SOURCE TERM IDENTIFICATION IN SEMIDIFFERENTIAL EQUATIONS[1]

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Abstract. In this paper we propose a numerical method for the source term identification in semidifferential equations form noisy data. Our method employ a mollification technique to stabilize (regularize) the inverse solution. We prove convergence results for both the continuous and discretized problems. Numerical examples are provided to validate the effectiveness of the proposed approach.

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1. Introduction

Let a, b, c be constants, and $\alpha \in (0,2)$. Consider the semidifferential equation

$$aD^2y(t) + bD_*^\alpha y(t) + cy(t) = f(t), \ \ 0 < t < 1, \tag{1.1}$$

subject to the initial conditions

$$y(0) = y_0, \ \ y'(0) = y_0', \tag{1.2}$$

where D^2 stands for the second derivative operator, while D_*^α denotes the Caputo fractional differential operator of order α defined by

$$D_*^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} D^n y(s) ds, \ \ n-1 < \alpha < n, \ \ n \in \mathbb{N}^+,$$

which reduces to the ordinary derivative when α is a positive integer; see [1]–[4] for more about fractional calculus and its applications.

Recently, the initial value problem (1.1)–(1.2) has found many applications in physics and mechanics. For instance, when $\alpha = 0.5$ it models the motion of a single degree-of-freedom spring-mass-damper system where in this case a, b, and

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c represent the mass, damping coefficient, and stiffness, respectively, and $f(t)$ is the externally applied force. For $\alpha = 1.5$, (1.1) reduces to the Bagley-Torvik equation which models, for example, the motion of a rigid thin-plate immersed in a Newtonian fluid, where a is the mass, c is the stiffness, b is a constant related to the area of the plate and fluid viscosity, and f is the loading force. For the numerical and analytical treatment of (1.1) and general semidifferential equations, we refer the reader to [5]–[7] and references therein.

In many practical applications, it is desirable to determine the source term f (loading force) from noisy observations y^ϵ of the displacement y. Therefore, we propose the important

Inverse Problem. *Estimate f from noisy observation y^ϵ satisfying $\|y^\epsilon - y\|_\infty \leq \epsilon$.*
This problem is ill-posed as it can be demonstrated by the following example.

Example 1.1. Let $b = c = 0$ in (1.1), and take $y^\epsilon(t) = y(t) + \epsilon \sin(\epsilon^{-1} t)$. Then

$$\|y^\epsilon - y\|_\infty \leq \epsilon \to 0, \text{ while } \|f^\epsilon - f\|_\infty \leq a/\epsilon \to \infty,$$

as $\epsilon \to 0$, showing the instability behavior of the inverse problem.

A consequence of the ill-posedness of the inverse problem is that standard analytical and numerical methods will fail to produce stable solutions regardless of how small the perturbations in the data is. Therefore, to tackle the instability issue, one needs to regularize the problem using, for instance, the Tikhonov regularization; see Engl et al. [8] for more about the theory of regularization and ill-posed problems.

To overcome the instability issue indicated above, we employ the mollification method to smooth out the given noisy data, then we compute the solution using equation (1.1). More precisely, let $y^{\epsilon,\delta}$ denotes the mollified (smoothed) data, then the source term f is approximated by the function

$$f^{\epsilon,\delta}(t) = aD^2 y^{\epsilon,\delta}(t) + bD_*^\alpha y^{\epsilon,\delta}(t) + cy^{\epsilon,\delta}(t),$$

where $\delta > 0$ is a parameter that controls the degree of smoothing. In practice only discrete data is available and therefore equation (1.1) must be discretized using appropriate finite-difference schemes. We will prove convergence results and provide error bounds for both the continuous and discretized problems.

The rest of this article proceeds as follows. In Section we introduce some definitions and preliminary results, in Section we present the mollification approach for the proposed inverse problem and prove the main results, numerical examples are given in Section .

