ON NEARLY CAP-EMBEDDED SUBGROUPS OF FINITE GROUPS

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Abstract. We introduce a new subgroup embedding property of a finite group called nearly CAP-embedded subgroup. Using this subgroup property, we determine the structure of finite groups with some nearly CAP-embedded subgroups of Sylow subgroups. Our results unify and generalize some recent theorems on p-nilpotency and supersolvability of finite groups.

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1. Introduction

In this paper, all groups considered are finite. Let $\pi(G)$ stand for the set of all prime divisors of the order of a group G. Let $\mathcal{F}$ denote a formation, $\mathcal{U}$ the class of supersolvable groups. $H\ Char\ G$ means that H is a characteristic subgroup of G. The other notations and terminology are standard (see[9]).

Let H be a subgroup of G, and A/B be a G-chief factor. We say that H covers A/B if $HA = HB$; and H avoids A/B if $H \cap A = H \cap B$. H is said to have cover-avoiding property in G, in brevity, H is a CAP-subgroup of G, if H either covers or avoids any G-chief factor. In 1962, Gaschütz[5] introduced a certain conjugacy class of subgroups of a solvable group called the pre-Frattini subgroups. These subgroups have cover-avoidance property. Thereafter, many

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authors devoted to find some kind of subgroups of a solvable group having this property, for example, Gillam[6] and Tomkinson[14]. In 1993, Ezquerro[4] considered the converse questions, he gave some characterizations for a group G to be p-supersolvable and supersolvable based on the assumption that all maximal subgroups of some subgroups of G are CAP-subgroups. Asaad in [1] obtained further results within the framework of formation theory. As a generalization of CAP-subgroups, Guo and Guo in[7] introduced CAP-embedded subgroups. A subgroup H of G is said to have the CAP-embedded property in G or is called a CAP-embedded subgroup of G if, for each prime p dividing the order of H, there exists a CAP subgroup K of G such that a Sylow p-subgroup of H is also a Sylow p-subgroup of K. Moreover, they presented some conditions for a finite group to be p-nilpotent and supersolvable under the condition that some subgroups of Sylow subgroup are CAP-embedded.

In recent years, it has been of interest to use some supplemented properties of subgroups to determine the structure of a group. For example, Wang in [15] introduced the concept of c-normal subgroups. A subgroup H of G is c-normal in G if there is a normal subgroup K_1 of G such that $G = HK_1$ and $H \cap K_1 \leq H_G = Core_G(H)$. As applications, he gave some criteria for the solvability and supersolvability of groups.

We provide examples in Section 2 to show that CAP-embedded property and c-normality cannot imply from one to the other one. In this paper, we will try an attempt to unify the two concepts and introduce a new subgroup embedding property of a finite group called nearly CAP-embedded subgroup. As applications, we study the influence of nearly CAP-embedded subgroups on the structure of finite groups. We present some sufficient conditions for a group to be p-nilpotent, p-supersolvable and supersolvable.

2. Basic definitions and preliminary results

When we recall the concepts of a $c-$normal subgroup and a CAP-embedded subgroup, it is easy to see that a normal subgroup N of G is both c-normal and CAP-embedded. The following examples show that c-normal and CAP-embedded are different properties:

Example 2.1. Let $G = A_5$, the alternative group of degree 5. Then all Sylow subgroups of G are CAP-embedded subgroups of G, but every Sylow subgroup is not a c-normal subgroup of G.

Example 2.2. Let A_4 be the alternative group of degree 4 and $C = \langle c \rangle$ be a cyclic group of order 2. Let $G = C \times A_4$. Then $A_4 = [K_4]C_3$, where $K_4 = \langle a, b \rangle$ is the Klein Four Group with generators a and b of order 2 and C_3 is the cyclic group of order 3. Take $H = \langle ac \rangle$ be the cyclic subgroup of order 2 of G. Then $G = HA_4$ and $H \cap A_4 = 1$. By definition, H is c-normal in G. However, H is not a CAP-embedded subgroup of G, if not, then there exists a CAP-subgroup B of G such that $H \in Syl_2(B)$, so B covers or avoids $(C \times K_4)/C$, it is impossible.

