ)$ is called a regular hyperideal of R if $I + S \circ M \circ L = S \cap M \cap L$ for any right hyperideal $S \supseteq I$, lateral hyperideal $M \supseteq I$ and left hyperideal $L \supseteq I$.

Based on the above definition, we have the following propostions concerning the regular hyperideals of a multipllicative ternary hyperrings.

Proposition 4.2. *Let $(R, +, \circ)$ be a multiplicative ternary hyperring and I be a regular hyperideal of R. If T is an hyperideal of R containing the regular hyperideal I, then T is also regular.*

Proof. Let S be a right hyperideal containing T, M a lateral hyperideal containing T and L a left hyperideal containing T. Then $S \supset T \supset I$, $M \supset T \supset I$ and $S \supset T \supset I$. Since I is regular, $I + S \circ M \circ L = S \cap M \cap L$. Now, we deduce that $T + S \circ M \circ L \supseteq I + S \circ M \circ L = S \cap M \cap L$. Again, $T \subseteq S \cap M \cap L$ and $S \circ M \circ L \subseteq S \cap M \cap L$ and hence $T + S \circ M \circ L \subseteq S \cap M \cap L$. Thus $T + S \circ M \circ L = S \cap M \cap L$. Hence, we have shown that T is regular. ∎

Proposition 4.3. *A multiplicative ternary hyperring $(R, +, \circ)$ is a regular multiplicative ternary hyperring if and only if (0) is a regular hyperideal of R.*

Proof. $(R, +, \circ)$ be a regular multiplicative ternary hyperring
$\Leftrightarrow S \circ M \circ L = S \cap M \cap L$ for any right hyperideal S, lateral hyperideal M and left hyperideal L of R (by Proposition 3.12)
$\Leftrightarrow (0) + S \circ M \circ L = S \cap M \cap L$ for any right hyperideal $(0) \subseteq S$, lateral hyperideal $(0) \subseteq M$ and left hyperideal $(0) \subseteq L$
$\Leftrightarrow (0)$ is a regular hyperideal of R. ∎

Corollary 4.4. *Let $(R, +, \circ)$ be a regular multiplicative ternary hyperring and T be an hyperideal of R. Then the hyperideal T is regular.*

Let N be the intersection of all nonzero hyperideals of R, N_r the intersection of all nonzero right hyperideals of R, N_m the intersection of all nonzero lateral hyperideals of R and N_l the intersection of all nonzero left hyperideals of R. If $N = \{0\}$, then obviously we have $N = N_r = N_m = N_l = (0)$.

Finally, we give some theorems related to regular multiplicative ternary hyperrings and their regular hyperideals.

Theorem 4.5. *Let $(R, +, \circ)$ be a multiplicative ternary hyperring and $N = N_r = N_m = N_l$. Then R is a regular multiplicative ternary hyperring if and only if N is a regular hyperideal of R.*

Proof. Let R be a regular multiplicative ternary hyperring. If $N = N_r = N_m = N_l = \{0\}$, then the proof follows from Proposition 4.3. Hence, we may assume that $N = N_r = N_m = N_l \neq \{0\}$. By Proposition 4.3, it follows that $\{0\}$ is a regular hyperideal of R. Now, $\{0\} \subseteq N$ implies that N is a regular hyperideal of R, by Proposition 4.2.

Conversely, let N be a regular hyperideal of R. Let S be a right hyperideal, M a lateral hyperideal and L a left hyperideal of R. Then $N{=}N_r \subseteq S$, $N{=}N_m \subseteq M$ and $N = N_l \subseteq L$. Since N is a regular hyperideal of R, $N + S \circ M \circ L = S \cap M \cap L$. Since $N \circ N \circ N$ is a right hyperideal of R, $N = N_r \subseteq N \circ N \circ N \subseteq S \circ M \circ L$ and $N + S \circ M \circ L = S \circ M \circ L$. Consequently, $N + S \circ M \circ L = S \cap M \cap L$ implies that $S \circ M \circ L = S \cap M \cap L$. Hence, from Theorem 3.12, it follows that R is a regular multiplicative ternary hyperring. ∎

Theorem 4.6. *Let R be a multiplicative ternary hyperring and $N = N_r = N_m = N_l \neq (0)$. Then R is regular multiplicative ternary hyperring if and only if every nonzero hyperideals of R is regular.*

Proof. Let R be a regular multiplicative ternary hyperring and I be a nonzero hyperideal of R. Since R is regular, (0) is a regular hyperideal of R, by Proposition 4.3. Agai,n since $I \supseteq (0)$, by Proposition 4.2, we see that I is a regular hyperideal of R.