2. Mollification technique

In this section we give a short review of mollification theory and some auxiliary results related to the source term inverse problem. Then we introduce a regularization scheme based on the mollification method for handling the source term identification problem. We prove convergence results for both the continuous and discretized schemes.

2.1. Preliminaries

We use the notation $\|g\|_{\infty,K}$ to denote the uniform norm of the function g over the set K.

Let $\delta > 0$ and define $A = \int_{-3}^{3} \exp(-s^2)\, ds$. The δ-mollifier, denoted by ρ_δ, is defined by

$$\rho_\delta(t) = \frac{1}{A\delta} \begin{cases} \exp\left(-t^2/\delta^2\right), & |t| \leq 3\delta, \\ 0, & |t| > 3\delta. \end{cases}$$

The function ρ_δ is nonnegative, $C^\infty(-3\delta, 3\delta)$ satisfies the normalization property

$$\int_{t-3\delta}^{t+3\delta} \rho_\delta(t-s)ds = \int_{-3\delta}^{3\delta} \rho_\delta(s)ds = 1.$$

Let $I = [0,1]$ and set $I_\delta = [3\delta, 1-3\delta]$ for $\delta < 1/6$, we have the following definition.

Definition 2.1. The δ-mollification of a function $g \in L^1(I)$ on I_δ, denoted by $J_\delta g$, is given by

$$(J_\delta g)(t) = \int_{-3\delta}^{3\delta} \rho_\delta(s) g(t-s) ds = \int_{t-3\delta}^{t+3\delta} \rho_\delta(t-s) g(s) ds, \qquad t \in I_\delta.$$

Figure 1 shows the δ-mollification of the function $g(t) = |t - 0.5|$ for various values of δ. It is evident that δ represents a smoothing parameter; the larger the value of δ the more smoothing effect.

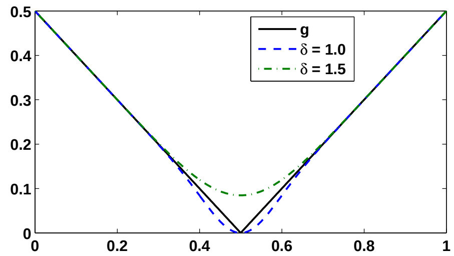


Figure 1: The function $g(t)=|x-0.5|$ and its δ-mollification for several values of δ.

The following result shows the convergence behavior for the δ-mollification.

Lemma 2.1. *Let $\|g^\epsilon - g\|_{\infty,I} \leq \epsilon$. There exists a constant C independent of δ and ϵ such that:*

(a) *If $g \in C(I)$, then $\|J_\delta g^\epsilon - g\|_{\infty,I_\delta} \leq C\left(\delta + \epsilon\right)$.*

(b) *If $g \in C^2(I)$, then $\|D_*^\alpha(J_\delta g^\epsilon) - D_*^\alpha g\|_{\infty,I_\delta} \leq C\left(\delta + \frac{\epsilon}{\delta^r}\right)$, where $r = 1$ if $\alpha \in (0,1]$, and $r = 2$ if $\alpha \in (1,2]$.*

Proof. cf. [9], [10]. ∎

The definition of the δ-mollification can be extended for discretized functions. To this end, let $K : 0 \leq t_1 < t_2 < \cdots < t_n \leq 1$ be uniform partition of $[0,1]$ and $G = \{G_1, G_2, \ldots, G_n\}$ be some discrete data.

Definition 2.2. The discrete δ-mollification of the data G is defined by

$$(J_\delta G)(t) = \sum_{i=1}^{n} \left(\int_{s_{i-1}}^{s_i} \rho_\delta(t-s)ds \right) G_i,$$

where $s_0 = 0$, $s_n = 1$, and $s_i = (t_i + t_{i+1})/2$ for $i = 1, \ldots, n-1$.