In the c-normal case, $G = HK_1$, if we let $K_2 = H_G K_1$, then $G = HK_2$ and $H \cap K_2 = H_G$; $H \cap K_2$ is, of course, a CAP-embedded subgroup of G. Based on the observation, we introduce the following:

Definition 2.3. A subgroup H of a group G is said to be nearly CAP-embedded in G if there are a subnormal subgroup T of G and a CAP-embedded subgroup H_{ce} of G contained in H such that $G = HT$ and $H \cap T \leq H_{ce}$.

If H is a CAP-embedded subgroup of G, taking $T = G$, we get H is a nearly CAP-embedded subgroup of G. Hence nearly CAP-embedded subgroup is a real uniform generalization of a c-normal subgroup and a CAP-embedded subgroup.

For the sake of convenience, we list here some known results which will be useful in the sequel.

Lemma 2.4 (([7, Lemma 1])). *Suppose that U is CAP-embedded in a group G and $N \trianglelefteq G$. Then UN/N is CAP-embedded in G/N.*

Lemma 2.5. ([19, Lemma 2.4]) *Let H be a normal subgroup of a group G such that G/H is p-nilpotent and let P be a Sylow p-subgroup of H, where p is a prime divisor of $|G|$. If $|P| \leq p^2$ and one of the following conditions holds, then G is p-nilpotent:*

(1) $(|G|, p-1) = 1$ *and* $|P| \leq p$;

(2) *G is A_4-free if $p = min\pi(G)$;*

(3) $(|G|, p^2 - 1) = 1$.

Lemma 2.6. ([20, Theorem 3.1]) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, and G a group with a normal subgroup N such that $G/N \in \mathcal{F}$. If all Sylow subgroups of $F^*(N)$ are cyclic, then $G \in \mathcal{F}$.*

Lemma 2.7. ([17, Theorem 3.1]) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, G a group with a solvable normal subgroup H such that $G/H \in \mathcal{F}$. If for any maximal subgroup M of G, either $F(H) \leq M$ or $F(H) \cap M$ is a maximal subgroup of $F(H)$, then $G \in \mathcal{F}$. The converse also holds, in the case where $\mathcal{F} = \mathcal{U}$.*

Lemma 2.8. *Let U be a nearly CAP-embedded subgroup and N a normal subgroup of a group G. Then*

(1) *If $N \leq U$, then U/N is nearly CAP-embedded in G/N.*

(2) *If $(|U|, |N|) = 1$, then UN/N is nearly CAP-embedded in G/N.*

Proof. By the hypotheses, there are a subnormal subgroup T of G and a CAP-embedded subgroup U_{ce} of G contained in U such that $G = UT$ and $U \cap T \leq U_{ce}$.

(1) $G/N = (U/N)(TN/N)$, $TN/N \lhd\lhd G/N$ by [3, Chap A, Lemma 14.1(b)], and $(U/N) \cap (TN/N) = (U \cap TN)/N = (U \cap T)N/N \leq (U_{ce}N)/N$. By Lemma 2.4, $(U_{ce}N)/N$ is CAP-embedded in G/N. Hence U/N is nearly CAP-embedded in G/N.

(2) Let π be the set of all prime divisors of $|U|$, then N is a normal π'-subgroup and U is a π-subgroup of G. Since $|G|_{\pi'} = |T|_{\pi'} = |TN|_{\pi'}$, we have that $|T \cap N| =$

$|T \cap N|_{\pi'} = |N|_{\pi'} = |N|$ and hence $N \le T$. Therefore, $G/N = (UN/N)(T/N)$, $T/N \lhd\lhd G/N$ by [3, Ch. A, Lemma 14.1(b)], and $(UN/N) \cap T/N = (U \cap T)N/N \le (U_{ce}N)/N$. By Lemma 2.4, we have $(U_{ce}N)/N$ is CAP-embedded in G/N. Hence, $(UN)/N$ is nearly CAP-embedded in G/N. ∎

3. Main results and their proofs

Theorem 3.1. *Let G be a group, N a normal subgroup of G such that G/N is p-nilpotent and P a Sylow p-subgroup of N, where $p \in \pi(G)$ with $(|G|, p-1) = 1$. If all maximal subgroups of P are nearly CAP-embedded in G, then G is p-nilpotent.*

Proof. Assume that the result is false. Let G be a minimal counterexample with least $|N| + |G|$.

(1) G has a unique minimal normal subgroup L contained in N, G/L is p-nilpotent and $L \nleq \Phi(G)$.