Conversely, assume that every nonzero hyperideals of R is regular and $N = N_r = N_m = N_l \neq (0)$. Then N is a regular hyperideal of R. Now, by Theorem 4.5, R is a regular ring. ∎

Lemma 4.7. *Let $(R, +, \circ)$ be a multiplicative ternary hyperring and I be an hyperideal of R. Then, the following conditions are equivalent:*

(i) $\langle a\rangle_r \cap \langle a\rangle_m \cap \langle a\rangle_l = \langle a\rangle_r \circ \langle a\rangle_m \circ \langle a\rangle_l$ *for any* $a \in R$,

(ii) $S \cap M \cap L = S \circ M \circ L$, *for any right hyperideal S, lateral hyperideal M and left hyperideal L.*

Proof. (i) $\Rightarrow$ (ii) Let $x \in S \cap M \cap L$. Now, $x \in \langle x\rangle_r \cap \langle x\rangle_m \cap \langle x\rangle_l = \langle x\rangle_r \circ \langle x\rangle_m \circ \langle a\rangle_l$ by (i) $\subseteq S \circ M \circ L$. Therefore, we have $S \cap M \cap L \subseteq S \circ M \circ L$. Obviously, $S \circ M \circ L \subseteq S \cap M \cap L$. Hence $S \cap M \cap L = S \circ M \circ L$.

Obviously, (ii) $\Rightarrow$ (i) holds. ∎

Theorem 4.8. *The following conditions are equivalent in a multiplicative ternary hyperring $(R, +, \circ)$.*

(i) *I is a regular hyperideal of R;*

(ii) *For $a, b, c \in R$,* $I + \langle a\rangle_r \circ \langle b\rangle_m \circ \langle c\rangle_l = I + \langle a\rangle_r \cap \langle b\rangle_m \cap \langle c\rangle_l$;

(iii) *For each $a \in R$,* $I + \langle a\rangle_r \circ \langle a\rangle_m \circ \langle a\rangle_l = I + (\langle a\rangle_r \cap \langle a\rangle_m \cap \langle a\rangle_l)$;

(iv) *For each $a \in R \backslash I = I^c$,* $a = x + \sum_{i=1}^{n} a \circ p_i \circ a \circ q_i \circ a + \sum_{i=1}^{n} a \circ r_i \circ s_i \circ a \circ u_i \circ v_i \circ a$, *for some $x \in I$ and $p_i, q_i, r_i, s_i, u_i, v_i \in R$.*

Proof. (i) $\Rightarrow$ (ii). Suppose that I is a regular hyperideal of R. For any $a, b, c \in R$, $I \subseteq (I + \langle a\rangle_r)$, $(I + \langle b\rangle_m)$ and $(I + \langle c\rangle_l)$. Now, we deduce that

$$\begin{aligned} & I + \langle a\rangle_r \cap \langle b\rangle_m \cap \langle c\rangle_l \subseteq (I + \langle a\rangle_r) \cap (I + \langle b\rangle_m) \cap (I + \langle c\rangle_l) \\ & = I + (I + \langle a\rangle_r) \circ (I + \langle b\rangle_m) \circ (I + \langle c\rangle_l) \text{ (since } I \text{ is regular)} \\ & \subseteq I + I \circ I \circ I + I \circ \langle b\rangle_m \circ I + I \circ \langle b\rangle_m \circ \langle c\rangle_l + I \circ I \circ \langle c\rangle_l + \langle a\rangle_r \circ I \circ I \\ & + \langle a\rangle_r \circ I \circ \langle c\rangle_l + \langle a\rangle_r \circ \langle b\rangle_m \circ I + \langle a\rangle_r \circ \langle b\rangle_m \circ \langle c\rangle_l \subseteq I + \langle a\rangle_r \circ \langle b\rangle_m \circ \langle c\rangle_l \end{aligned}$$