Let $G^\epsilon = \{G_1^\epsilon, \ldots, G_n^\epsilon\}$ be a perturbed version of the data $G = \{G_i = g(t_i) \mid t_i \in K\}$ satisfying $\|G^\epsilon - G\|_\infty \le \epsilon$, and set $\Delta t = 1/(n-1)$. We have the following results [9], [10]:

Lemma 2.2.
(a) *If $g \in C(I)$, then there exists a constant C independent of ϵ and δ such that*

$$\|J_\delta G^\epsilon - g\|_{\infty, I_\delta} \le C\left(\epsilon + \delta + \Delta t\right).$$

(b) *If $g \in C^2(I)$, then there exists a constant C independent of ϵ and δ such that*

$$\|D_*^\alpha\left(J_\delta G^\epsilon\right) - D_*^\alpha g\|_{\infty, I_\delta} \le C\left(\delta + \frac{\epsilon}{\delta^r} + \Delta t\right).$$

where $r = 1$ if $\alpha \in (0,1)$ and $r = 2$ if $\alpha \in (1,2)$. Moreover, there exists a constant C_δ independent of ϵ and Δt such that

$$\|\boldsymbol{D}^2(J_\delta G^\epsilon) - g''\|_{\infty, I_\delta} \le C\left(\delta + \frac{\epsilon + \Delta t}{\delta^2}\right) + C_\delta\left(\Delta t\right)^2,$$

where $\boldsymbol{D}^2$ denotes the centered difference approximation for the second derivative.

2.2. Error analysis

As indicated in the introduction of this paper, our strategy for stabilizing the source term problem is to replace the noisy data y^ϵ and its discrete version Y^ϵ by their mollifications $y^{\epsilon,\delta} = J_\delta y^\epsilon$ and $Y^{\epsilon,\delta} = J_\delta Y^\epsilon$. Then the continuous and discrete approximations to the source term f, denoted by $f^{\epsilon,\delta}$ and $F^{\epsilon,\delta}$, are computed respectively as

$$f^{\epsilon,\delta}(t) = aD^2 y^{\epsilon,\delta}(t) + bD_*^\alpha y^{\epsilon,\delta}(t) + cy^{\epsilon,\delta}(t), \quad F^{\epsilon,\delta} = a\mathbf{D}^2 Y^{\epsilon,\delta} + bD_*^\alpha Y^{\epsilon,\delta} + cY^{\epsilon,\delta}.$$

Assume $\|y^\epsilon - y\|_{\infty, I} \le \epsilon$, then we have the following main result:

Theorem 2.1. *If $y \in C^2(I)$, then there exists constant C independent of ϵ and δ such that*

$$\|f^{\epsilon,\delta} - f\|_{\infty, I_\delta} \le C\left(\delta + \epsilon + \frac{\epsilon}{\delta^2}\right),$$

$$\|F^{\epsilon,\delta} - f\|_{\infty, I_\delta} \le C\left(\delta + \epsilon + \Delta t + \frac{\epsilon + \Delta t}{\delta^2}\right) + C_\delta(\Delta t)^2.$$

Proof. Using Lemma 2.1 and the Triangle inequality, we have

$$\begin{aligned}\|f^{\epsilon,\delta}-f\|_{\infty,I_\delta} &\leq |a| \left\|D^2y^{\epsilon,\delta}-D^2y\right\|_{\infty,I_\delta} + |b| \left\|D^\alpha y^{\epsilon,\delta}-D^\alpha y\right\|_{\infty,I_\delta} + |c| \left\|y^{\epsilon,\delta}-y\right\|_{\infty,I_\delta} \\ &\leq C\left(\delta+\epsilon+\frac{\epsilon}{\delta^2}\right).\end{aligned}$$

The proof of the second bound is similar but using Lemma 2.2 instead. ■

Remark 2.1.
(i) If δ is chosen so that $\delta = O(\epsilon^\mu)$ for some $0 < \mu < 0.5$, then we obtain the convergence result $\|f - f^{\epsilon,\delta}\|_{\infty,I_\delta} \to 0$, as $\epsilon \to 0$.
(ii) If δ is chosen so that $\delta = O\left((\epsilon + \Delta t)^\mu\right)$ for some $0 < \mu < 0.5$, then we obtain the convergence result $\|F - F^{\epsilon,\delta}\|_{\infty,I_\delta} \to 0$, as $\epsilon, \Delta t \to 0$.