Let L be a minimal normal subgroup of G contained in N. Consider the factor group $\overline{G} = G/L$. Clearly $\overline{G}/\overline{N} \cong G/N$ is p-nilpotent and $\overline{P} = PL/L$ is a Sylow p-subgroup of $\overline{N}$, where $\overline{N} = N/L$. Now let $\overline{P_1} = P_1L/L$ be a maximal subgroup of $\overline{P}$. We may assume that P_1 is a maximal subgroup of P. Then $P_1 \cap L = P \cap L$ is a Sylow p-subgroup of L. By the hypothesis, there are a subnormal subgroup T of G and a CAP-embedded subgroup $(P_1)_{ce}$ contained in P_1 of G such that $G = P_1T$ and $P_1 \cap T \le (P_1)_{ce}$. Clearly $TL/L \lhd \lhd G/L$. Now we let $\pi(G) = \{p_1, p_2, \cdots p_n\}$ where $p_1 = p$, and T_{p_i} be a Sylow p_i-subgroup of T $(i = 2, \cdots, n)$. Then T_{p_i} is also a Sylow p_i-subgroup of G, hence $T_{p_i} \cap L$ is a Sylow p_i-subgroup of L $(i = 2, \cdots, n)$. Write $V = \langle L \cap T_{p_2}, \cdots, L \cap T_{p_n} \rangle$, then $V \le T \cap L$. Note that $(|L : P_1 \cap L|, |L : V|) = 1$, $L = (P_1 \cap L)V$, thus $P_1L \cap TL = (P_1L \cap T)L = (P_1V \cap T)L = (P_1 \cap T)VL = (P_1 \cap T)L$. It follows from Lemma 2.4 that $(P_1L/L) \cap (TL/L) = (P_1 \cap T)L/L \le (P_1)_{ce}L/L$ and $(P_1)_{ce}L/L$ is CAP-embedded in G/L. Therefore $\overline{P_1}$ is nearly CAP-embedded in $\overline{G}$. The choice of G implies that $\overline{G}$ is p-nilpotent. Since the class of p-nilpotent groups is a saturated formation, L is a unique minimal normal subgroup of G contained in N and $L \nleq \Phi(G)$.

(2) $O_{p'}(G) = 1$.

If $E = O_{p'}(G) \neq 1$, we consider $\overline{G} = G/E$. Clearly, $\overline{G}/\overline{N} \cong G/NE$ is p-nilpotent because G/N is, where $\overline{N} = NE/E$. Let $\overline{P_1} = P_1E/E$ be a maximal subgroup of PE/E. We may assume that P_1 is a maximal subgroup of P. Since P_1 is nearly CAP-embedded in G, P_1E/E is nearly CAP-embedded in G/E by Lemma 2.8 (2). The minimality of G yields that G is p-nilpotent, therefore G is p-nilpotent, a contradiction.

(3) $O_p(N) = 1$ and so L is not p-nilpotent.

If not, then by (1), $L \le O_p(N)$ and, there is a maximal subgroup M of G such that $G = LM$ and $L \cap M = 1$. Since $M_p < P$, where $M_p \in Syl_p(M)$, we may let P_1 be a maximal subgroup of P containing M_p. Because P_1 is a nearly CAP-embedded subgroup of G, there are a subnormal subgroup T of G and a CAP-embedded subgroup $(P_1)_{ce}$ contained in P_1 of G such that $G = P_1T$ and $P_1 \cap T \le (P_1)_{ce} \in Syl_p(K)$, where K is a CAP subgroup of G. If K covers

$L/1$, then $L \leq K$. It follows from $(P_1)_{ce} \in Syl_p(K)$ that $L \leq P_1$, thus $P = LM_p = LP_1 = P_1$, a contradiction. So K must avoids $L/1$, i.e., $K \cap L = 1$, hence $P_1 \cap T \cap L = 1$. Consequently, $|T \cap L| \leq p$. Since $T/L \cap T \cong TL/L \leq G/L$, $T/L \cap T$ is p-nilpotent. It follows that T is p-nilpotent by Lemma 2.5. Let $T_{p'}$ be the normal p-complement of T. Then $T_{p'}$ is a Hall p'-subgroup of G and $T_{p'}$ $Char$ $T \trianglelefteq \trianglelefteq G$, so $T_{p'} \trianglelefteq G$, contrary to $O_{p'}(G) = 1$.