Again $\langle a\rangle_r \circ \langle b\rangle_m \circ \langle c\rangle_l \subseteq \langle a\rangle_r \cap \langle b\rangle_m \cap \langle c\rangle_l$ implies that

$$I + \langle a\rangle_r \circ \langle b\rangle_m \circ \langle c\rangle_l \subseteq I + \langle a\rangle_r \cap \langle b\rangle_m \cap \langle c\rangle_l.$$

Hence $I + \langle a\rangle_r \circ \langle b\rangle_m \circ \langle c\rangle_l = I + \langle a\rangle_r \cap \langle b\rangle_m \cap \langle c\rangle_l$.

Obviously, (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (iv). We first observe that $\langle I+\langle a\rangle_r\rangle_r = I+\langle a\rangle_r = I+\langle a\rangle_r \cap R\cap R = I+\langle a\rangle_r \circ R\circ R =$ (by Lemma 4.7) $I+(a\circ R\circ R+na)\circ R\circ R \subseteq I+a\circ R\circ R$. Obviously, $I+a\circ R\circ R \subseteq \langle I+\langle a\rangle_r\rangle_r$, and hence, $\langle I+\langle a\rangle_r\rangle_r = I+a\circ R\circ R$. Similarly, we have $\langle I+\langle a\rangle_m\rangle_m = I+R\circ a\circ R+R\circ R\circ a\circ R\circ R$ and $\langle I+\langle a\rangle_l\rangle_l = I+R\circ R\circ a$. Now, we have

$$\begin{aligned}
&\langle a\rangle_r \cap \langle a\rangle_m \cap \langle a\rangle_l \subseteq I+\langle I+\langle a\rangle_r\rangle_r \cap \langle I+\langle a\rangle_m\rangle_m \cap \langle I+\langle a\rangle_l\rangle_l\\
&= I+\langle I+\langle a\rangle_r\rangle_r \circ \langle I+\langle a\rangle_m\rangle_m \circ \langle I+\langle a\rangle_l\rangle_l \text{ (by Lemma 4.7)}\\
&= I+(I+a\circ R\circ R)\circ(I+R\circ a\circ R+R\circ R\circ a\circ R\circ R)\circ(I+R\circ R\circ a)\\
&\subseteq I+(a\circ R\circ R\circ R\circ a\circ R\circ R\circ R\circ a\\
&+a\circ R\circ R\circ R\circ R\circ a\circ R\circ R\circ R\circ R\circ a)\\
&\subseteq I+a\circ R\circ a\circ R\circ a+a\circ R\circ R\circ a\circ R\circ R\circ a.
\end{aligned}$$

Since $a\in \langle a\rangle_r \cap \langle a\rangle_m \cap \langle a\rangle_l$, there exist elements $x\in I$ and $p_i,q_i,r_i,s_i,u_i,v_i\in R$ such that

$$a = x+\sum_{i=1}^{n} a\circ p_i\circ a\circ q_i\circ a+\sum_{i=1}^{n} a\circ r_i\circ s_i\circ a\circ u_i\circ v_i\circ a.$$

(iv) $\Rightarrow$ (i). Let S,M and L be any right, lateral and left hyperideal of R respectively such that $S\supseteq I$, $M\supseteq I$ and $L\supseteq I$. Then, obviously, $I+S\circ M\circ L\subseteq S\cap M\cap L$. Again, let $a\in S\cap M\cap L$. Then, by condition (iv), we have $a = x+\sum_{i=1}^{n} a\circ p_i\circ a\circ q_i\circ a+\sum_{i=1}^{n} a\circ r_i\circ s_i\circ a\circ u_i\circ v_i\circ a$ for some $x\in I$ and $p_i,q_i,r_i,s_i,u_i,v_i\in R$. Since $\sum_{i=1}^{n} a\circ p_i\circ a\circ q_i\circ a$, $\sum_{i=1}^{n} a\circ r_i\circ s_i\circ a\circ u_i\circ v_i\circ a\in S\circ M\circ L$. $a\in I+S\circ M\circ L$ and hence $S\cap M\cap L\subseteq I+S\circ M\circ L$. Thus $S\cap M\cap L = I+S\circ M\circ L$. Consequently, I is a regular hyperideal. ∎

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