3. Numerical experiments

In this section, we present numerical examples to test the feasibility and validity of the proposed algorithm.

In the experiments below, we take $\Delta t = 0.01$. The noisy data Y^ϵ is computed according to the formula

$$Y_i^\epsilon = y(t_i) + \gamma u_i, \quad i = 1, \dots, 101,$$

where u_i is a uniformly distributed random number in $[-1, 1]$. Here the number γ determines the (percentage) noise level ϵ which is traditionally defined as

$$\epsilon = \frac{\|Y^\epsilon - Y\|}{\|Y\|} \times 100\%,$$

where $\|\cdot\|$ is the usual Euclidean norm. To assess the quality of the approximations, we use the relative root-mean-square error (RES) given by

$$\text{RES} = \frac{\sqrt{\sum_{i=1}^{101}\left[f(t_i) - f^{\delta,\epsilon}(t_i)\right]^2}}{\sqrt{\sum_{i=1}^{101}\left[f(t_i)\right]^2}},$$

which is the discrete version of the relative L^2-error. The mollification parameter δ is determined by the Principle of Generalized Cross Validation as described in [11], where as the discretization of the Caputo fractional derivative is computed using the algorithm described in [9].

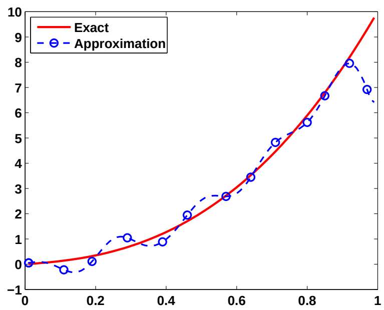
Example 3.2. Consider the semidifferential equation

$$D^2y(t) + 15\sqrt{\pi}D_*^{1/2}y(t) + 6y(t) = t^3 + t + 8t^{5/2}, \ y(0) = 0, \ y'(0) = 0.$$

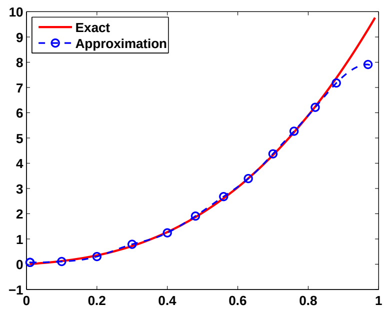
Here the solution is $y(t) = t^3/6$. The RES computed for different noise levels are presented in Table 1. Plots comparing the exact and approximated source term are shown in Figure 2.

Table 1: Errors measured by the RES for Example 3.2.

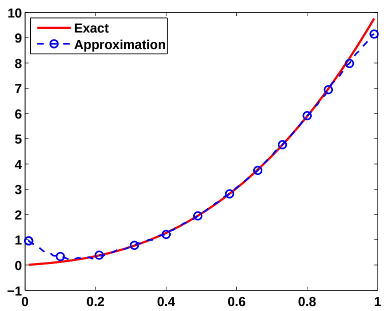
Noise Level	0.1	0.02	0.004	0.0008
RES	0.1509	0.0902	0.0526	0.0098



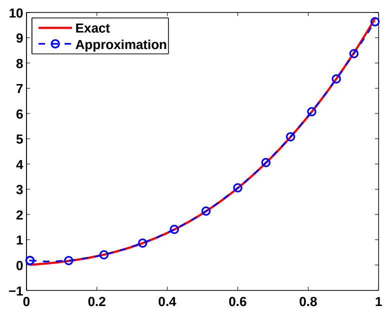
(a) $\epsilon = 0.1$



(b) $\epsilon = 0.02$



(c) $\epsilon = 0.004$



(d) $\epsilon = 0.0008$

Figure 2: Exact and approximated source term f in Example 3.2 for different noise levels.