If L is p-nilpotent, then $L_{p'}$ $Char$ $L \trianglelefteq N$, so $L_{p'} \leq O_{p'}(N) \leq O_{p'}(G) = 1$ by (2). Thus L is a p-group, $L \leq O_p(N) = 1$, a contradiction. Hence (3) holds.

(4) The final contradiction.

If $P \cap L \leq \Phi(P)$, then L is p-nilpotent by Tate's theorem [9, IV, Th 4.7], contrary to (3). Consequently, there exists a maximal subgroup P_1 of P such that $P = (L \cap P)P_1$. Let T be a subnormal supplement of P_1 in G, we have $P_1 \cap T \leq (P_1)_{ce} \in Syl_p(K)$, where K is a CAP subgroup of G. If K covers $L/1$, then $L \leq K$. It follows from $(P_1)_{ce} \in Syl_p(K)$ that $P_1 \cap K = (P_1)_{ce} \in Syl_p(K)$, then $P_1 \cap L \in Syl_p(L)$. Thus $L \cap P = L \cap P_1$. We obtain $P = (L \cap P)P_1 = P_1$, a contradiction. So K must avoids $L/1$, i.e., $K \cap L = 1$, hence $P_1 \cap T \cap L = 1$. Consequently, $|P \cap T \cap L| \leq p$. Since $T/L \cap T \cong TL/L \leq G/L$, $T/L \cap T$ is p-nilpotent. It follows that T is p-nilpotent by Lemma 2.5. Let $T_{p'}$ be the normal p-complement of T. Then $T_{p'}$ is a Hall p'-subgroup of G and $T_{p'}$ $Char$ $T \trianglelefteq \trianglelefteq G$, so $T_{p'} \trianglelefteq G$, contrary to $O_{p'}(G) = 1$. This contradiction completes the proof. ∎

Theorem 3.2. *Let p be a prime dividing the order of the group G and let N be a p-solvable normal subgroup of G such that G/N is p-supersolvable. If there exists a Sylow p-subgroup P of N such that every maximal subgroup of P is nearly CAP-embedded in G, then G is p-supersolvable.*

Proof. Assume that the result is false and let G be a counterexample of minimal order. Now, arguing as in the proof of Theorem 3.1, the following statements (1) and (2) about G are true.

(1) G has a unique minimal normal subgroup L contained in N, G/L is p-supersolvable and $L \nleq \Phi(G)$.

(2) $O_{p'}(G) = 1$.

Since G is p-solvable and $O_{p'}(G) = 1$, L is a p-group and $L \leq P$. If $L \leq \Phi(P)$, by [12, Theorem 5.2.13], $L \leq \Phi(G)$, a contradiction. Consequently, there exists a maximal subgroup P_1 of P such that $P_1L = P$. Since P_1 is a nearly CAP-embedded subgroup of G, there are a subnormal T of G and a CAP-embedded subgroup $(P_1)_{ce}$ of G such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{ce} \in Syl_p(K)$, where K is a CAP subgroup of G. If K covers $L/1$, then $L \leq K$. It follows from $(P_1)_{ce} \in Syl_p(K)$ that $L \leq P_1$, thus $P = LP_1 = P_1$, a contradiction. So K must avoids $L/1$, i.e., $K \cap L = 1$, hence $P_1 \cap T \cap L = 1$. Consequently, $|T \cap L| \leq p$. Noting that G/T_G is p-group, so $N \cap T_G \neq 1$. If not, then $G = G/N \cap T_G \lesssim G/N \times G/T_G$ is p-supersolvable, a contradiction. So $L \leq N \cap T_G$ by (1). Hence $|L| = |T \cap L| = p$. The p-supersolvability of G/L implies that G is p-supersolvable, final contradiction. ∎

Remark 3.3. The hypothesis that N is p-solvable in Theorem 3.2 is essential. For example, if we let G be the alternating group A_5 of degree 5, $N = G$ and $p = 3$, then it is clear that the statement of Theorem 3.2 does not hold.

Theorem 3.4. *Let G be a group. Then G is supersolvable if and only if there exists a normal subgroup N such that G/N is supersolvable and all maximal subgroups of any Sylow subgroup of N are nearly CAP-embedded in G.*

Proof. The necessity part can be obtained if we let $N = G$ and apply a result due to Ezquerro[4]. So we need to prove the sufficiency part.

Let p be the smallest prime divisor of $|G|$. The supersolvability of G/N implies that G/N is p-nilpotent. By Theorem 3.1, G is p-nilpotent. Furthermore G is solvable. Applying Theorem 3.2, it is easy to see that G is supersolvable. ∎

Theorem 3.5. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup N such that $G/N \in \mathcal{F}$. If all maximal subgroups of any Sylow subgroup of N are nearly CAP-embedded in G, then $G \in \mathcal{F}$.*

Proof. Let G be a minimal counterexample. With similar arguments as in the proof of Theorem 3.1, we have the following claim (1).

(1) G has a unique minimal normal subgroup L contained in N such that $G/L \in \mathcal{F}$ and $L \nleq \Phi(G)$.

(2) L is an elementary abelian p-group for some prime p.

Let q be the smallest prime divisor of $|N|$, Q a Sylow q-subgroup of N. If $Q \cap L \nleq \Phi(Q)$, then there exists a maximal subgroup Q_1 of Q such that $Q = (Q \cap L)Q_1$. By the hypotheses, there are a subnormal T of G and a CAP-embedded subgroup $(Q_1)_{ce}$ of G such that $G = Q_1T$ and $Q_1 \cap T \leq (Q_1)_{ce} \in Syl_q(K)$, where K is a CAP subgroup of G. If K covers $L/1$, then $L \leq K$. It follows from $(Q_1)_{ce} \in Syl_q(K)$ that $L \cap Q_1 = L \cap Q$, thus $Q = (Q \cap L)Q_1 = (Q_1 \cap L)Q_1 = Q_1$, a contradiction. So K must avoids $L/1$, i.e., $K \cap L = 1$, hence $Q_1 \cap T \cap L = 1$. Consequently, $|T \cap L| \leq q$. Noting that G/T_G is q-group, so $N \cap T_G \neq 1$. If not, then $G = G/N \cap T_G \lesssim G/N \times G/T_G$ belongs to $\mathcal{F}$, a contradiction. So $L \leq N \cap T_G$ by (1). Hence $|L| = |T \cap L| = q$. By applying Lemma 2.6, we obtain $G \in \mathcal{F}$, a contradiction. Therefore, $Q \cap L \leq \Phi(Q)$, then L is q-nilpotent by Tate's theorem [9, IV, Th 4.7] and, by the Odd Order Theorem, L is solvable, statement (2) is true.

(3) A final contradiction.

From (1) and (2), there exists a maximal subgroup M of G such that $G = LM$ and $L \cap M = 1$. Let P be a Sylow p-subgroup of N. Then $P = LM_p$ where $M_p \in Syl_p(G)$. Since $M_p < P$, we may let P_1 be a maximal subgroup of P such that $M_p \leq P_1$. By the hypotheses, there are a subnormal T of G and a CAP-embedded subgroup $(P_1)_{ce}$ of G such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{ce} \in Syl_p(K)$, where K is a CAP subgroup of G. If K covers $L/1$, then $L \leq K$. It follows from $(P_1)_{ce} \in Syl_p(K)$ that $L \leq P_1$, thus $P = LM_p \leq P_1$, a contradiction. So K must avoids $L/1$, i.e., $K \cap L = 1$, hence $P_1 \cap T \cap L = 1$. Consequently, $|T \cap L| \leq p$. Noting that G/T_G is p-group, so $N \cap T_G \neq 1$. If not, then $G = G/N \cap T_G \lesssim G/N \times G/T_G$

belongs to $\mathcal{F}$, a contradiction. So $L \leq N \cap T_G$ by (1). Hence $|L| = |T \cap L| = q$. By applying Lemma 2.6, we obtain $G \in \mathcal{F}$, final contradiction. We are done. ∎

Theorem 3.6. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let N be a solvable normal subgroup of G such that $G/N \in \mathcal{F}$. If all maximal subgroups of any Sylow subgroup of $F(N)$ are nearly CAP-embedded subgroups of G, then $G \in \mathcal{F}$.*