Example 3.3. Consider the semidifferential equation

$$D^2 y(t) + 5D_*^{3/2} y(t) + 3y(t) = f(t), \;\; y(0) = -15/16, \; y'(0) = -1/2.$$

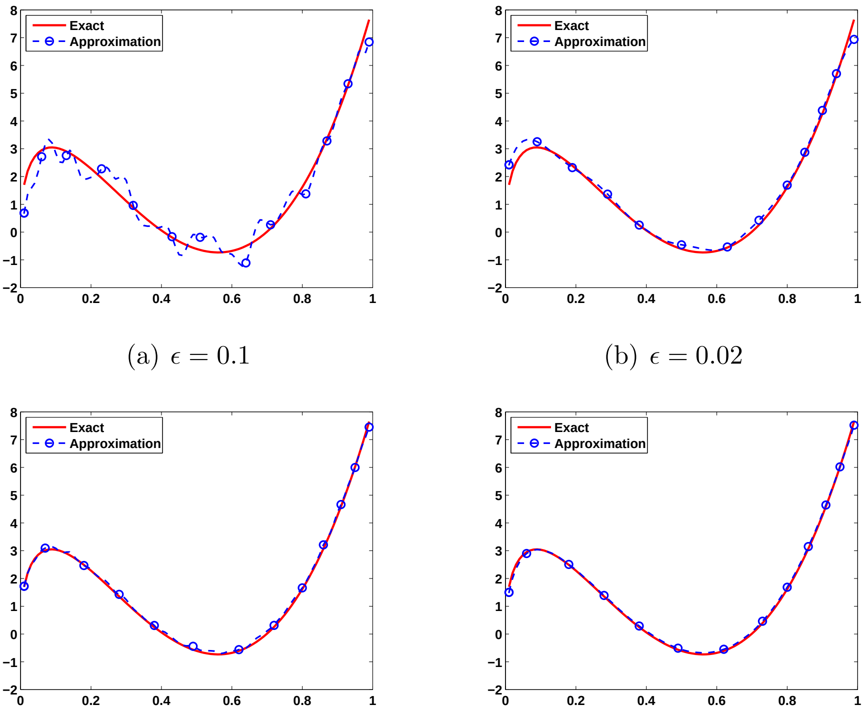
Here the solution is $y(t) = \left(t - \frac{1}{2}\right)^4 - 1$, whereas the exact source term is

$$f(t) = 3t^4 - 6t^3 + \frac{33t^2}{2} - \frac{27t}{2} + \frac{30\sqrt{t}}{\sqrt{\pi}} - \frac{80t^{3/2}}{\sqrt{\pi}} + \frac{64t^{5/2}}{\sqrt{\pi}} + \frac{3}{16}.$$

The RES computed for different noise levels are presented in Table 2. Plots comparing the exact and approximated source term are shown in Figure 3.

Table 2: Errors measured by the RES for Example 3.3.

Noise Level	0.1	0.02	0.004	0.0008
RES	0.1617	0.0735	0.0291	0.0256



(a) $\epsilon = 0.1$

(b) $\epsilon = 0.02$

(c) $\epsilon = 0.004$

(d) $\epsilon = 0.0008$

Figure 3: Exact and approximated source term f in Example 3.3 for different noise levels.

4. Conclusions and future work

We investigated the possibility of recovering the source term in semidifferential equations from noisy observed data. We proposed a regularization scheme based on the mollification method. We provided convergence results for both the continuous and discrete data. Our numerical experiments showed noteworthy results. We look forward to generalize our approach for systems of fractional differential equations; we defer these investigations for a sequel to this paper.

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