Proof. Assume that the result is false and let G be a counterexample of minimal order. First we have $\Phi(G) = 1$. Suppose that $\Phi(G) \neq 1$ and take a prime p dividing $|\Phi(G)|$. Denote $D = O_p(\Phi(G)) \neq 1$. Clearly $D \trianglelefteq G$. Let $F(ND/D) = L/D$. By L/D $Char$ $ND/D \trianglelefteq G/D$, $L/D \trianglelefteq G/D$. Hence $L \trianglelefteq G$. Since L/D is a normal nilpotent subgroup of G/D and $D \leq \Phi(G)$, applying a result due to Gaschütz[9, III, Theorem 3.5] , we have that L is a normal nilpotent subgroup of ND. Thus $L \leq F(ND)$. Consequently $F(ND/D) = F(ND)/D = L/D$. By [2, Lemma 3.1], $F(ND/D) = F(N)D/D$. It is clear that $(G/D)/(ND/D) \cong G/ND \cong (G/N)/(ND/N)$ belongs to $\mathcal{F}$. Now, by Lemma 2.8(1), the hypotheses of the theorem hold in G/D. By the minimality of G, $G/D \in \mathcal{F}$. Since $\mathcal{F}$ is saturated, $G \in \mathcal{F}$, a contradiction. We obtain $\Phi(N) \leq \Phi(G) = 1$. Let M be a maximal subgroup of G such that $F(N) \nleq M$. Then there exists a prime p such that $O_p(N) \nleq M$. It follows that $G = O_p(N)M$. Clearly, $O_p(N) \cap M < O_p(N)$, so we may take a maximal subgroup P_1 of $O_p(N)$ containing $O_p(N) \cap M$. Then $P_1 \cap M = O_p(N) \cap M \trianglelefteq G$, therefore $P_1 \cap M \leq (P_1)_G$. If $(P_1)_G M = G$, then $O_p(N) = O_p(N) \cap (P_1)_G M = (P_1)_G(O_p(N) \cap M) = (P_1)_G$, a contradiction. Thus $(P_1)_G M < G$, so $(P_1)_G \leq O_p(N) \cap M$ and $P_1 \cap M = O_p(N) \cap M = (P_1)_G$. Let $O_p(N)/K$ be a chief factor of G with $O_p(N) \cap M \leq K$. Then $O_p(N) \cap M = K \cap M$. If $KM = G$, then $O_p(N) = O_p(N) \cap KM = K(O_p(N) \cap M) = K$, a contradiction. Thus $KM < G$, so $K \leq M$ and $K = O_p(N) \cap M = (P_1)_G$. Since P_1 is a nearly CAP-embedded subgroup of G, there are a subnormal T of G and a CAP-embedded subgroup $(P_1)_{ce}$ of G such that $G = P_1 T$ and $P_1 \cap T \leq (P_1)_{ce} \in Syl_p(B)$, where B is a CAP subgroup of G. Clearly $(P_1)_G(O_p(N) \cap T)$ is normal in G. From the fact that $O_p(N)/(P_1)_G$ is a G-chief factor, we know that either $(P_1)_G = (P_1)_G(O_p(N) \cap T)$ or $(P_1)_G(O_p(N) \cap T) = O_p(N)$. If the former holds, then $O_p(N) \cap T \leq (P_1)_G$. Furthermore, $O_p(N) \cap T = P_1 \cap T$ and $O_p(N) = P_1$ as $P_1 T = O_p(N)T = G$, a contradiction. So $(P_1)_G(O_p(N) \cap T) = O_p(N)$, we obtain $O_p(N) \leq (P_1)_G T$. Thus $G = P_1 T = (P_1)_G T$. Noting that B is a CAP subgroup of G. If B covers $O_p(N)/(P_1)_G$, then $O_p(N) \leq B(P_1)_G$. It follows from $(P_1)_{ce} \in Syl_p(B)$ that $O_p(N) \leq P_1$, a contradiction. So B must avoids $O_p(N)/(P_1)_G$, i.e., $(P_1)_{ce} = B \cap O_p(N) = B \cap (P_1)_G$, hence $(P_1)_{ce} \leq (P_1)_G$. Consequently $(P_1)_G \cap T = P_1 \cap T$, we have $P_1 = (P_1)_G = O_p(N) \cap M$. Therefore $|G : M| = |O_p(N) : O_p(N) \cap M| = p$. By Lemma 2.7, we get $G \in \mathcal{F}$, a final contradiction. ∎

Remark 3.7. The hypothesis that N is solvable in Theorem 3.6 cannot be removed. For example, if we let $G = SL(2, 5)$ and $N = G$, then $F(N)$ is a group of order 2. Thus all maximal subgroups of any Sylow subgroup of $F(N)$ have the nearly CAP-embedded property in G, but G is not supersolvable.

4. Some applications

Since many relevant families of subgroups, such as normal subgroups, c-normal subgroups, CAP subgroups, CAP-embedded subgroups and $c^\sharp$-normal subgroups, enjoy the nearly CAP-embedded property, a lot of nice results can be obtained according to our theorems.

Recall first the concept of $c^\sharp$-normal subgroups mentioned above. Let H be a subgroup of G. We call H a $c^\sharp$-normal subgroup of G if there exists a normal subgroup T of G such that $G = HT$ and $H \cap T$ is a CAP subgroup of G (see[16]).

Now, we here list special cases of our theorems which can be found in the literature.

Theorem 3.1 immediately implies:

Corollary 4.1. ([7, Theorem 3.1]) *Let p be a prime dividing the order of the group G with $(|G|, p-1) = 1$ and let H be a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that P is cyclic or every maximal subgroup of P is CAP-embedded in G, then G is p-nilpotent.*

Proof. If P is a cyclic group, by [9, p. 420, Theorem 2.8], we have G is p-nilpotent. So every maximal subgroup of P has the CAP-embedded property in G. Hence G is p-nilpotent by Theorem 3.1. ∎

Corollary 4.2. ([8, Theorem 3.4]) *Let p be the smallest prime number dividing the order of a group G and P a Sylow p-subgroup of G. If every maximal subgroup of P is c-normal in G, then G is p-nilpotent.*

Corollary 4.3. ([16, Theorem 3.1]) *Let G be a group, H a normal subgroup of G such that G/H is p-nilpotent and P a Sylow p-subgroup of H, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If all maximal subgroups of P are $c^\sharp$-normal in G, then G is p-nilpotent. In particular, G is p-supersolvable.*

From Theorem 3.2 we obtain:

Corollary 4.4. ([7, Theorem 4.1]) *Let p be a prime dividing the order of the group G and let H be a p-solvable normal subgroup of G such that G/H is p-supersolvable. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is CAP-embedded in G, then G is p-supersolvable.*

Corollary 4.5. ([16, Theorem 3.4]) *Let G be a p-solvable group, H a normal subgroup of G such that G/H is p-supersolvable and P a Sylow p-subgroup of H, where p is a prime. If all maximal subgroups of P are $c^\sharp$-normal in G, then G is p-supersolvable.*

By Theorem 3.5 we have:

Corollary 4.6. ([13, Theorem 1]) *If the maximal subgroups of the Sylow subgroups of G are normal in G, then G is supersoluble.*

Corollary 4.7. ([11, Theorem 3.5]) *Assume that G/H is supersolvable and all maximal subgroups of the Sylow subgroups of H are normal in G. Then G is supersolvable.*

Corollary 4.8. ([15, Theorem 4.1]) *If the maximal subgroups of the Sylow subgroups of G are c-normal in G, then G is supersoluble.*

Corollary 4.9. ([16, Theorem 4.1]) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of any Sylow subgroup of H are $c^\sharp$-normal in G, then $G \in \mathcal{F}$.*

As immediate corollaries of Theorem 3.6, we have the following:

Corollary 4.10. ([11, Theorem 3.1]) *Assume that G is solvable and every maximal subgroup of the Sylow subgroups of $F(G)$ is normal in G. Then G is supersolvable.*

Corollary 4.11. [7, Theorem 4.3] *Let G be a group. Then G is supersolvable if and only if there exists a solvable normal subgroup H such that G/H is supersolvable and all maximal subgroups of any Sylow subgroup of $F(H)$ have the CAP-embedded property in G.*

Corollary 4.12. ([10, Theorem 2]) *Let G be a group and E a soluble normal subgroup of G such that G/E is supersoluble. If all maximal subgroups of the Sylow subgroups of $F(E)$ are c-normal in G, then G is supersoluble.*

Corollary 4.13. [1, Theorem 4.4] *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a solvable group with a normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are CAP-subgroups of G, then $G \in \mathcal{F}$*

Corollary 4.14. ([18, Theorem 1]) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a soluble normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are c-normal in G, then $G \in \mathcal{F}$.*

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